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BÄCKLUND TRANSFORMATIONS FOR SYSTEMS OF CONSERVATION LAWS*

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Abstract. A class of Bäcklund transformations is introduced for systems of conservation laws. Applications are given in non-steady gasdynamics and two-phase filtration through a porous medium.

1. Introduction. Haar [1], in a paper on adjoint variational problems, presented a remarkable invariance property of the two-dimensional gasdynamics equations. Subsequently, Bateman [2] constructed an associated but less restrictive class of invariant transformations later to become known as reciprocal relations [3, 4]. The application of the latter in the approximation of subsonic gas flows was first noted by Tsien [5].

That the adjoint and reciprocal transformations are both of the Bäcklund-type may be readily seen in terms of the Martin formulation of the gasdynamics equations [6, 7]. Alternatively, both kinds of transformation may be shown to lie within a class of matrix Bäcklund transformations, linear in a hodograph plane, introduced by Loewner [8, 9].

Since the work of Bateman, numerous papers on such Bäcklund transformations have appeared with developments in both steady and non-steady gasdynamics and non-dissipative magnetogasdynamics [10–17]. This research has been surveyed in a recent monograph on Bäcklund transformations [18].

Here, a class of Bäcklund transformations is introduced for systems of conservation laws. Applications are given in non-steady gasdynamics and two phase filtration in a porous medium.

2. A class of Bäcklund transformations. Bäcklund transformations are investigated for systems of conservation laws given by

$$d\mathbf{\Phi} = P(\partial/\partial\mathbf{X};\, \boldsymbol{\xi})\, d\mathbf{X}, \tag{2.1}$$

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where

$$d\mathbf{\Phi} = [d\phi_1, \ldots, d\phi_n]^T, \qquad d\mathbf{X} = [dx_1, dx_2]^T,$$

$$\boldsymbol{\xi} = [\xi_1, \ldots, \xi_m]^T, \ \frac{\partial}{\partial \mathbf{X}} = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]^T, \tag{2.2}$$

and P is an $n \times 2$ matrix whose components are functions of m dependent variables ξ_i, $i = 1, \ldots, m$, together with their derivatives with respect to the two independent variables $x_j, j = 1, 2$.

Thus, transformations of the system (2.1) are introduced in the form

$$d\mathbf{X}' = Ad\mathbf{X} + Bd\mathbf{\Phi}, \tag{2.3}$$

$$d\mathbf{\Phi}' = Cd\mathbf{X} + Dd\mathbf{\Phi}, \tag{2.4}$$

wherein A, B are 2×2 and $2 \times n$ constant matrices respectively, while C, D are, in turn, $n \times 2$ and $n \times n$ constant matrices. Combination of (2.1) and (2.3)–(2.4) now yields

$$d\mathbf{X}' = (A + BP)d\mathbf{X} \tag{2.5}$$

and

$$d\mathbf{\Phi}' = (C + DP)d\mathbf{X}, \tag{2.6}$$

where the independence of the new variables x_i', $i = 1, 2$, requires that

$$|A + BP| \neq 0. \tag{2.7}$$

The relation (2.5) shows that

$$\partial/\partial\mathbf{X}' = (A + BP)^{T^{-1}} \partial/\partial\mathbf{X}, \tag{2.8}$$

while (2.3) and (2.6) combine to give

$$d\mathbf{\Phi}' = P'd\mathbf{X}', \tag{2.9}$$

where

$$P' = (C + DP)(A + BP)^{-1}. \tag{2.10}$$

If (2.9) represents a new system of conservation laws with new dependent variables $\boldsymbol{\xi}'$ of the $\mathbf{X}'$ then (2.10) gives its relationship with the original system (2.1) in the form

$$P'(\partial/\partial\mathbf{X}'; \boldsymbol{\xi}') = (C + DP(\partial/\partial\mathbf{X}; \boldsymbol{\xi}))(A + BP(\partial/\partial\mathbf{X}; \boldsymbol{\xi}))^{-1} \tag{2.11}$$

In particular, if invariance of the conservation laws is required then (2.8) together with (2.11) provide the condition

$$P\left((A + BP)^{T^{-1}} \partial/\partial\mathbf{X}; \boldsymbol{\xi}'\right) = (C + DP(\partial/\partial\mathbf{X}; \boldsymbol{\xi}))(A + BP(\partial/\partial\mathbf{X}; \boldsymbol{\xi}))^{-1} \tag{2.12}$$

and this functional matrix equation provides the relationship between the ξ_i and ξ_i' and their derivatives necessary for such invariance. On the other hand, it may be required to link the original system (2.1) with a target canonical form (2.9) with a specified $P'(\partial/\partial\mathbf{X}'; \boldsymbol{\xi}')$. Then (2.12) together with a prescribed $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-relation produces the $P(\partial/\partial\mathbf{X}, \boldsymbol{\xi}')$ for which the reduction to canonical form may be achieved.

