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A NEW MODEL FOR THIN PLATES WITH RAPIDLY VARYING THICKNESS. II: A CONVERGENCE PROOF*

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Abstract. Our recent paper [6] presented a model for thin plates with rapidly varying thickness, distinguishing between thickness variation on a length scale longer than, on the order of, or shorter than the mean thickness. We review the model here, and identify the case of long scale thickness variation as an asymptotic limit of the intermediate case, where the scales are comparable. We then present a convergence theorem for the intermediate case, showing that the model correctly represents the solution of the equations of linear elasticity on the three-dimensional plate domain, asymptotically as the mean thickness tends to zero.

1. Introduction. In [6] we presented a model for the bending of symmetric, linearly elastic plates with rapidly varying thickness. We considered plates with thickness of order ε varying on a length scale of order ε^a, $a > 0$, and we distinguished between three cases: "$a < 1$", in which the thickness varies on a scale much larger than the mean thickness; "$a = 1$", in which the variation is on a scale comparable to the mean thickness; and "$a > 1$", in which the length scale of thickness variation is much shorter than the mean. Our model yields, in each case, a fourth-order equation for the vertical midplane displacement,

$$\partial_{\alpha\beta}\left(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w\right) = \mathscr{F};$$

the formula for the effective rigidity tensor $M_{\alpha\beta\gamma\delta}$ depends, however, on whether $a < 1$, $a = 1$ or $a > 1$. Secs. 2 and 3 give a review of the model, thus making the present exposition self-contained.

The main goal of this paper is to show the validity of the model in the intermediate case "$a = 1$". Attention is restricted to plates with clamped edges and periodic thickness variation. Our main result, Theorem 6.1, shows that the model approximates the true

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three-dimensional elastostatic displacement in energy norm on the thin plate domain, asymptotically as the thickness tends to zero. The corresponding result for flat, homogeneous plates is well known [8, 9]; an analogous one for flat plates with rapidly varying composition has recently been proved by Caillerie [2, 3].

Similar convergence theorems remain to be proved for the other two scalings, involving relatively slow ("$a < 1$") or very fast ("$a > 1$") variation. Proposition 3.1 represents a step in this direction: it identifies the effective rigidities in the "$a < 1$" case as a limit of those from the "$a = 1$" case, as the period of the thickness variation tends to infinity. One can, at least formally, obtain the "$a > 1$" effective rigidities from the "$a = 1$" case in the opposite limit as the period of the thickness variation tends to zero. That analysis is not presented here, because we are unable to make it completely rigorous. We mention it, however, in combination with Proposition 3.1, as an indication that the case "$a = 1$"—when the period is comparable to the mean thickness—is in a certain weak sense universal.

The present problem combines the difficulties inherent in plate theory with those arising from the rapid variation of the surface geometry, and our analysis therefore has aspects in common with both [3, 9] and [4] (more complete references to the extensive literature on periodic homogenization and plate theory are found in [6]). Our analysis is organized around three integral estimates: (i) a Korn-type inequality, (ii) a weak form of Kirchhoff's hypothesis, and (iii) an "averaging lemma".

The Korn-type inequality is Proposition 4.1: it estimates an arbitrary displacement in the H^1 norm on the three-dimensional plate domain in terms of its elastic energy, with a constant whose dependence on ε is made explicit. For flat plates this estimate can be proved by rescaling Korn's inequality for a cylinder of height 1. A related result is proved in [4] for domains with holes, by means of an extension argument. Those methods seem not to apply in the present context; instead we divide the plate domain into $\mathcal{O}(\varepsilon^{-2})$ subdomains each with diameter of order ε. By rescaling Korn's inequality for a unit-sized domain one can relate the displacement on each subdomain to a suitable rigid motion. The variation of these rigid motions can in turn be estimated by the local elastic energy; since the rigid motions vanish at the clamped edge this leads to Proposition 4.1.

Kirchhoff's hypothesis asserts that the displacement $\underline{u}$, to leading order has the form

$$\underline{u} \sim \left(-x_3 \frac{\partial w}{\partial x_1}, -x_3 \frac{\partial w}{\partial x_2}, w \right).$$

From Theorem 6.1 it follows directly that the solution to the three dimensional elastostatic boundary value problem does indeed satisfy this. Proposition 4.2, as a preliminary result, establishes the weaker assertion that

$$\int \left| u_\alpha + x_3 \frac{\partial u_3}{\partial x_\alpha} \right|^2 d\underline{x} < C\varepsilon^2 \int |e(\underline{u})|^2 \, d\underline{x} \qquad (\alpha = 1, 2)$$

for any $\underline{u}$ satisfying certain symmetry conditions. The symmetry conditions require that $\underline{u}$ lie in the space

$$X_\varepsilon = H^1 \cap \{ \underline{u} \colon \underline{u} = 0 \text{ at the plate edge; } u_1, u_2 \text{ are odd and } u_3 \text{ is even as functions of } x_3 \}.$$

The proof is similar to that of Proposition 4.1, although considerably simpler: we again apply Korn's inequality on subdomains of diameter ε to approximate $\underline{u}$ by rigid motions. The symmetry conditions restrict the rigid motions that can arise, and this leads Proposition 4.2.

Our "averaging lemma" Proposition 5.1, is an adaptation to plate theory of the following simple result for functions in $\mathbf{R}^n$:

Consider a function $f(x; \xi)$ defined for $x \in \Omega \subset \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$, and periodic in each ξ_i with period 1. *Assume moreover that f and each of the derivatives $\partial f/\partial x_i$ are uniformly bounded, and set*

$$f_\varepsilon(x) = f(x; x/\varepsilon), \qquad \bar{f}(x) = \int_{[0,1]^n} f(x; \xi)\, d\xi.$$

Then

$$\|f_\varepsilon - \bar{f}\|_{H^{-1}(\Omega)} \leqslant C\varepsilon.$$

We call this an "averaging lemma", because it approximates the rapidly varying function f_ε by its "local average" $\bar{f}$. In Proposition 5.1, $H^{-1}(\Omega)$ (the dual of $\mathring{H}^1(\Omega)$) is replaced by the dual of X_ε, where X_ε is equipped with the energy norm.

The main convergence argument, presented in Sec. 6, is partly inspired by Nordgren's article [9]. The Ansatz for the displacement contained in our model provides a stress field τ, defined by (6.10), which is almost statically admissible (Proposition 6.1) and at the same time is almost kinematically admissible (cf. (6.21)). Therefore the Ansatz is a good approximation to the true linearly elastostatic displacement, and that is the assertion of Theorem 6.1. Proving that τ is nearly statically admissible is by far the more difficult, and it is here the three aformentioned integral estimates are used. As a corollary to Theorem 6.1 we also conclude that the first term of the Ansatz correctly estimates the mean vertical displacement (in a weighted L_2 norm on the midplane).

