TWO-DIMENSIONAL ALIGNED-FIELD MAGNETOFLUIDDYNAMIC FLOW. PART I: STEADY INCOMPRESSIBLE FLOW WITH NON-ZERO CHARGE DENSITY*

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Abstract. It is shown that, in the case of non-zero charge density, the class of steady, plane, incompressible, aligned-fluid magnetofluiddynamic flows contains no rotational motions. Therefore, this class of flows is exhausted by the irrotational solutions of Kingston and Power.

1. Introduction. Using complex variable techniques, Kingston and Power [1] gave an exact analysis of steady two-dimensional magnetogasdynamic flows in which the magnetic field is aligned with the velocity vector. They considered compressible and incompressible fluids with finite electric conductivity. However, their results were largely restricted to the special case where the magnet field is a *constant* multiple of the mass flux. For the general case of aligned-field flows with non-zero charge density, these authors obtained the general representations of the various unknown quantities which satisfy all the governing equations except the linear momentum equation. The last equation would impose certain restrictions on the arbitrary functions contained in the representations, as they had indicated, but the details were not worked out. Their analysis of aligned-field flow with zero charge density was entirely based on assuming the magnetic field to be a *constant* multiple of the mass flux.

Recently, Chandna et al. [2] studied the problem for *incompressible* fluids in the case of non-zero charge density and without assuming the proportionality between the magnet field and the aligned velocity field. They reduced the system of governing equations to the linear momentum equation, the continuity equation, and two integrability conditions for the velocity field which ensure the existence of the charge density distribution and the magnetic field. With a view to study "some rotational and irrotational viscous incompressible aligned plane flows", these authors introduced additional kinematical assumptions to obtain special families of velocity fields that satisfy the continuity equation and the two integrability conditions. However, since the linear momentum equation was not

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used to delimit their irrotational flows except in one special case, the results obtained by these authors generally do not qualify as true solutions.

In this paper, the general solution of the continuity equation and the two integrability conditions for the velocity field is obtained without introducing special kinematical assumptions. This general solution consists of one simple family of *irrotational* flows obtained previously by Kingston and Power (with the magnetic field *proportional* to the velocity field), and two special families of rotational motions. However, when the last two families of rotational motions are substituted into the linear momentum equation, it is found that the equation cannot be satisfied except for fluids with zero charge density.

The analysis yields the following conclusion: in the case of non-zero charge density, all steady, incompressible, aligned-field magnetofluiddynamic plane flows are irrotational motions. Such irrotational flows are either steady *rigid* translational motions or radial flows symmetric with respect to a singular point, and in these flows the magnetic field is a constant multiple of the velocity field. Therefore, for incompressible fluids with non-zero charge density, aligned-field solutions having non-proportional magnetic and velocity fields do not exist; nor are there any rotational solutions. The class of rotational flows investigated in [2] is therefore vacuous.

In a subsequent paper (Part II), the corresponding problem for the case of zero-charge density will be solved.

2. Governing equations and integrability conditions. Following mainly the notation of [2], in the case of non-zero electric charge density the governing equations for steady, plane *aligned* flow of a viscous, incompressible electromagneto-fluid are given in the following

$$\operatorname{div} \mathbf{v} = 0, \tag{1}$$

$$\operatorname{div} \mathbf{H} = 0, \tag{2}$$

$$\operatorname{curl} \mathbf{H} = \mu\sigma \mathbf{v} \times \mathbf{H} = \mathbf{0}, \tag{3}$$

$$\operatorname{div}(q\mathbf{v}) = -\sigma q/\varepsilon, \tag{4}$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{5}$$

$$\operatorname{grad} p = -\rho\mathbf{v}\cdot\operatorname{grad}\mathbf{v} + \nu\Delta\mathbf{v} - (q^2/\sigma)\mathbf{v}, \tag{6}$$

where the unknown variables $\mathbf{v}$, $\mathbf{H}$, q and p stand for the velocity field, the magnetic field, the charge density and the pressure field, and where the material constants ρ, σ, ν, μ and ε represent the fluid density, the electric conductivity, the kinematic viscosity, the magnetic permeability, and the permitivity, respectively. The symbol Δ denotes the Laplacian operator.