3. Reciprocal relations.

THEOREM 1. The *reciprocal relations*

$$\begin{aligned} d\mathbf{X}' &= Ad\mathbf{X} + Bd\boldsymbol{\Phi}, \\ d\boldsymbol{\Phi}' &= Cd\mathbf{X} + Dd\boldsymbol{\Phi}, \end{aligned}$$

with

$$\begin{aligned} A^2 + BC &= I, & AB + BD &= 0, \\ CA + DC &= 0, & CB + D^2 &= I, \end{aligned}$$

transform each system of conservation laws

$$\begin{aligned} d\boldsymbol{\Phi} &= P(\partial/\partial\mathbf{X}; \xi)d\mathbf{X} \\ d\boldsymbol{\Phi}' &= P'(\partial/\partial\mathbf{X}'; \xi')d\mathbf{X}' \end{aligned}$$

into the other, where

$$P'(\partial/\partial\mathbf{X}'; \xi') = (C + DP(\partial/\partial\mathbf{X}; \xi))(A + BP(\partial/\partial\mathbf{X}; \xi))^{-1}.$$

Proof. We seek the conditions under which the transformation defined by (2.3)–(2.4) applied to the new system $d\boldsymbol{\Phi}' = P'd\mathbf{X}'$ produces the system $d\boldsymbol{\Phi}'' = P''d\mathbf{X}''$ where

$$d\mathbf{X}'' = d\mathbf{X}, \quad d\boldsymbol{\Phi}'' = d\boldsymbol{\Phi}, \quad P'' = P.$$

That is, a second application of the transformation reproduces the original system of conservation laws.

In this case,

$$d\mathbf{X}'' = Ad\mathbf{X}' + Bd\boldsymbol{\Phi}' = (A^2 + BC)d\mathbf{X} + (AB + DB)d\boldsymbol{\Phi} = d\mathbf{X}$$

if and only if

$$A^2 + BC = I, \tag{3.1}$$

$$AB + BD = 0 \tag{3.2}$$

Further,

$$d\boldsymbol{\Phi}'' = Cd\mathbf{X}' + Dd\boldsymbol{\Phi}' = (CA + DC)d\mathbf{X} + (CB + D^2)d\boldsymbol{\Phi} = d\boldsymbol{\Phi}$$

if and only if

$$CA + DC = 0, \tag{3.3}$$

$$CB + D^2 = I. \tag{3.4}$$

Finally, on use of the above conditions (3.1)–(3.4) it is seen that

$$\begin{aligned} P''(\partial/\partial\mathbf{X}''; \xi'') &= (C + DP'(\partial/\partial\mathbf{X}'; \xi'))(A + BP'(\partial/\partial\mathbf{X}'; \xi')) \\ &= \left(C + D(C + DP)(A + BP)^{-1}\right)\left(A + B(C + DP)(A + BP)^{-1}\right)^{-1} = P. \end{aligned}$$

Accordingly, the systems of conservation laws $d\boldsymbol{\Phi} = Pd\mathbf{X}$ and $d\boldsymbol{\Phi}' = P'd\mathbf{X}'$ are reciprocal under the above reciprocal relations. □

It is noted that the reciprocal relations have the additional reciprocal property that they may be rewritten as

$$d\mathbf{X} = A\,d\mathbf{X}' + B\,d\mathbf{\Phi}', \tag{3.5}$$

$$d\mathbf{\Phi} = C\,d\mathbf{X}' + D\,d\mathbf{\Phi}', \tag{3.6}$$

while the relation (2.10) reduces to

$$(A + BP') = (A + BP)^{-1}. \tag{3.7}$$

Accordingly, the relationship (2.8) yields

$$\partial/\partial\mathbf{X} = (A + BP')^{T^{-1}}\partial/\partial\mathbf{X}' \tag{3.8}$$

and this expression allows the functional form of P' to be determined from (2.11).

4. Invariant transformations in non-steady gasdynamics. The governing equations of inviscid, one-dimensional, non-steady homentropic gasdynamics in the absence of heat conduction and radiation are

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho u^2) &= 0, \end{aligned} \tag{4.1}$$

together with an appropriate equation of state which relates the gas pressure p and the gas density ρ. In (4.1), $u(x, t)$ represents gas speed.

