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SOME MONOTONICITY RESULTS FOR RATIOS OF MODIFIED BESSEL FUNCTIONS*

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Abstract. We consider the functions $v_\alpha(t) \equiv tI_\alpha(t)/I_{\alpha+1}(t)$ where I_α are the modified Bessel functions of the first kind of order $\alpha \geqslant 0$. We prove that v_α is strictly monotone and strictly convex on $\mathbf{R}^+$. These results have application in finite elasticity[1].

1. Introduction. For $\alpha \geqslant 0$ the modified Bessel functions of the first kind[2], I_α, are the globally analytic solutions of the differential equation

$$t^2\ddot{I}_\alpha(t) + t\dot{I}_\alpha(t) - (t^2 + \alpha^2)I_\alpha(t) = 0.$$

These Bessel functions have infinite series expansions

$$I_\alpha(t) = \sum_{r=0}^{\infty} \frac{(t/2)^{\alpha+2r}}{r!\Gamma(\alpha + r + 1)} \tag{1}$$

where Γ denotes the *gamma function*.

The infinite series formula (1) can be used to show that I_α satisfy the recurrence relations

$$t\dot{I}_{\alpha+1}(t) = tI_\alpha(t) - (\alpha + 1)I_{\alpha+1}(t), \qquad t\dot{I}_\alpha(t) = tI_{\alpha+1}(t) + \alpha I_\alpha(t). \tag{2}$$

The results in this paper concern the functions

$$v_\alpha(t) \equiv \frac{tI_\alpha(t)}{I_{\alpha+1}(t)}.$$

A simple consequence of the recurrence relations (2) and the infinite series (1) is that v_α satisfies

$$t\dot{v}_\alpha(t) = t^2 + 2(\alpha + 1)v_\alpha(t) - v_\alpha^2(t), \tag{3}$$

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[1]Simpson & Spector [10] use these results, with $\alpha = 0$, to prove that a nonlinearly elastic cylinder eventually becomes unstable in uniaxial compression. We anticipate that for general $\alpha > 0$ these results will have application in the buckling and necking of such cylinders, cf., e.g., Cheng, Ariaratnam & Dubey [2], Green & Spencer [4], Miles [7], Sensenig [9], and Wilkes [13].

[2]Cf., e.g., Watson [12], chapter 3.71 for a discussion of these functions.

$$v_\alpha(0) = 2(\alpha + 1). \tag{4}$$

To simplify notation we will drop the subscript α on v_α for the remainder of this paper.

2. Results. We now prove that $v(t)$ is strictly monotone and strictly convex for all $t \geqslant 0$.

THEOREM 1[3]. $\dot{v}(t)$ is strictly positive for $t \in (0, \infty)$.

Proof. If we differentiate (3) we discover that

$$t\ddot{v}(t) = 2t + (2\alpha + 1)\dot{v}(t) - 2v(t)\dot{v}(t), \tag{5}$$

$$t\dddot{v}(t) = 2 + 2\alpha\ddot{v}(t) - 2\dot{v}^2(t) - 2v(t)\ddot{v}(t), \tag{6}$$

and hence we find, with the aid of (4), that

$$\dot{v}(0) = 0, \qquad \ddot{v}(0) = 1/(\alpha + 2). \tag{7}$$

Since $\ddot{v}$ is continuous, we conclude that $\dot{v}$ is strictly positive on some interval $(0, s)$.

Suppose, for the sake of contradiction, that $\dot{v}$ is not strictly positive on $(0, \infty)$ and define

$$s_* = \inf\{t \in (0, \infty) : \dot{v}(t) = 0\}.$$

We note from above that by the continuity of $\dot{v}$,

$$\dot{v}(t) > 0, \qquad t \in (0, s_*). \tag{8}$$

At $t = s_*$ we conclude from (5) that $\ddot{v}(s_*) = 2$ and hence that $\dot{v}$ is strictly negative on an interval (s, s_*). This contradicts (8). Thus $\dot{v}$ is strictly positive on $(0, \infty)$. □

THEOREM 2. $v^2(t) - (2\alpha + 1)v(t) - (t^2 + \alpha + 1/2) > 0$.