Chandna et al. showed that, for $q \neq 0$, Eqs. (4) and (5) are equivalent to a single equation

$$\operatorname{grad}(\ln q) = \frac{-\sigma}{\varepsilon}\frac{\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \frac{(\operatorname{curl}\mathbf{v})\times\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}. \tag{7}$$

Furthermore, Eqs. (1), (2) and (3) yield

$$\mathbf{H} = K(x, y)\mathbf{v}, \tag{8}$$

$$\operatorname{grad}(\ln K) = \frac{(\operatorname{curl}\mathbf{v}) \times \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}. \tag{9}$$

Consequently, when the variables H and q are eliminated from the five equations (1)–(5), the following over-determined system of equations is obtained for the remaining field $\mathbf{v}$:

$$\begin{gathered} \operatorname{div}\mathbf{v} = 0, \\ \operatorname{curl}\left\{\frac{(\operatorname{curl}\mathbf{v}) \times \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right\} = \mathbf{0}, \\ \operatorname{curl}\left(\frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) = \mathbf{0}. \end{gathered} \tag{10}$$

According to Eq. (9), irrotational flows correspond to the case $K \equiv$ constant. This class of solutions was completely determined by Kingston and Power [1, p. 856]. The class consists only of constant velocity flows (i.e., steady rigid translational motions) and the radial flows whose velocity fields are symmetric with respect to a singular point:

$$\mathbf{v} = \begin{cases} \mathbf{v}_0 & \text{(constant velocity flows)}, \\ c\dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)} & \text{(radial flows)}, \end{cases} \tag{11}$$

where c is an arbitrary constant. For simplicity, we have chosen a coordinate system (x, y) whose origin coincides with the singular point.

Instead of attempting to find special solutions by introducing additional kinematical assumptions, as Chandna et al. did, we obtain the general solution of the system of equations (10) in the following analysis. Corresponding to every particular solution $\mathbf{v}$ derivable from this general solution, there exist fields $\mathbf{H}$ and q which, together with $\mathbf{v}$, satisfy the system of equations (1)–(5). The linear momentum equation, Eq. (6), remains yet to be satisfied, and this equation imposes further restriction upon the velocity field. The restriction turns out be be so strong that the only flows satisfying Eqs. (1)–(6) are steady rigid translational motions and the radial flows represented by Eq. (11).

3. General solution of Eq. (10). In the case of incompressible plane flow, the equation of continuity (Eq. (10a)) implies the existence of a stream function $\psi(x, y)$. Equation (10b) holds if and only if the stream function is functionally dependent on some plane harmonic function $\xi(x, y)$, as will be shown in the following. Using the method of analytic functions, we next show that Eq. (10c) restricts the harmonic function ξ to assume a special form. Furthermore, Eq. (10c) imposes a corresponding restriction upon the functional relation between ξ and the stream function ψ. The results obtained in this section constitute the general solution of the system of equations (10).

Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors along the x- and y-directions and let $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. Then

$$\mathbf{v} = \psi_y\mathbf{i} - \psi_x\mathbf{j}, \tag{12}$$

$$\operatorname{curl}\mathbf{v} = \omega\mathbf{k}, \tag{13}$$

where

$$\omega = -\Delta\psi = -(\psi_{xx} + \psi_{yy}). \tag{14}$$

Equation (10b) becomes

$$\operatorname{curl}\left(\frac{\omega \operatorname{grad} \psi}{\mathbf{v} \cdot \mathbf{v}}\right) = -\operatorname{grad}\left(\frac{\Delta\psi}{|\operatorname{grad} \psi|^2}\right) x \operatorname{grad} \psi = \mathbf{0}.$$

It follows that $\Delta\psi/|\operatorname{grad} \psi|^2$ is functionally dependent on the stream function ψ. We write

$$-\Delta\psi/|\operatorname{grad} \psi|^2 = p'(\psi). \tag{15}$$

At this stage, it is convenient to introduce the formalism of complex variables. Let

$$z = x + iy, \qquad \bar{z} = x - iy.$$

Then, for any scalar function

$$A = \hat{A}(x, y) = \hat{A}\left(\frac{z + \bar{z}}{2}, -i\frac{z - \bar{z}}{2}\right) = \mathring{A}(z, \bar{z}),$$

we obtain

$$A_x = A_z + A_{\bar{z}}, \qquad A_y = i(A_z - Z_{\bar{z}}),$$

or,

$$A_z = \tfrac{1}{2}(A_x - iA_y), \qquad A_{\bar{z}} = \tfrac{1}{2}(A_x + iA_y). \tag{16}$$