The system (4.1) may be reformulated as

$$d\mathbf{\Phi} = P(\xi)\,d\mathbf{X}, \tag{4.2}$$

where

$$\begin{aligned} d\mathbf{\Phi} &= [d\phi_1, d\phi_2]^T, \\ \mathbf{X} &= [x, t]^T, \qquad \xi = [\rho, u, p]^T, \\ P &= \begin{pmatrix} \rho & -\rho u \\ \rho u & -(p + \rho u^2) \end{pmatrix}. \end{aligned} \tag{4.3}$$

Particular Bäcklund transformations of the type described in Theorem 1 may now be applied to the system (4.2)–(4.3) to generate invariance properties of the gasdynamics system (4.1). Certain such invariant transformations are presented below:

(i) *The Movsesian transformation*. The specializations

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.4}$$

in (2.10) lead to

$$\begin{pmatrix} \rho' & -\rho' u' \\ \rho' u' & -(p' + \rho' u'^2) \end{pmatrix} = \begin{pmatrix} \rho & -\rho u \\ \rho u & -(p + \rho u^2) \end{pmatrix} \begin{pmatrix} \dfrac{1}{1 + k\rho} & \dfrac{k\rho u}{1 + k\rho u} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\rho}{1 + k\rho} & \dfrac{-\rho u}{1 + k\rho u} \\ \dfrac{\rho u}{(1 + k\rho)} & -\rho - \dfrac{\rho u^2}{(1 + k\rho)} \end{pmatrix}$$

whence,

$$\rho' = \frac{\rho}{1 + k\rho}, \quad u' = u, \quad p' = p. \tag{4.5}$$

The relations (4.5) together with the $(x, t) \to (x', t')$ transformation given by

$$dx' = (1 + k\rho)dx - k\rho u dt, \tag{4.6}$$

$$dt' = dt, \tag{4.7}$$

$$0 < |1 + k\rho| < \infty,$$

is of a type originally introduced by Movsesian [19] who exploited such mappings in the analysis of the flow of a compressible gas behind a piston. It is noted that the Movsesian transformation is non-reciprocal.

(ii) *The Ustinov transformation*. In this case, the system of conservation laws for the gasdynamics system is written in the alternative form

$$d\mathbf{\Phi} = \Pi(\xi) d\mathbf{X}, \tag{4.8}$$

where

$$\Pi = \begin{pmatrix} \rho u & -(p + \rho u^2) \\ \rho & -\rho u \end{pmatrix}. \tag{4.9}$$

The specializations

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

in (2.10) lead to

$$\begin{pmatrix} \rho' & -\rho' u' \\ \rho' u' & -(p' + \rho' u'^2) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{p} & -\dfrac{u}{p} \\ \dfrac{u}{p} & \dfrac{-(p + \rho u^2)}{p\rho} \end{pmatrix}$$

so that

$$\rho' = \frac{1}{p}, \quad u' = u, \quad p' = \frac{1}{\rho}. \tag{4.10}$$

These relations, together with the $(x, t) \to (x', t')$-transformation given by

$$dx' = \rho u dx - (p + \rho u^2) dt, \tag{4.11}$$

$$dt' = \rho dx - \rho u dt, \qquad 0 < |p\rho| < \infty, \tag{4.12}$$

provide the Ustinov mapping [20]. Again, the transformation is seen to be non-reciprocal.

(iii) *The adjoint relations.* If we set

$$\begin{array}{ll} A = 0, & ,B = I, \\ C = I, & D = 0, \end{array}$$

then (2.10) show that

$$P' = \begin{pmatrix} \rho' & -\rho' u' \\ \rho' u' & -(p' + \rho' u'^2) \end{pmatrix} = P^{-1}$$

$$= \frac{1}{\rho p} \begin{pmatrix} p + \rho u^2 & -\rho u \\ \rho u & -\rho \end{pmatrix}.$$

Accordingly,

$$\rho' = \frac{p + \rho u^2}{\rho p}, \quad u' = \frac{\rho u}{(p + \rho u^2)}, \quad p' = \frac{1}{(p + \rho u^2)}. \tag{4.13}$$

These relations, together with the $(x, t) \to (x', t')$-transformation

$$dx' = \rho dx - \rho u dt, \tag{4.14}$$

$$dt' = \rho u dx - (p + \rho u^2) dt, \tag{4.15}$$

$$0 < |p\rho| < \infty,$$

constitute the adjoint transformation introduced in [11]. In this case, the conditions (3.1)–(3.4) hold so that the transformation is reciprocal.

(iv) *The reciprocal relations.* The specializations

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & -a_1^2 a_2 \end{pmatrix}, \quad D = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where the $a_i \in \mathbb{R}$, $i = 1, 2$, lead to

$$\begin{pmatrix} \rho' & -\rho' u' \\ \rho' u' & -(p' + \rho' u'^2) \end{pmatrix} = \frac{a_2}{(p + \rho u^2)} \begin{pmatrix} p\rho & a_1 \rho u \\ -a_1 \rho u & a_1^2 \end{pmatrix}$$

so that

$$\rho' = \frac{a_2 p \rho}{(p + \rho u^2)}, \quad u' = \frac{-a_1 u}{p}, \quad p' = \frac{-a_2 a_1^2}{p} \tag{4.16}$$

together with

$$dx' = a_1 dx, \tag{4.17}$$

$$dt' = \rho u dx - (p + \rho u^2) dt, \tag{4.18}$$

$$0 < |a_1(p + \rho u^2)| < \infty.$$

If $a_1^2 = +1$, $a_2 = -1$ then the transformation (4.16)–(4.18) is seen to be reciprocal [10, 11].