The analysis presented in this paper applies only to the case "$a = 1$" and only to plates with periodic thickness variation and clamped edges. The method appears, however, to be more general. We believe it could be applied with other boundary conditions at the plate edge, and with plates whose thickness is "locally periodic" or "quasiperiodic" in the sense of [6]. An analysis of the cases "$a < 1$" and "$a > 1$" could perhaps be done following a similar outline.

Structural engineers are interested in plates of the type studied here, because they may be stronger per unit weight than uniform or slowly varying ones in certain design contexts. Some references to the literature on structural design optimization are found in [6]. It is natural to ask which scaling—$a < 1$, $a = 1$ or $a > 1$—produces the most rigid structure; we intend to address this issue in a forthcoming paper.

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2. Preliminaries. We shall write $\underline{x} = (x_1, x_2, x_3)$ for vectors in $\mathbf{R}^3$ and $\underset{\sim}{x} = (x_1, x_2)$ for vectors in $\mathbf{R}^2$. Latin indices will usually range from 1 to 3, and Greek ones from 1 to 2; the summation convention applies whenever indices are repeated. We write $\partial_i = \partial/\partial x_i$ and $\partial_{ij} = \partial^2/\partial x_i \partial x_j$.

2A. *Constitutive laws.* Associated with any displacement $\underline{u} = (u_1, u_2, u_3)$ of $\mathbf{R}^3$ is its linear strain tensor

$$e_{ij}(\underline{u}) = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \tag{2.1}$$

and the corresponding stress tensor

$$\sigma_{ij}(\underline{u}) = B_{ijkl} e_{kl}(\underline{u}). \tag{2.2}$$

The fourth-order tensor B_{ijkl} satisfies

$$B_{ijkl} = B_{jikl} = B_{ijlk} = B_{klij};$$

we assume that the elastic energy

$$B_{ijkl} e_{ij} e_{kl}$$

is positive definite on symmetric tensors.

We shall always assume that the horizontal planes are planes of elastic symmetry; this means [7]

$$B_{\alpha\beta\gamma 3} = 0, \qquad B_{\alpha 333} = 0.$$

Finally, we define the positive definite fourth-order tensor

$$\tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - \frac{B_{\alpha\beta 33} B_{\gamma\delta 33}}{B_{3333}}. \tag{2.3}$$

2B. *Plate geometry.* The plate geometry is determined by

a smoothly bounded domain Ω in the $x_1 - x_2$ plane, representing the midplane; (2.4a)

a real parameter a, $0 < a < \infty$, determining the length scale of the thickness variation, and (2.4b)

a bounded function $h(\underset{\sim}{\eta}) \geqslant 0$, defined for any $\underset{\sim}{\eta} \in \mathbf{R}^2$ and periodic in η_α with period L_α, $\alpha = 1, 2$. (2.4c)

The three-dimensional region occupied by the plate is

$$R(\varepsilon) = \{\underline{x} : \underset{\sim}{x} \in \Omega, |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon^a)\};$$

$\tilde{R}(\varepsilon)$ denotes its natural periodic extension

$$\tilde{R}(\varepsilon) = \{\underline{x} : \tilde{x} \in \mathbf{R}^2, |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon^a)\}. \tag{2.5}$$

We assume throughout that $\tilde{R}(\varepsilon)$ is a connected, $C^{2,\alpha}$ domain, for some Hölder exponent $\alpha > 0$. The function h may nonetheless have discontinuities—*i.e.* parts of $\partial\tilde{R}(\varepsilon)$ may be vertical; and h may vanish on a set of positive measure—*i.e.* our plates may have holes. (In Sec. 3D, where we study an asymptotic limit of the $a = 1$ case, we shall impose additional smoothness assumptions on h.)

We denote by $\partial_0 R(\varepsilon)$ the outer edge of the plate,

$$\partial_0 R(\varepsilon) = \{\underline{x} : \underset{\sim}{x} \in \partial\Omega, |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon^a)\};$$

$\partial_+ R(\varepsilon)$ and $\partial_- R(\varepsilon)$ are the remaining parts of $\partial R(\varepsilon)$ above and below Ω, respectively; $\underset{\sim}{\nu}^\varepsilon$ is the outward unit normal to $\partial R(\varepsilon)$.

When, in the following, we call a function "periodic in $\underset{\sim}{\eta}$" we shall always mean that it has the same periods $\underset{\sim}{L} = (L_1, L_2)$ as h. It will often be necessary to average a periodic function $f(\underset{\sim}{\eta})$ with respect to $\underset{\sim}{\eta}$:

$$\mathscr{M} f = \frac{1}{L_1 L_2} \int_0^{L_1} \int_0^{L_2} f(\underset{\sim}{\eta})\, d\eta_1\, d\eta_2 .$$

We shall use the norm

$$\|g\|_{2,\varepsilon} = \left(\int_{R(\varepsilon)} |g|^2\, d\underset{\sim}{x} \right)^{1/2} ;$$

The same notation will be used for tensors, in which case $|g|^2$ denotes the sum of the squares of the components.

2C. *Loads and equations of equilibrium.* The following discussion applies for $a = 1$; when $a \neq 1$, it is more natural to work with the load per unit projected surface area, see [6].

We suppose that the plate is loaded along its faces $\partial_\pm R(\varepsilon)$ by forces $\varepsilon^3(0, 0, f(\underset{\sim}{x}; \underset{\sim}{x}/\varepsilon))$ per unit surface area, and that the body force is $\varepsilon^2(0, 0, F(\underset{\sim}{x}; \underset{\sim}{x}/\varepsilon))$ per unit volume, where

$$f(\underset{\sim}{x}; \underset{\sim}{\eta}) \text{ and } F(\underset{\sim}{x}; \underset{\sim}{\eta}) \text{ are bounded, periodic in } \underset{\sim}{\eta}, \text{ and even with respect to } \eta_3. \tag{2.6}$$

