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A REMARK ON THE EQUATIONS OF AGE-DEPENDENT POPULATION DYNAMICS*

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The aim of this note is to improve and complete the estimates given in [1]. We deal with the problem as introduced by M. E. Gurtin and R. C. MacCamy in [2], that is to say, we are looking for a function $\rho(a, t)$ satisfying

$$\begin{cases} \rho_a(a,t) + \rho_t(a,t) + \mu(a,t,P(t))\rho(a,t) = 0 & \text{for } a, t > 0, \\ \rho(0,t) = \int_0^{+\infty} \beta(a,t,P(t))\rho(a,t)\,da, & t > 0, \\ \rho(a,0) = \varphi(a), & a > 0, \end{cases} \tag{1}$$

where

$$P(t) = \int_0^{+\infty} \rho(a,t)\,da.$$

As quoted in [1], the problem reduces to finding a function $\rho(a, t)$ which satisfies, for a, t positive.

$$\rho(a,t) = \begin{cases} \left[\int_0^{+\infty} \beta(\xi,t,P(t-a))\rho(\xi,t-a)\,d\xi\right] \\ \qquad \cdot \exp - \int_0^a \mu(s,s+t-a,P(s+t-a))\,ds & \text{if } a < t, \\ \varphi(a-t) \cdot \exp - \int_{a-t}^a \mu(s,s+t-a,P(s+t-a))\,ds & \text{if } a > t, \end{cases} \tag{2}$$

the data being μ, β and φ. So, if $F\rho(a, t)$ denotes the right side of (2), the problem is to find a fixed point for F. In order to do this we will solve (2) for $(a, t) \in \mathbf{R}^+ \times [0, T]$ and if $C(T)$ denotes a constant depending on T only, we will assume:

$\mu(a, t, P)$ is a positive measurable function such that the mapping $s \to \mu(s, s+u, P)$ belongs to $L^1_{\mathrm{Loc}}([0, +\infty))$ for almost all (u, P) in $\mathbf{R}^2$ and there exists a constant $C(T)$ such that, for all $P, P' \in \mathbf{R}$, (3)

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$$
\begin{aligned}
&|\mu(a,t,P)-\mu(a,t,P')|\leqslant C(T)|P-P'| \quad \text{a.e}\,(a,t)\in \mathbf{R}^+\times[0,T].\\
&\beta(a,t,P) \text{ is a measurable function and, for all } P, P'\in\mathbf{R},\\
&|\beta(a,t,P)-\beta(a,t,P')|\leqslant C(T)|P-P'| \quad \text{a.e.}\,(a,t)\in \mathbf{R}^+\times[0,T].
\end{aligned}
\tag{4}
$$

Now, the growth of β has to be limited (see [1], [2]) and we assume: There exists a function $\bar{\beta}(P)$ continuous on $[0,+\infty)$ such that

$$
\begin{aligned}
&\bar{\beta}\,(P) \text{ is nondecreasing},\\
&\int_A^{+\infty}\frac{ds}{s\bar{\beta}\,(s)}=+\infty \quad \text{for some } A>0 \text{ such that } \bar{\beta}\,(A)>0,\\
&|\beta(a,t,P)|\leqslant C(T)\bar{\beta}\,(|P|) \quad \text{a.e.}\,(a,t)\in\mathbf{R}^+\times[0,T].
\end{aligned}
\tag{5}
$$

Under these assumptions, which are better than in [1], we can prove

THEOREM. If (3), (4), (5) hold for some T, then for each $\varphi\in L^1(\mathbf{R}^+)$ there exists a unique $\rho\in L^\infty([0,T], L^1(\mathbf{R}^+))$ satisfying (2) a.e. on $\mathbf{R}^+\times[0,T]$ (T can be taken arbitrarily if (3), (4), (5) hold for T).

The proof follows the same steps as in [1] and thus we only sketch it. First set

$$
G(t)=\int_A^t\frac{ds}{s\bar{\beta}\,(s)}. \tag{6}
$$

It follows from (5) that G is a one-to-one mapping from $[A,+\infty)$ to $\mathbf{R}^+$. We will denote by G^{-1} the inverse of G. Under the above assumptions, if $|\ \ |_1$ denotes the usual $L^1(\mathbf{R}^+)$ norm and if $M=\operatorname{Max}(A,|\varphi|_1)$, we have

LEMMA. If $\rho(a,t)\in L^\infty([0,T], L^1(\mathbf{R}^+))$ is a solution of (2), then

$$
|\rho(\cdot,t)|_1\leqslant G^{-1}[C(T)t+G(M)] \quad \text{a.e. } t\in[0,T].
$$