5. Reduction to a canonical form in two-phase filtration. In a recent paper by Fokas and Yortsos [21], a description of two phase (oil/water) filtration in a semi-infinite porous reservoir subject to constant injection at its boundary was reduced to consideration of a nonlinear boundary value problem for the equation

$$\frac{\partial S}{\partial \tau} = \frac{\partial}{\partial \xi}\left[\frac{1}{(\beta S + \gamma)^2}\frac{\partial S}{\partial \xi} - \frac{\alpha}{\beta(\beta S + \gamma)}\right], \tag{5.1}$$

where S represents oil saturation, ξ, τ are dimensional space and time measures and α, β are parameters which arise in the assumed class of capillary pressure-saturation laws.

A Bäcklund transformation was introduced which maps the nonlinear equation (5.1) to Burgers' equation. A subsequent application of the Cole-Hopf transformation [22, 23] reduces the nonlinear boundary vlaue problem investigated in [21] to a linear problem solvable by established integral transform methods. Here it is shown that the Bäcklund transformation used in [21] is, in fact, a member of a class of transformations as described in the following corollary of Theorem 1:

THEOREM 2. The nonlinear equation

$$\frac{\partial S}{\partial \tau} - \frac{\partial}{\partial \xi}\left\{g(S)\frac{\partial S}{\partial \xi} + k(S)\right\} = 0 \tag{5.2}$$

is mapped to

$$\frac{\partial S'}{\partial \tau'} - \frac{\partial}{\partial \xi'}\left\{g'(S')\frac{\partial S'}{\partial \xi'} + k'(S')\right\} = 0 \tag{5.3}$$

under the Bäcklund transformation

$$\begin{aligned}
\frac{\partial S'}{\partial \xi'} &= \frac{-\beta}{(\beta S + \gamma)^3}\frac{\partial S}{\partial \xi},\\
\frac{\partial S'}{\partial \tau'} &= \frac{\beta}{(\beta S + \gamma)^2}\left[-\frac{\partial S}{\partial \tau} + \frac{\beta}{(\beta S + \gamma)}\left\{g(S)\frac{\partial S}{\partial \xi} + k(S)\right\}\frac{\partial S}{\partial \xi}\right],\\
d\xi' &= (\beta S + \gamma)d\xi + \beta\left[g(S)\frac{\partial S}{\partial \xi} + k(S)\right]d\tau,\\
\tau' &= \tau,
\end{aligned} \tag{5.4}$$

where

$$\begin{gathered}
S' = \frac{1}{(\beta S + \gamma)}, \qquad k'(S') = \frac{-\beta k(S)}{(\beta S + \gamma)},\\
g'(S') = (\beta S + \gamma)^2 g(S), \qquad 0 < |\beta S + \gamma| < \infty.
\end{gathered} \tag{5.5}$$

Proof. This result may either be established as a special case of Theorem 1 or, more conveniently, 'ab initio'.

Thus, it is seen that

$$S'd\xi' + \left[g'(S')S'_{\xi'} + k'(S')\right] d\tau' = S'\left[(\beta S + \gamma)\, d\xi + \beta\{g(S)S_\xi + k(S)\}\, d\tau\right]$$

$$+\left[\frac{-\beta g'(S)}{(\beta S + \gamma)^3} S_\xi + k'(S')\right] d\tau' = d\xi,$$

whence

$$\frac{\partial S'}{\partial \tau'} = \frac{\partial}{\partial \xi'}\left\{g'(S')\frac{\partial S'}{\partial \xi'} + k'(S')\right\} = 0$$

and the result is established. □

If

$$g(S) = \frac{1}{(\beta S + \gamma)^2}, \tag{5.6}$$

$$k(S) = \frac{-\alpha}{\beta(\beta S + \gamma)}, \tag{5.7}$$

as in (5.1), then (5.5) shows that

$$g'(S') = +1, \tag{5.8}$$

$$k'(S') = \alpha S'^2, \tag{5.9}$$

so that (5.1) is reduced via the Bäcklund transformation (5.4) to the Burgers canonical form

$$\frac{\partial S'}{\partial \tau'} - \frac{\partial^2 S'}{\partial \xi'^2} - 2\alpha S'\frac{\partial S'}{\partial \xi'} = 0. \tag{5.10}$$

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