Hence,

$$\begin{aligned} 4A_z A_{\bar{z}} &= A_x^2 + A_y^2, \\ 4A_{z\bar{z}} &= A_{xx} + A_{yy} = \Delta A. \end{aligned} \tag{17}$$

Furthermore, for any pair of scalar functions A and B, we have

$$4A_z B_{\bar{z}} = A_x B_x + A_y B_y + i(A_x B_y - B_x A_y). \tag{18}$$

If g is a complex analytic function with the real part ξ and the imaginary part η, then

$$\begin{aligned} g' &= \xi_x + i\eta_x = \xi_x - i\xi_y = 2\xi_z = 2i\eta_z, \\ \bar{g}' &= 2\xi_{\bar{z}} = -2i\eta_{\bar{z}}. \end{aligned} \tag{19}$$

In view of the expression (17), Eq. (15) reduces to

$$\psi_{z\bar{z}}/\psi_z = -p'(\psi)\psi_{\bar{z}}.$$

Integrating with respect to $\bar{z}$, we obtain

$$\ln(\psi_z) = -p(\psi) + \ln\{g'(z)/2\}$$

or

$$[2e^{p(\psi)}]\psi_z = g'(z).$$

Where g is an arbitrary complex analytic function. Applying (19a) to the last equation, we have

$$\int e^{p(\psi)}\, d\psi = \operatorname{Re}[g(z)] \equiv \xi(x, y).$$

It follows that the stream function ψ is functionally dependent on the harmonic function ξ, i.e.,

$$\psi = \Psi(\xi), \qquad \Delta\xi = \Delta\,\mathrm{Re}[g] = 0, \tag{20}$$

where the functional form Ψ is determined by the function p as introduced in Eq. (15).

Equation (10c) now yields

$$\left(\frac{-\psi_x}{\psi_x^2+\psi_y^2}\right)_x - \left(\frac{\psi_y}{\psi_x^2+\psi_y^2}\right)_y = 0.$$

Substituting (20) into the last equation, we have

$$\xi_x\left\{\frac{1}{\Psi'(\xi)\left(\xi_x^2+\xi_y^2\right)}\right\}_x + \xi_y\left\{\frac{1}{\Psi'(\xi)\left(\xi_x^2+\xi_y^2\right)}\right\}_y = 0,$$

or, using the notation of complex variables,

$$4\,\mathrm{Re}\left[\xi_{\bar z}\left\{\frac{1}{\Psi'(\xi)g'\bar g'}\right\}_z\right] = 2\,\mathrm{Re}\left[\bar g'\left\{\frac{1}{\Psi'(\xi)g'}\right\}_z\frac{1}{\bar g'}\right]$$
$$= \frac{2}{\Psi'}\,\mathrm{Re}\left[\left(\frac{1}{g'}\right)'\right] - \frac{\Psi''}{(\Psi')^2} = 0.$$

Consequently,

$$\mathrm{Re}\left[\left(\frac{1}{g'}\right)'\right] = \frac{\Psi''(\xi)}{2\Psi'(\xi)}. \tag{21}$$

Being the real part of an analytic function $(1/g')'$, the scalar function $\Psi''/(2\Psi')$ is a plane harmonic function. Since it is well known (and can be easily proved) that a harmonic function which is functionally dependent on another harmonic function ξ must be *linearly* related to the latter, we have

$$\frac{\Psi''(\xi)}{2\Psi'(\xi)} = 2\alpha\xi + \beta = \mathrm{Re}[2\alpha g + \beta] \tag{22}$$

where α and β are real constants. Equations (21) and (22) now yield

$$\mathrm{Re}\left[\left(\frac{1}{g'}\right)' - 2\alpha g - \beta\right] = 0.$$

Another well-known theorem of complex variables states that if the real part of an analytic function vanishes identically, then that analytic function assumes a *constant* imaginary value. Hence the last equations delivers