The equations of elastostatic equilibrium for the clamped, ε-dependent, three-dimensional plate are

$$-\partial_j \left[\sigma_{ij}(\underset{\sim}{u}^\varepsilon) \right] = \begin{cases} 0, & i = 1, 2, \\ \varepsilon^2 F, & i = 3 \end{cases} \quad \text{in } R(\varepsilon); \tag{2.7}$$

$$\sigma_{ij}(\underset{\sim}{u}^\varepsilon) \nu_j^\varepsilon = \begin{cases} 0, & i = 1, 2 \\ \varepsilon^3 f, & i = 3 \end{cases} \quad \text{on } \partial_\pm R(\varepsilon); \tag{2.8}$$

$$\underset{\sim}{u}^\varepsilon = 0 \quad \text{on } \partial_0 R(\varepsilon). \tag{2.9}$$

This scaling of the loads ensures that $\underset{\sim}{u}^\varepsilon$ stays bounded as $\varepsilon \to 0$. Notice that

$$\begin{aligned} &u_1^\varepsilon, u_2^\varepsilon \text{ are odd}; \quad u_3^\varepsilon \text{ is even}, \\ &\sigma_{\alpha\beta}^\varepsilon, \sigma_{33}^\varepsilon \text{ are odd}; \quad \sigma_{\alpha 3}^\varepsilon \text{ is even} \end{aligned} \tag{2.10}$$

with respect to η_3, as a consequence of (2.6); X_ε will denote the space of all admissible displacements that obey these symmetries:

$$X_\varepsilon = \left\{ \underset{\sim}{u} \in H^1(R(\varepsilon)) : \underset{\sim}{u}|_{\partial_0 R(\varepsilon)} = 0;\ u_1, u_2 \text{ are odd and } u_3 \text{ is even in } x_3 \right\} \tag{2.11}$$

where $H^1(R(\varepsilon))$ is the space of (vector-valued) functions with square-integrable first derivatives.

The restriction to even loads is merely a matter of technical convenience. If F and f are odd in η_3, then the solution of (2.7)–(2.9) satisfies

$$\|e(\underset{\sim}{u}^\varepsilon)\|_{2,\varepsilon} \leqslant C\varepsilon^{7/2}. \tag{2.12}$$

In case $|\nu_3^\varepsilon| \geqslant c > 0$ on $\partial_\pm R(\varepsilon)$ (*i.e.* $\partial_\pm R(\varepsilon)$ has no vertical parts), one can prove (2.12) by taking the inner product of $\underset{\sim}{u}^\varepsilon$ with (2.7), integrating by parts, and using a Poincaré inequality on each vertical line. The proof in the general case is similar, but it requires the methods of section 4. We shall show here that for even loads

$$\|e(\underset{\sim}{u}^\varepsilon)\|_{2,\varepsilon} \sim \varepsilon^{3/2}, \tag{2.13}$$

whenever the "mean load" is nonzero. Since the problem is linear, any load can be decomposed into its even and odd components; by (2.12) and (2.13), the even part is the one that produces the dominant strain.

3. Review of the model. The model presented in [6] provides the initial terms of an asymptotic expansion for the displacement vector, and—most importantly—an equation for the limiting vertical displacement of the midplane. The equation has in each case the form

$$\partial_{\alpha\beta}\left(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w\right) = \mathscr{F}, \tag{3.1}$$

where $\mathscr{F} = \mathscr{F}(F, f)(\underset{\sim}{x})$ is the rescaled mean vertical load (see (5.1) for the precise definition of $\mathscr{F}$ when $a = 1$; for $a \neq 1$ see [6]). The tensor $M_{\alpha\beta\gamma\delta}$ represents the "effective rigidity" of the plate; it satisfies the usual symmetries

$$M_{\alpha\beta\gamma\delta} = M_{\beta\alpha\gamma\delta} = M_{\alpha\beta\delta\gamma} = M_{\gamma\delta\alpha\beta},$$

and it is positive definite in the sense that

$$M_{\alpha\beta\gamma\delta}\xi_{\alpha\beta}\xi_{\gamma\delta} \geqslant c|\xi|^2$$

for symmetric tensors $\xi_{\alpha\beta}$. *The formula for $M_{\alpha\beta\gamma\delta}$ depends on whether $a > 1$, $a = 1$, or $a < 1$*; in each case it is determined by h through the solution of certain "cell problems" with periodic boundary conditions.

3A. *The case $a < 1$.* Let H^2_{per} denote the set of functions which are periodic with period $\underset{\sim}{L} = (L_1, L_2)$, with square integrable derivatives of order $\leqslant 2$. The auxiliary functions $\phi^{\alpha\beta}(\underset{\sim}{\eta})$ are in this space, and are characterized (modulo a constant) by

$$\begin{aligned}\mathscr{M}&\left[h^3\tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2}{\partial\eta_\gamma\partial\eta_\delta}\phi^{\alpha\beta}\frac{\partial^2}{\partial\eta_\rho\partial\eta_\sigma}\psi\right]\\ &= -\mathscr{M}\left[h^3\tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2}{\partial\eta_\gamma\partial\eta_\delta}\left(\frac{1}{2}\eta_\alpha\eta_\beta\right)\frac{\partial^2}{\partial\eta_\rho\partial\eta_\sigma}\psi\right] \quad \forall\psi \in H^2_{\text{per}}.\end{aligned} \tag{3.2}$$

The tensor $M_{\alpha\beta\gamma\delta}$ is

$$\begin{aligned}M_{\alpha\beta\gamma\delta} &= \mathscr{M}\left[\frac{2}{3}h^3\tilde{B}_{\lambda\mu\rho\sigma}\frac{\partial^2}{\partial\eta_\lambda\partial\eta_\mu}\left(\phi^{\alpha\beta} + \frac{1}{2}\eta_\alpha\eta_\beta\right)\frac{\partial^2}{\partial\eta_\rho\partial\eta_\sigma}\left(\phi^{\gamma\delta} + \frac{1}{2}\eta_\gamma\eta_\delta\right)\right]\\ &= \mathscr{M}\left[\frac{2}{3}h^3\tilde{B}_{\alpha\beta\gamma\delta}\right] + \mathscr{M}\left[\frac{2}{3}h^3\tilde{B}_{\alpha\beta\rho\sigma}\frac{\partial^2}{\partial\eta_\rho\partial\eta_\sigma}\phi^{\gamma\delta}\right].\end{aligned} \tag{3.3}$$

The lowest order terms in the displacement vector are

$$\begin{aligned} u_\gamma^* &= -x_3\partial_\gamma w - \varepsilon^a x_3 \frac{\partial}{\partial\eta_\gamma}(\phi^{\alpha\beta})\partial_{\alpha\beta}w - \varepsilon^{2a}x_3\partial_\gamma\left(\phi^{\alpha\beta}\partial_{\alpha\beta}w\right), \\ u_3^* &= w + \varepsilon^{2a}\phi^{\alpha\beta}\partial_{\alpha\beta}w + \frac{1}{2}(x_3)^2\frac{B_{33\gamma\delta}}{B_{3333}}\frac{\partial^2}{\partial\eta_\gamma\partial\eta_\delta}\left(\phi^{\alpha\beta} + \frac{1}{2}\eta_\alpha\eta_\beta\right)\partial_{\alpha\beta}w, \end{aligned} \tag{3.4}$$

where w solves (3.1), with the appropriate boundary condition. The right side of (3.4) must be evaluated at $\eta = \underset{\sim}{x}/\varepsilon^a$ *after* differentiation.