Proof. Define

$$h(t) = \left\{(2\alpha + 1) + \left[(2\alpha + 1)^2 + 4\left(t^2 + \alpha + 1/2\right)\right]^{1/2}\right\}/2, \tag{9}$$

the larger root of the quadratic

$$h^2 - (2\alpha + 1)h - \left(t^2 + \alpha + 1/2\right) = 0. \tag{10}$$

We will show that v is strictly greater than h to prove this theorem. By (4) and (9)

$$\begin{aligned} v(0) - h(0) &= \left[(2\alpha + 3) - (4\alpha^2 + 8\alpha + 3)^{1/2}\right]/2 \\ &= \frac{2\alpha + 3}{(2\alpha + 3) + (4\alpha^2 + 8\alpha + 3)^{1/2}} > 0, \end{aligned}$$

and hence v is strictly greater than h in a neighborhood of zero.

Suppose, for the sake of contradiction, that v is not strictly greater than h on $[0, \infty)$ and define

$$s \equiv \inf\{t \in [0, \infty) : v(t) = h(t)\}.$$

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[3] This theorem can be obtained from a result of Amos [1]. It is also a trivial consequence of a much deeper result of Ismail and Kelker [5]. We present an alternative elementary proof of this result.

It is clear (since $v(t) > h(t)$ for $t \in [0, s)$) that $\dot{v}(s) \leqslant \dot{h}(s)$. However (putting $H = h(s)$ and using (3) and (10))

$$\dot{v}(s) - \dot{h}(s) = \frac{1}{s}\left[s^2 + 2(\alpha+1)H - H^2\right] - \frac{s}{H - (\alpha + 1/2)}$$

$$= \frac{[H - (\alpha + 1/2)]^2 - s^2}{s[H - (\alpha + 1/2)]}. \tag{11}$$

A straightforward computation, using the fact that H is a solution of the quadratic (10) shows that the numerator of the last expression is equal to

$$(\alpha + 1/2)^2 + (\alpha + 1/2) > 0.$$

Finally, by (9) the denominator in (11) is strictly positive and hence so is (11). This is a contradiction. Thus no such s can exist. □

THEOREM 3. $\ddot{v}(t)$ is strictly positive on $[0, \infty)$.

Proof. By (5)

$$\ddot{v}(t) = w(t)(2v(t) - 2\alpha - 1)/t,$$

where

$$w(t) \equiv -\dot{v}(t) + 2t/(2v(t) - 2\alpha - 1). \tag{12}$$

Since $\dot{v}$ is positive and $v(0) = 2(\alpha + 1)$, it is clear that

$$\operatorname{sgn} w(t) = \operatorname{sgn} \ddot{v}(t), \qquad t > 0.$$

Now $\ddot{v}(0) = 1/(\alpha + 2)$, so that both $\ddot{v}$ and w are strictly positive on an interval $(0, s)$. Suppose, for the sake of contradiction, that $\ddot{v}$ is not strictly positive on $[0, \infty)$ and define

$$s_* \equiv \inf\{t \in [0, \infty): \ddot{v}(t) = 0\}.$$

We note that $\ddot{v}(s_*) = w(s_*) = 0$ and that w must be strictly positive on the interval $(0, s_*)$. Thus we conclude, with the aid of (12), that

$$0 \geqslant \dot{w}(s_*) = \frac{4v_* - 4s_*\dot{v}_* - 4\alpha - 2}{(2v_* - 2\alpha - 1)^2},$$

where $v_* \equiv v(s_*)$ and $\dot{v}_* \equiv \dot{v}(s_*)$. If we combine the last equation with (3) we find that

$$0 \geqslant v_*^2 - (2\alpha + 1)v_* - (t^2 + \alpha + 1/2).$$

This contradicts Theorem 2. Thus $\ddot{v}$ is strictly positive on $[0, \infty)$. □

Finally, we note that the following corollary, which is used in [5], is an immediate consequence of Theorem 1, Theorem 2, and equation (4).

COROLLARY. $v^2(t) > t^2 + (2\alpha + 1)(2\alpha + 2) + \alpha + 1/2$.

3. Discussion. Related results on ratios of modified Bessel functions of the first kind are due to Soni [11], Jones [6], Cochran [3], Amos [1], Nasell [8], and Ismail and Kelker [5].

In particular (in our notation) Soni has shown that $v(t) > t$; Nasell has proven that

$$\frac{t^2 + 2(\alpha + 1)t + 2(\alpha + 1)(\alpha + 3/2)}{t + \alpha + 3/2} > v(t) > t + \alpha;$$

and Amos has concluded that $v(t) > t\dot{v}(t)$.

Finally, we note that Ismail and Kelker [5] have proven that the function $[v(t^{1/2})]^{-1}$ is a completely monotone function. Theorem 1 is an immediate consequence of this result.

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