$$(1/g')' - 2\alpha g - (\beta + i\gamma) = 0, \tag{23}$$

where the constant γ is real. We obtain by integration

$$\ln g' + \alpha g^2 + (\beta + i\gamma)g = \ln C_1$$

or

$$e^{\alpha g^2 + (\beta+i\gamma)g} g' = C_1,$$

where C_1 is a complex constant. Further integration introduces another complex constant which, without loss of generality, may be set to zero by redefining the origin of the complex plane:

$$\int e^{\alpha g^2+(\beta+i\gamma)g}\,dg = C_1 z. \tag{24}$$

Furthermore, we have from Eq. (22)

$$\tfrac{1}{2}\ln\Psi' = \alpha\xi^2 + \beta\xi + \tfrac{1}{2}\ln\delta,$$

where δ is a scalar constant. We then obtain the stream function

$$\psi = \Psi(\xi) = \delta\int e^{2(\alpha\xi^2+\beta\xi)}\,d\xi,\ \xi = \mathrm{Re}[g]. \tag{25}$$

Equations (24), (25) and (12) represent the general solution of the system of equations (10).

4. Non-existence of rotational flows. The general solution of Eq. (10) obtained in the last section consists of three distinct families of steady motions corresponding to the following three cases respectively:

Case (i): $\alpha = \beta = 0$. For this case Eq. (22) implies that Ψ' is a constant, so that the stream function ψ is linearly related to the harmonic function ξ:

$$\psi = \psi_0\xi$$

where ψ_0 is a scalar constant. Therefore ψ itself is also harmonic and the solution represents an irrotational flow. The analytic function g is obtained from Eq. (24):

$$g(z) = \begin{cases} -(i/\gamma)\{\ln z + \ln(i\gamma C_1)\} & \text{if } \gamma \neq 0, \\ C_1 z & \text{if } \gamma = 0. \end{cases} \tag{26}$$

The complex velocity of the flow is given by

$$\psi_y + i\psi_x = i\psi_0 g' = \begin{cases} -i\psi_0/(\gamma z) & \text{if } \gamma \neq 0, \\ i\psi_0 C_1 & \text{if } \gamma = 0, \end{cases} \tag{27}$$

which represents either a symmetric radial flow (away from a source or toward a sink) or a rigid-body translation. Since these motions are irrotational, they satisfy the linear momentum equation for whatever values of viscosity ν and electrical conductivity σ. As was mentioned earlier, these solutions were already obtained by Kingston and Power [1].

Case (ii): $\alpha = 0$, $\beta \neq 0$. For this case Eqs. (24) and (25) yield

$$\begin{aligned} g(z) &= (\beta + i\gamma)^{-1}\ln(C_2 z), \\ \psi &= (\delta/2\beta)e^{2\beta\xi}, \end{aligned} \tag{28}$$

where $C_2 = (\beta + i\gamma)C_1$. Hence the complex velocity is given by

$$\begin{aligned} \psi_y + i\psi_x &= 2i\psi_z = 2i\delta e^{2\beta\xi}\xi_z = i\delta e^{2\beta\xi}g' \\ &= \frac{i\delta}{(\beta+i\gamma)z}\exp\left\{2\beta\,\mathrm{Re}\left[(\beta+i\gamma)^{-1}\ln(C_2 z)\right]\right\}. \end{aligned} \tag{29}$$

For this family of steady motions, we find that the expression

$$2\psi_{z\bar{z}\bar{z}}\psi_z = \beta\delta^2 e^{4\beta\xi}\left(\overline{g''}\, g'^2 + \beta \overline{g'}^2 g'^2\right) = \frac{i\gamma\beta\delta^2 e^{4\beta\xi}}{\left(\beta^2+\gamma^2\right)^2 |z|^4}$$

is purely imaginary. Hence the streamlines (ψ = constant) are orthogonal to the lines of constant vorticity ($\psi_{z\bar{z}}$ = constant). Therefore, this family of steady motions is identical to one of the special cases considered by Chandna et al. [2, Sec. 4 case (c)].