3B. *The case* $\underset{\sim}{a} = 1$. For any function $\underline{\phi}(\underline{\eta})$ we define

$$E_{ij}(\underline{\phi}) = \frac{1}{2}\left(\frac{\partial\phi_i}{\partial\eta_j} + \frac{\partial\phi_j}{\partial\eta_i}\right) \tag{3.5}$$

and

$$\Sigma_{ij}(\underline{\psi}) = B_{ijkl}E_{kl}(\underline{\phi}). \tag{3.6}$$

Let Q denote the rescaled period cell determined by h,

$$Q = \left\{\underline{\eta}\colon |\eta_\alpha| < L_\alpha/2,\ |\eta_3| < h(\underset{\sim}{\eta})\right\}; \tag{3.7}$$

and let $\underline{\Gamma}^{\alpha\beta}$ denote the vector

$$\begin{aligned} \underline{\Gamma}^{\alpha\beta} = \Bigg(&-\eta_3\frac{\partial}{\partial\eta_1}\left(\frac{1}{2}\eta_\alpha\eta_\beta\right), \\ &-\eta_3\frac{\partial}{\partial\eta_2}\left(\frac{1}{2}\eta_\alpha\eta_\beta\right), \frac{1}{2}\eta_\alpha\eta_\beta + \frac{1}{2}\eta_3^2\frac{B_{33\gamma\delta}}{B_{3333}}\frac{\partial^2}{\partial\eta_\gamma\partial\eta_\delta}\left(\frac{1}{2}\eta_\alpha\eta_\beta\right)\Bigg). \end{aligned} \tag{3.8}$$

The auxiliary functions $\underline{\phi}^{\alpha\beta} \in H^1(Q)$ are periodic in $\underset{\sim}{\eta}$, and they satisfy

$$\int_Q \Sigma_{ij}(\underline{\phi}^{\alpha\beta})E_{ij}(\underline{\psi})\,d\underline{\eta} = -\int_Q \Sigma_{ij}(\underline{\Gamma}^{\alpha\beta})E_{ij}(\underline{\psi})\,d\underline{\eta} \tag{3.9}$$

for any $\underline{\psi} \in H^1(Q)$ which is periodic in $\underset{\sim}{\eta}$. The tensor $M_{\alpha\beta\gamma\delta}$ is given by

$$\begin{aligned} M_{\alpha\beta\gamma\delta} &= \frac{1}{L_1L_2}\int_Q \Sigma_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right)E_{ij}\left(\underline{\phi}^{\gamma\delta} + \underline{\Gamma}^{\gamma\delta}\right)d\underline{\eta} \\ &= \mathscr{M}\left[\frac{2}{3}h^3\tilde{B}_{\alpha\beta\gamma\delta}\right] - \frac{1}{L_1L_2}\int_Q \eta_3\tilde{B}_{\alpha\beta\lambda\mu}E_{\lambda\mu}\left(\underline{\phi}^{\gamma\delta}\right)d\underline{\eta}. \end{aligned} \tag{3.10}$$

The lowest order terms in the displacement vector are

$$\begin{aligned} u_\gamma^* &= -x_3\partial_\gamma w + \varepsilon^2\phi_\gamma^{\alpha\beta}(\underline{x}/\varepsilon)\partial_{\alpha\beta}w, \\ u_3^* &= w + \frac{1}{2}(x_3)^2\frac{B_{33\alpha\beta}}{B_{3333}}\partial_{\alpha\beta} + w\varepsilon^2\phi_3^{\alpha\beta}(\underline{x}/\varepsilon)\partial_{\alpha\beta}w. \end{aligned} \tag{3.11}$$

3C. *The case* $a > 1$. We define a tensor C_{ijkl}, for the use in this section only, by

$$\begin{aligned} C_{\alpha 3\beta 3} = C_{3\alpha\beta 3} = C_{\alpha 33\beta} &= C_{3\alpha 3\beta} = 0, \\ C_{ijkl} = B_{ijkl} &\quad \text{otherwise}. \end{aligned}$$

For any function $\underline{\phi}(\underline{\eta})$, $\hat{\Sigma}(\underline{\phi})$ will denote the associated "stress",

$$\hat{\Sigma}_{ij}(\underline{\phi}) = C_{ijkl}E_{kl}(\underline{\phi}). \tag{3.12}$$

Let Q and $\underline{\Gamma}^{\alpha\beta}$ be as in (3.7), (3.8), and let $\mathscr{Y}$ be the space of functions $\underline{\psi} \in L^2(Q)$ such that

$$\underline{\psi} \text{ is periodic in } \underset{\sim}{\eta},\ \psi_3 \text{ depends only on } \eta_3 \text{ and}$$

$$\int_Q \hat{\Sigma}_{ij}(\underline{\psi}) E_{ij}(\underline{\psi})\, d\underline{\eta} < \infty.$$

When $a > 1$, the auxiliary functions $\underline{\phi}^{\alpha\beta}$ are in $\mathscr{Y}$, and they satisfy

$$\int_Q \hat{\Sigma}_{ij}(\underline{\phi}^{\alpha\beta}) E_{ij}(\underline{\psi})\, d\underline{\eta} = -\int_Q \hat{\Sigma}_{ij}(\underline{\Gamma}^{\alpha\beta}) E_{ij}(\underline{\psi})\, d\underline{\eta} \tag{3.13}$$

for every $\underline{\psi} \in \mathscr{Y}$. The tensor $M_{\alpha\beta\gamma\delta}$ is given by

$$M_{\alpha\beta\gamma\delta} = \frac{1}{L_1L_2}\int_Q \hat{\Sigma}_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right) E_{ij}\left(\underline{\phi}^{\gamma\delta} + \underline{\Gamma}^{\gamma\delta}\right) d\underline{\eta}, \tag{3.14}$$

and the lowest order terms in the displacement vector are

$$\begin{aligned} u_\gamma^* &= -x_3\partial_\gamma w + \varepsilon^{1+a}\phi_\gamma^{\alpha\beta}\partial_{\alpha\beta}w, \\ u_3^* &= w + \frac{1}{2}(x_3)^2\frac{B_{33\alpha\beta}}{B_{3333}}\partial_{\alpha\beta}w + \varepsilon^2\phi_3^{\alpha\beta}\partial_{\alpha\beta}w. \end{aligned} \tag{3.15}$$

The right hand side of (3.15) is evaluated at $\underline{\eta} = (\underset{\sim}{x}/\varepsilon^a, x_3/\varepsilon)$.