Proof. If ρ satisfies (2), we get (using the fact that $\mu\geqslant 0$)

$$
|\rho(\cdot,t)|_1\leqslant\int_0^t\left|\int_0^{+\infty}\beta(\xi,t,P(t-a))\rho(\xi,t-a)\,d\xi\right|da+\int_t^{+\infty}|\varphi(a-t)|\,da.
$$

Hence after an easy computation using (5), we obtain

$$
|\rho(\cdot,t)|_1\leqslant|\varphi|_1+C(T)\int_0^t\bar{\beta}\,(|\rho(\cdot,s)|_1)|\rho(\cdot,s)|_1\,ds
$$

and thus

$$
|\rho(\cdot,t)|_1\leqslant M+C(T)\int_0^t\bar{\beta}\,(|\rho(\cdot,s)|_1)|\rho(\cdot,s)|_1\,ds=\psi(t). \tag{7}
$$

As in [1, Lemma 1], there is no loss of generality in assuming $t\mapsto|\rho(\cdot,t)|_1$ continuous and thus from (7) and (5), we get

$$
\frac{d\psi(t)}{dt}=C(T)\cdot\bar{\beta}\,(|\rho(\cdot,t)|_1)\cdot|\rho(\cdot,t)|_1\leqslant C(T)\cdot\bar{\beta}\,(\psi(t))\cdot\psi(t)
$$

i.e.

$$
\frac{d}{dt}G[\psi(t)]\leqslant C(T).
$$

Integrated between 0 and t leads to

$$G(\psi(t)) - G(M) \leqslant C(T)t.$$

Hence by (7) and the fact that G^{-1} is nondecreasing

$$|\rho(\cdot, t)|_1 \leqslant \psi(t) \leqslant G^{-1}[C(T)t + G(M)] \quad \text{a.e. } t \in [0, T].$$

This concludes the proof of the lemma.

The lemma entitles us to look for a solution ρ of (2) in

$$C = \{\rho \in L^\infty([0, T], L^1(\mathbf{R}^+)) \,|\, |\rho(\cdot, t)|_1 \leqslant G^{-1}[C(T)t + G(M)] \quad \text{a.e. } t \in [0, T]\} \tag{8}$$

and to conclude with the same proof as in [1]. That is to say, to prove that F has a unique fixed point in C, we only need to prove that F maps C into C. So let $\rho \in C$. From (2) and with the same inequalities that we need in order to derive (7), we get for $t \in [0, T]$

$$|F\rho(\cdot, t)|_1 \leqslant M + C(T)\int_0^t \bar{\beta}\left(|\rho(\cdot, s)|_1\right)|\rho(\cdot, s)|_1 \, ds.$$

Thus by (5) and (8)

$$\begin{aligned} |F\rho(\cdot, t)|_1 &\leqslant M + C(T)\int_0^t \bar{\beta}\,\{G^{-1}[C(T)s + G(M)]\}G^{-1}[C(T)s + G(M)]\, ds \\ &= M + \int_0^t \frac{C(T)\, ds}{G'[G^{-1}(C(T)s + G(M))]} \\ &= M + \int_0^t (G^{-1})[C(T)s + G(M)]'\, ds = G^{-1}[C(T)t + G(M)]. \end{aligned}$$

This concludes the proof.

Remark. This result is the best possible in the sense that one cannot weaken the assumptions on the growth of β and still get a global solution of (1). Indeed if we choose $\beta(a, t, P) = \bar{\beta}(P)$ with

$$\int_A^{+\infty} \frac{ds}{s\bar{\beta}\,(s)} < +\infty$$

then (see [1, 2]) the resolution of (2) is equivalent, when $\mu \equiv 0$, $\varphi \geqslant 0$, to finding $P(t)$, a solution of

$$\frac{dP(t)}{dt} = \bar{\beta}\,(P(t))P(t), \qquad P(0) = |\varphi|_1,$$

which in turn is equivalent to

$$G(P(t)) = t + G(|\varphi|_1).$$

But the solution of this problem blows up when $t + G(|\varphi|_1)$ goes out of the range of G, which is $[0, \int_A^{+\infty} ds/s\bar{\beta}(s))$.

Added in proofs: The above results can be improved limitating only the growth of $\beta - \mu$ (in P). See: M. C. in Proceedings of Journées d'Analyse Nonlineaire, Lille, France, May 1983.

References

[1] M. Chipot, *On the equations of age-dependent population dynamics* Arch. Rat. Mech. Anal. 82, 1, 1983, pp. 13–25.

[2] M. E. Gurtin and R. C. MacCamy, *Nonlinear age-dependent population dynamics* Arch. Rat. Mech. Anal. **54** (1974), 281–300