Case (iii): $\alpha \neq 0$. For this case we obtain from Eqs. (24) and (25):

$$g(z) = -\frac{\beta+i\gamma}{2\alpha} + (-\alpha)^{-1/2}\operatorname{inverf}\left\{2zC_1(-\alpha/\pi)^{1/2}\exp\left[\frac{(\beta+i\gamma)^2}{4\alpha}\right]\right\},$$

$$\psi = \frac{\delta}{2}e^{-\beta/(2\alpha)}\{-\pi/\ (2\alpha)\}^{1/2}\operatorname{erf}\left\{(-2\alpha)^{1/2}\left(\operatorname{Re}[g]+\frac{\beta}{2\alpha}\right)\right\}, \tag{30}$$

where the symbol "inverf" represents the inverse complex error function. The complex error function is defined by

$$\operatorname{erf}\zeta = (4/\pi)^{1/2}\int_0^{\zeta}\exp(-t^2)\,dt.$$

We now show that, in the case of non-zero charge density, the steady motions represented by Eqs. (28) and (30) are inconsistent with the linear momentum equation.

Applying the curl operator to Eq. (6), and using Eqs. (5) and (12)–(14), we obtain

$$\operatorname{curl}\left\{\rho\Delta\psi\operatorname{grad}\psi + \nu\Delta\mathbf{v} - (q^2/\sigma)\mathbf{v}\right\}$$
$$= \rho\{(\Delta\psi)_x\psi_y - (\Delta\psi)_y\psi_x\}\mathbf{k} - \nu\Delta\Delta\psi\mathbf{k} - \frac{q^2}{\sigma}\operatorname{grad}(\ln q)\times\mathbf{v} = \mathbf{0}.$$

Substituting (7) into the last equation, we have

$$\rho\{(\Delta\psi)_x\psi_y - (\Delta\psi)_y\psi_x\} - \nu\Delta\Delta\psi - \frac{q^2}{\sigma}\Delta\psi = 0.$$

Hence, for rotational motions ($\Delta\psi \neq 0$), we obtain an expression for the charge density

$$\frac{q^2}{\sigma} = \rho\{(\Delta\psi)_x\psi_y - (\Delta\psi)_y\psi_x\}/\Delta\psi - \nu\Delta\Delta\psi/\Delta\psi. \tag{31}$$

Since

$$2\psi_{\bar{z}} = \Psi'(\xi)\bar{g}'$$
$$\Delta\psi = 4\psi_{z\bar{z}} = 2(\Psi'\bar{g}')_z = 2\Psi''\xi_z\bar{g}' = \Psi''g'\bar{g}',$$
$$(\Delta\psi)_z = \frac{\Psi'''}{2}g'^2\bar{g}' + \Psi''g''\bar{g}',$$
$$\tfrac{1}{4}\Delta\Delta\psi = (\Delta\psi)_{z\bar{z}} = \frac{\Psi''''}{4}g'^2\bar{g}'^2 + \frac{\Psi'''}{2}\left(g'^2\overline{g''} + \overline{g'}^2 g''\right) + \Psi''g''\overline{g''},$$

we have

$$\rho\{(\Delta\psi)_x\psi_y - (\Delta\psi)_y\psi_x\}/\Delta\psi = \frac{4\rho}{\Delta\psi}\,\mathrm{Im}[(\Delta\psi)_z\psi_{\bar z}]$$
$$= \frac{4\rho}{\Psi''g'\bar g'}\,\mathrm{Im}\left[\frac{\Psi'''\Psi'}{4}g'^2\bar g'^2 + \frac{1}{2}\Psi'\Psi''g''\bar g'^2\right]$$
$$= 2\rho g'\bar g'\Psi'\,\mathrm{Im}\left[\frac{g''}{(g')^2}\right] = -2\rho g'\bar g'\Psi'\,\mathrm{Im}[2\alpha g + \beta + i\gamma]$$
$$= -2\rho\Psi' g'\bar g'(2\alpha\eta + \gamma) \tag{32}$$

where Eq. (23) has been used. Furthermore,

$$-\nu\Delta\Delta\psi/\Delta\psi = -\nu g'\overline{g'}\left[\frac{\Psi''''}{\Psi''} - \frac{2\Psi'''}{\Psi''}\left\{\left(\frac{1}{g'}\right)' + \overline{\left(\frac{1}{g'}\right)'}\right\} + 4\left(\frac{1}{g'}\right)'\overline{\left(\frac{1}{g'}\right)'}\right] \tag{33}$$