In [6] we wrote (3.13)–(3.15) in a slightly different form, to emphasize the connection with homogenization of a rough surface. The functions $g^{\alpha\beta}(\eta_3)$, $\underset{\sim}{\psi}^{\alpha\beta}(\underline{\eta})$, and $\underset{\sim}{\psi}^{33}(\underline{\eta})$ used in [6] correspond to the decomposition

$$\begin{aligned} \phi_3^{\alpha\beta}(\eta_3) &= g^{\alpha\beta}(\eta_3) - \frac{1}{2}\eta_3^2\frac{B_{33\alpha\beta}}{B_{3333}}, \\ \phi_\gamma^{\alpha\beta}(\underline{\eta}) &= -\eta_3\psi_\gamma^{\alpha\beta}(\underline{\eta}) + \frac{\partial g^{\alpha\beta}}{\partial \eta_3}(\eta_3)\cdot\psi_\gamma^{33}(\underline{\eta}). \end{aligned} \tag{3.16}$$

One can characterize $\underset{\sim}{\psi}^{ij}(\cdot, \eta_3)$ as the solutions of certain cell problems on the horizontal slices of Q; $g^{\alpha\beta}(\eta_3)$ may be expressed in terms of certain averages of $\underset{\sim}{\psi}^{ij}$.

3D. *An asymptotic limit of the case $a = 1$.* For a given periodic function $h(\underset{\sim}{\eta})$, let $M_{\alpha\beta\gamma\delta}^{a<1}$ be the effective rigidity of the associated "$a < 1$" plate defined by (3.3). Let $\tilde{M}_{\alpha\beta\gamma\delta}^{1,s}$ denote the effective rigidity of the "$a = 1$" plate with thickness variation

$$h_s(\underset{\sim}{\eta}) = h(\underset{\sim}{\eta}/s), \qquad 0 < s < \infty,$$

i.e., $M_{\alpha\beta\gamma\delta}^{1,s}$ is as defined by (3.10) with h replaced by h_s. We show here that $M^{1,s} \to M^{a<1}$ as $s \to \infty$, if h is smooth enough; the proof is similar to Nordgren's convergence argument [9]. A related result in the context of Laplace's equation can be found in [2].

PROPOSITION 3.1. If $h \geqslant c > 0$ is smooth enough then

$$\lim_{s \to \infty} M^{1,s}_{\alpha\beta\gamma\delta} = M^{a<1}_{\alpha\beta\gamma\delta}. \tag{3.16}$$

Proof. Let $\underline{\phi}^{\alpha\beta}_{s}$ denote the solution of (3.9) with h replaced by h_s, which is periodic with period $s\underset{\sim}{L}$. We introduce the rescaled variables

$$\eta'_\alpha = \eta_\alpha / s \qquad (\alpha = 1,2), \quad \eta'_3 = \eta_3, \tag{3.17}$$

which range over the s-independent cell

$$Q' = \left\{ \underline{\eta}' \colon |\eta'_\alpha| < \tfrac{1}{2} L_\alpha,\ \alpha = 1,2;\ |\eta'_3| < h\left(\underset{\sim}{\eta}'\right) \right\} \tag{3.18}$$

and note that

$$\begin{aligned} E_{\alpha\beta}(\underline{\psi}) &= \frac{1}{2s}\left(\frac{\partial \psi_\alpha}{\partial \eta'_\beta} + \frac{\partial \psi_\beta}{\partial \eta'_\alpha} \right), \\ E_{\alpha 3}(\underline{\psi}) &= \frac{1}{2s}\frac{\partial \psi_3}{\partial \eta'_\alpha} + \frac{1}{2}\frac{\partial \psi_\alpha}{\partial \eta'_3}, \\ E_{33}(\underline{\psi}) &= \frac{\partial \psi_3}{\partial \eta'_3}. \end{aligned} \tag{3.19}$$

Let $\psi^{\alpha\beta}(\underset{\sim}{\eta}')$ be the solution of (3.2) with thickness $h(\underset{\sim}{\eta}')$, and define

$$\begin{aligned} \underline{\phi}^{\alpha\beta}_{*}(\underline{\eta}') = {} & s^2(0,0,\psi^{\alpha\beta}) + s\left(-\eta'_3 \frac{\partial \psi^{\alpha\beta}}{\partial \eta'_1}, -\eta'_3 \frac{\partial \psi^{\alpha\beta}}{\partial \eta'_2}, 0 \right) \\ & + \left(0, 0, \frac{1}{2}(\eta'_3)^2 \frac{B_{33\gamma\delta}}{B_{3333}} \frac{\partial^2 \psi^{\alpha\beta}}{\partial \eta'_\gamma \partial \eta'_\delta} \right). \end{aligned} \tag{3.20}$$

We shall show that

$$\lim_{s \to \infty} \int_{Q'} \left| E\left(\underline{\phi}^{\alpha\beta}_{*} - \underline{\phi}^{\alpha\beta}_{s} \right) \right|^2 d\underline{\eta}' = 0, \tag{3.21}$$

provided that

$$\psi^{\alpha\beta}(\underset{\sim}{\eta}') \text{ has bounded derivatives of order} \leqslant 4, \tag{3.22}$$

which is true for sufficiently regular h. Assertion (3.16) follows immediately from (3.21), since

$$M^{1,s}_{\alpha\beta\gamma\delta} = \frac{1}{L_1 L_2} \int_{Q'} \Sigma_{ij}\left(\underline{\phi}^{\alpha\beta}_{s} + \underline{\Gamma}^{\alpha\beta} \right) E_{ij}\left(\underline{\phi}^{\gamma\delta}_{s} + \underline{\Gamma}^{\gamma\delta} \right) d\underline{\eta}' \tag{3.23}$$

and

$$\lim_{s \to \infty} \frac{1}{L_1 L_2} \int_{Q'} \Sigma_{ij}\left(\underline{\phi}^{\alpha\beta}_{*} + \underline{\Gamma}^{\alpha\beta} \right) E_{ij}\left(\underline{\phi}^{\gamma\delta}_{*} + \underline{\Gamma}^{\gamma\delta} \right) d\underline{\eta}' = M^{a<1}_{\alpha\beta\gamma\delta}.$$

For fixed α and β, define a tensor $\tau_{ij}(\underline{\eta}')$ as follows: for $\gamma, \delta \in \{1,2\}$,

$$\tau_{\gamma\delta} = -\tilde{B}_{\alpha\beta\gamma\delta}\eta'_3 - \tilde{B}_{\gamma\delta\rho\sigma} \frac{\partial^2 \psi^{\alpha\beta}}{\partial \eta'_\rho \partial \eta'_\sigma} \eta'_3;$$

for $\gamma = 1, 2$,

$$\tau_{\gamma 3} = \mathscr{s}^{-1}\left\{\frac{1}{2}\left[(\eta_3')^2 - h^2(\underset{\sim}{\eta}')\right]\tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^3\psi^{\alpha\beta}}{\partial\eta_\rho'\partial\eta_\sigma'\partial\eta_\delta'} - \tilde{B}_{\alpha\beta\gamma\delta}h\frac{\partial h}{\partial\eta_\delta'} - \tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2\psi^{\alpha\beta}}{\partial\eta_\rho'\partial\eta_\sigma'}h\cdot\frac{\partial h}{\partial\eta_\delta'}\right\};$$

and τ_{33} is the solution of

$$\frac{-\partial\tau_{33}}{\partial\eta_3'} = \mathscr{s}^{-1}\frac{\partial\tau_{\gamma 3}}{\partial\eta_\gamma'}, \tag{3.24}$$

$$\tau_{33}\big|_{\eta_3' = \pm h(\underset{\sim}{\eta}')} = \pm\mathscr{s}^{-2}\left(-\tilde{B}_{\alpha\beta\gamma\delta}h\frac{\partial h}{\partial\eta_\gamma'}\frac{\partial h}{\partial\eta_\delta'} - \tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2\psi^{\alpha\beta}}{\partial\eta_\rho'\partial\eta_\sigma'}h\frac{\partial h}{\partial\eta_\gamma'}\frac{\partial h}{\partial\eta_\delta'}\right).$$

One verifies the consistency condition for (3.24),

$$\int_{-h}^{+h}\frac{\partial\tau_{\gamma 3}}{\partial\eta_\gamma'}\,d\eta_3' = 2\mathscr{s}^{-1}\left(\tilde{B}_{\alpha\beta\gamma\delta}h\frac{\partial h}{\partial\eta_\gamma'}\frac{\partial h}{\partial\eta_\delta'} + \tilde{B}_{\gamma\delta\rho\sigma}\frac{\partial^2\psi^{\alpha\beta}}{\partial\eta_\rho'\partial\eta_\sigma'}h\frac{\partial h}{\partial\eta_\gamma'}\frac{\partial h}{\partial\eta_\delta'}\right)$$

by means of (3.2). A straightforward computation shows that

$$\Sigma_{ij}\left(\underset{\sim}{\phi}_*^{\alpha\beta} + \underset{\sim}{\Gamma}^{\alpha\beta}\right) = \tau_{ij} + \mathscr{O}(\mathscr{s}^{-1}), \tag{3.25}$$

and that

$$\frac{\partial}{\partial\eta_j}\tau_{ij} = \mathscr{s}^{-1}\frac{\partial\tau_{i\alpha}}{\partial\eta_\alpha'} + \frac{\partial\tau_{i3}}{\partial\eta_3'} = 0 \quad \text{in } Q',$$

$$\tau_{ij}\nu_j = \left(\frac{1 + |\nabla h|^2}{1 + \mathscr{s}^{-2}|\nabla h|^2}\right)^{1/2}\left(\mathscr{s}^{-1}\tau_{i\alpha}\nu_\alpha' + \tau_{i3}\nu_3'\right) = 0 \quad \text{on } \partial_\pm Q', \tag{3.26}$$

where $\partial_\pm Q'$ are the upper and lower faces of Q', and $\underset{\sim}{\nu}'$ is the outward unit vector normal to $\partial_\pm Q'$.

Let $\underset{\sim}{\chi} = \underset{\sim}{\phi}_*^{\alpha\beta} - \underset{\sim}{\phi}_{\mathscr{s}}^{\alpha\beta}$; using (3.9), (3.25), (3.26), and Green's formula, we see that

$$\begin{aligned}\int_{Q'}\Sigma_{ij}(\underset{\sim}{\chi})E_{ij}(\underset{\sim}{\chi})\,d\underset{\sim}{\eta}' &= \int_{Q'}\Sigma_{ij}\left(\underset{\sim}{\phi}_*^{\alpha\beta} + \underset{\sim}{\Gamma}^{\alpha\beta}\right)E_{ij}(\underset{\sim}{\chi})\,d\underset{\sim}{\eta}' \\ &= \int_{Q'}\tau_{ij}E_{ij}(\underset{\sim}{\chi})\,d\underset{\sim}{\eta}' + \mathscr{O}\left(\mathscr{s}^{-1}\|E(\underset{\sim}{\chi})\|_{L^2(Q')}\right) \\ &= \mathscr{O}\left(\mathscr{s}^{-1}\|E(\underset{\sim}{\chi})\|_{L^2(Q')}\right),\end{aligned} \tag{3.27}$$

which implies (3.21).

4. Integral estimates. This section establishes certain integral inequalities for $\underset{\sim}{u} \in H^1(R(\varepsilon))$. We consider only the case $a = 1$, i.e.

$$R(\varepsilon) = \left\{\underset{\sim}{x} : \underset{\sim}{x} \in \Omega,\ |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon)\right\}$$

(see, however, Remark 4.2 at the end of the section). Our method is to decompose $R(\varepsilon)$ into $\mathscr{O}(\varepsilon^{-2})$ subdomains, each with diameter of order ε, and to apply Korn's inequality on each subdomain.

We begin by reviewing Korn's inequality on the unit-sized domain[1]

$$Q = \{\underline{x}\colon |x_\alpha| < L_\alpha/2, |x_3| < h(\underset{\sim}{x})\}.$$

$\mathscr{R}$ is the space of rigid motions,

$$\underline{\gamma} \in \mathscr{R} \Leftrightarrow \gamma_i(\underline{x}) = c_{ij}x_j + d_i, \text{ for some}$$
$$\underline{d} \in \mathbf{R}^3 \text{ and some skew-symmetric matrix } \underline{c}.$$

$\nabla \underline{u}$ denotes the (nonsymmetric tensor) $\partial_j u_i$, and $e(\underline{u})$ denotes the (symmetric) strain tensor $\frac{1}{2}(\partial_j u_i + \partial_i u_j)$.