Repeated differentiation of Eq. (25) yields

$$\Psi'(\xi) = \delta\exp\{2(\alpha\xi^2 + \beta\xi)\},$$
$$\Psi''(\xi) = 2(2\alpha\xi + \beta)\Psi'(\xi),$$
$$\Psi'''(\xi) = 4\{\alpha + (2\alpha\xi + \beta)^2\}\Psi'(\xi),$$
$$\Psi''''(\xi) = 8(2\alpha\xi + \beta)\{3\alpha + (2\alpha\xi + \beta)^2\}\Psi'(\xi).$$

Substituting these expressions and Eq. (23) into Eq. (33), we find that

$$-\nu\Delta\Delta\psi/\Delta\psi = -4\nu g'\overline{g'}\{(2\alpha\eta + \gamma)^2 + \alpha\}. \tag{34}$$

With (32) and (34), the expression (31) becomes

$$\frac{-q^2}{2\sigma} = g'\overline{g'}\left[\rho\Psi'(2\alpha\eta + \gamma) + 2\nu\{(2\alpha\eta + \gamma)^2 + \alpha\}\right]. \tag{35}$$

Differentiating, we obtain

$$-qq_z/\sigma = \frac{g''}{g'}\left(-\frac{q^2}{2\sigma}\right)$$
$$+g'\overline{g'}\left[\rho\frac{\Psi''}{2}g'(2\alpha\eta + \gamma) - \rho\Psi'\alpha i g' - 4\nu(2\alpha\eta + \gamma)\alpha i g'\right].$$

Hence

$$(\ln q)_z = \frac{g''}{2g'}$$
$$+\frac{g'}{2}\left[\frac{\Psi'\rho\{(2\alpha\xi + \beta)(2\alpha\eta + \gamma) - i\alpha\} - 4i\nu\alpha(2\alpha\eta + \gamma)}{\Psi'\rho(2\alpha\eta + \gamma) + 2\nu\{(2\alpha\eta + \gamma)^2 + \alpha\}}\right] \tag{36}$$

Writing Eq. (7) in the complex form, we find that

$$2(\ln q)_z = -\frac{i\sigma}{\varepsilon}\frac{g'}{\Psi'g'\bar g'} - \frac{\Psi''}{\Psi'}g'$$
$$= -\frac{i\sigma}{\varepsilon}\frac{g'}{\Psi'g'\bar g'} - 2(2\alpha\xi + \beta)g'. \tag{37}$$

Equating the right hand sides of (36) and (37), we obtain for non-stagnant motions ($g' \neq 0$)

$$-\left(\frac{1}{g'}\right)' + \frac{\Psi'\rho(2\alpha\xi + \beta)(2\alpha\eta s + \gamma) - i\alpha\{\rho\Psi' + 4\nu(2\alpha\eta + \gamma)\}}{\Psi'\rho(2\alpha\eta + \gamma) + 2\nu\{(2\alpha\eta + \gamma)^2 + \alpha\}}$$
$$+2(2\alpha\xi + \beta) + \frac{i\sigma}{\varepsilon}\frac{1}{\Psi' g' \bar{g}'} = 0.$$

Consider the real part of the last equality

$$(2\alpha\xi + \beta)\left[1 + \frac{\rho\Psi'(\xi)(2\alpha\eta + \gamma)}{\rho\Psi'(\xi)(2\alpha\eta + \gamma) + 2\nu\{(2\alpha\eta + \gamma)^2 + \alpha\}}\right] = 0.$$

This relation implies that either the harmonic function ξ assumes a constant value or it is functionally dependent on its harmonic conjugate η. In the latter case they are linearly related and it can be easily shown that they both have constant values. Hence Eq. (12) yields

$$\mathbf{v} = \Psi'(\xi)(\xi_y \mathbf{i} - \xi_x \mathbf{j}) = \mathbf{0}.$$

In the course of proof we have tacitly assumed that $\Psi'' \neq 0$. If $\Psi'' \equiv 0$, then Eq. (22) implies that $\alpha = \beta = 0$ and the solutions reduce to the irrotational flows of case (i).

References

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