LEMMA 4.1. For any $\underline{u} \in H^1(Q)$ there exists $\gamma \in \mathscr{R}$ such that

$$\int_Q |\nabla(\underline{u} - \underline{\gamma})|^2 \, d\underline{x} \leqslant C \int_Q |e(\underline{u})|^2 \, d\underline{x} \tag{4.1}$$

and

$$\int_Q |\underline{u} - \underline{\gamma}|^2 \, d\underline{x} \leqslant C \int_Q |e(\underline{u})|^2 \, d\underline{x}. \tag{4.2}$$

The constant C depends only on Q.

Proof. This follows, for any Lipschitz domain, from the results in [5]. □

Recall that $\tilde{R}(\varepsilon)$ is defined by (2.5). For each pair of integers (k, l) let $R_{kl} = R_{kl}(\varepsilon)$ denote the period cell centered around $(k\varepsilon L_1, l\varepsilon L_2)$,

$$R_{kl} = \left\{\underline{x}\colon |x_1 - k\varepsilon L_1| < \frac{\varepsilon L_1}{2}, |x_2 - l\varepsilon L_2| < \frac{\varepsilon L_2}{2}, |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon)\right\}.$$

Rescaling (4.1) and (4.2) yields the following result.

LEMMA 4.2. For any $\underline{u} \in H^1_{\text{loc}}(\tilde{R}(\varepsilon))$ and any pair (k, l) there exists $\underline{\gamma}^{kl} \in \mathscr{R}$ such that

$$\int_{R_{kl}} \left|\nabla\left(\underline{u} - \underline{\gamma}^{kl}\right)\right|^2 d\underline{x} \leqslant C \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x} \tag{4.3}$$

and

$$\int_{R_{kl}} \left|\underline{u} - \underline{\gamma}^{kl}\right|^2 d\underline{x} \leqslant C\varepsilon^2 \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x}. \tag{4.4}$$

The constant C in (4.3), (4.4) depends only on h.

Let

$$\underline{\gamma}^{kl}(\underline{x}) = \underline{c}^{kl}\underline{x} + \underline{d}^{kl},$$

where $\underline{d}^{kl} \in \mathbf{R}^3$ and $\underline{c}^{kl}$ is a skew-symmetric matrix. Clearly

$$\begin{aligned} &\varepsilon^4\left|\underline{c}^{k+1,l} - \underline{c}^{kl}\right|^2 + \varepsilon^2\left|\underline{d}^{k+1,l} - \underline{d}^{kl}\right|^2 \\ &\quad \leqslant C \int \left|\underline{\gamma}^{k+1,l}(\underline{x}) - \underline{\gamma}^{kl}(\underline{x})\right|^2 dx_2\, dx_3 \\ &\quad \leqslant C \int \left(\left|\underline{\gamma}^{k+1,l} - \underline{u}\right|^2 + \left|\underline{u} - \underline{\gamma}^{kl}\right|^2\right) dx_2\, dx_3, \end{aligned} \tag{4.5}$$

---

[1] We always assume that the rescaled period cell Q is a Lipschitz domain.

where the integrals are over the interface between R_{kl} and $R_{k+1,l}$:

$$\left\{ \underset{\sim}{x}\colon x_1 = (k+\tfrac{1}{2})\varepsilon L_1, |x_2 - l\varepsilon L_2| < \frac{\varepsilon L_2}{2}, |x_3| < \varepsilon h(\underset{\sim}{x}/\varepsilon) \right\}.$$

One has the trace estimate (on any Lipschitz domain)

$$\int_{\partial Q} |\underline{w}|^2 \, d\sigma \leqslant C \int_Q \left(|\nabla \underline{w}|^2 + |\underline{w}|^2 \right) d\underline{x} \tag{4.6}$$

for all $\underline{w} \in H^1(Q)$. Rescaling (4.6), and combining the result with (4.5), we obtain

$$\begin{aligned} &\varepsilon^4 |\underline{c}^{k+1,l} - c^{kl}|^2 + \varepsilon^2 |\underline{d}^{k+1,l} - \underline{d}^{kl}|^2 \\ &\quad \leqslant C\left(\varepsilon \int_{R_{k+1,l}} \left|\nabla\left(\underline{u} - \underline{\gamma}^{k+1,l}\right)\right|^2 d\underline{x} + \varepsilon^{-1} \int_{R_{k+1,l}} \left|\underline{u} - \underline{\gamma}^{k+1,l}\right|^2 d\underline{x} \right. \\ &\qquad \left. + \varepsilon \int_{R_{kl}} \left|\nabla\left(\underline{u} - \underline{\gamma}^{kl}\right)\right|^2 d\underline{x} + \varepsilon^{-1} \int_{R_{kl}} \left|\underline{u} - \underline{\gamma}^{kl}\right|^2 d\underline{x} \right). \end{aligned} \tag{4.7}$$

A combination of (4.3), (4.4) and (4.7) gives

$$\begin{aligned} &\varepsilon^2 |\underline{c}^{k+1,l} - \underline{c}^{kl}|^2 + |\underline{d}^{k+1,l} - \underline{d}^{kl}|^2 \\ &\quad \leqslant C\varepsilon^{-1} \int_{R_{k+1,l} \cup R_{kl}} |e(\underline{u})|^2 \, d\underline{x}. \end{aligned} \tag{4.8}$$

Similarly, we have

$$\begin{aligned} &\varepsilon^2 |\underline{c}^{k,l+1} - \underline{c}^{kl}|^2 + |\underline{d}^{k,l+1} - \underline{d}^{kl}|^2 \\ &\quad \leqslant C\varepsilon^{-1} \int_{R_{k,l+1} \cup R_{kl}} |e(\underline{u})|^2 \, d\underline{x}. \end{aligned} \tag{4.9}$$

PROPOSITION 4.1. For any $\underline{u} \in H^1(R(\varepsilon))$ with $\underline{u} = 0$ on $\partial_0 R(\varepsilon)$,

$$\|\underline{u}\|_{2,\varepsilon} + \|\nabla \underline{u}\|_{2,\varepsilon} \leqslant C\varepsilon^{-1} \|e(\underline{u})\|_{2,\varepsilon}. \tag{4.10}$$

Proof. Extend $\underline{u}$ to $\tilde{R}(\varepsilon)$ by letting it be zero outside $R(\varepsilon)$, and let $\{\underline{\gamma}^{kl}\}$ be the rigid motions introduced in Lemma 4.2; notice that $\underline{\gamma}^{kl} = 0$ if $R_{kl} \cap R(\varepsilon) = \varnothing$. Let $\underline{\sigma}(\underset{\sim}{x})$ and $\underline{\delta}(\underset{\sim}{x})$ denote the piecewise bilinear interpolants to $\underline{c}^{kl}$ and $\underline{d}^{kl}$, *i.e.*

$\sigma_{ij}(k\varepsilon L_1, l\varepsilon L_2) = c_{ij}^{kl}$,

$\delta_i(k\varepsilon L_1, l\varepsilon L_2) = d_i^{kl}$, and

$\sigma_{ij}(\underset{\sim}{x}), \delta_i(\underset{\sim}{x})$ are bilinear functions on

$$\left\{ \underset{\sim}{x}\colon \left|x_1 - \left(k + \frac{1}{2}\right)\varepsilon L_1\right| < \frac{\varepsilon L_1}{2}, \left|x_2 - \left(l + \frac{1}{2}\right)\varepsilon L_2\right| < \frac{\varepsilon L_2}{2} \right\}$$

for each pair of integers (k, l).

It is standard that

$$\int_{\mathbf{R}^2} |\nabla \underline{\delta}|^2 \, d\underset{\sim}{x} \leqslant C \sum_{k,l} \left[|\underline{d}^{k+1,l} - \underline{d}^{kl}|^2 + |\underline{d}^{k,l+1} - \underline{d}^{kl}|^2 \right]$$

and an analogous inequality holds for $\int |\nabla \underline{\sigma}|^2 \, d\underset{\sim}{x}$. It follows, using (4.8)–(4.9), that

$$\varepsilon^2 \int_{\mathbf{R}^2} |\nabla \underline{\sigma}|^2 \, d\underset{\sim}{x} + \int_{\mathbf{R}^2} |\nabla \underline{\delta}|^2 \, d\underset{\sim}{x} \leqslant C\varepsilon^{-1} \int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x}.$$

Since $\underline{\sigma}$ and $\underline{\delta}$ are compactly supported, we conclude by Poincaré's inequality that

$$\varepsilon^2 \int_{\mathbf{R}^2} |\underline{\sigma}|^2 \, d\underset{\sim}{x} + \int_{\mathbf{R}^2} |\underline{\delta}|^2 \, d\underset{\sim}{x} \leqslant C\varepsilon^{-1} \int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x},$$

and hence

$$\sum_{k,l} \left[\varepsilon^2 |\underline{c}^{kl}|^2 + |\underline{d}^{kl}|^2 \right] \leqslant C\varepsilon^{-3} \int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x}. \tag{4.11}$$

Since $\nabla \underline{\gamma}^{kl} = \underline{c}^{kl}$, (4.3) may be rewritten

$$\int_{R_{kl}} |\nabla \underline{u} - \underline{c}^{kl}|^2 \, d\underline{x} \leqslant C \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x},$$

which leads immediately to

$$\int_{R(\varepsilon)} |\nabla \underline{u}|^2 \, d\underline{x} \leqslant C \left(\int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x} + \sum_{k,l} \varepsilon^3 |\underline{c}^{kl}|^2 \right). \tag{4.12}$$

Similarly, since $|\underline{\gamma}^{kl}(\underline{x})| \leqslant C(|\underline{c}^{kl}| + |\underline{d}^{kl}|)$ for every $\underline{x} \in R(\varepsilon)$, (4.4) leads to

$$\int_{R(\varepsilon)} |\underline{u}|^2 \, d\underline{x} \leqslant C \left[\varepsilon^2 \int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x} + \varepsilon^3 \sum_{k,l} \left(|\underline{c}^{kl}|^2 + |\underline{d}^{kl}|^2 \right) \right]. \tag{4.13}$$

A combination of (4.11), (4.12) and (4.13) gives

$$\int_{R(\varepsilon)} \left(|\underline{u}|^2 + |\nabla \underline{u}| \right)^2 \, d\underline{x} \leqslant C\varepsilon^{-2} \int_{R(\varepsilon)} |e(\underline{u})|^2 \, d\underline{x},$$

which is equivalent to (4.10). □

Recall that the space X_ε is defined by (2.11).

Proposition 4.2. For any $\underline{u} \in X_\varepsilon$,

$$\sum_{\alpha=1}^{2} \| u_\alpha + x_3 \partial_\alpha u_3 \|_{2,\varepsilon} \leqslant C\varepsilon \| e(\underline{u}) \|_{2,\varepsilon}. \tag{4.14}$$

Proof. When $\underline{u} \in X_\varepsilon$, one may choose the rigid motions $\underline{\gamma}^{kl}$ of Lemma 4.2 to have the same symmetry properties, *i.e.*

$$d_1^{kl} = d_2^{kl} = c_{12}^{kl} = 0 \tag{4.15}$$

for each k, l. By (4.4) and (4.15),

$$\int_{R_{kl}} |u_\alpha - c_{\alpha 3}^{kl} x_3|^2 \, d\underline{x} = \int_{R_{kl}} |u_\alpha - \gamma_\alpha^{kl}|^2 \, d\underline{x} \leqslant C\varepsilon^2 \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x}.$$

By (4.3), on the other hand,

$$\int_{R_{kl}} |\partial_\alpha u_3 - c_{3\alpha}^{kl}|^2 \, d\underline{x} = \int_{R_{kl}} |\partial_\alpha (u_3 - \gamma_3^{kl})|^2 \, d\underline{x}$$

$$\leqslant C \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x}.$$

Since $|x_3| \leqslant C\varepsilon$ and $c_{3\alpha}^{kl} = -c_{\alpha 3}^{kl}$, this implies that

$$\int_{R_{kl}} |u_\alpha + x_3 \partial_\alpha u_3|^2 \, d\underline{x} \leqslant C\varepsilon^2 \int_{R_{kl}} |e(\underline{u})|^2 \, d\underline{x}. \tag{4.16}$$

Adding (4.16) over all k, l and over $\alpha = 1, 2$ we get (4.14). □

Remark 4.1. Inequalities (4.10) and (4.14) are sharp in their dependence on ε. For (4.10), one sees this by considering $\underline{u} = (-x_3\partial_1 w, -x_3 \partial_2 w, w)$, where $w = w(x_1, x_2)$. For (4.14), one uses $\underline{u} = (0, 0, w)$. □

Remark 4.2. The estimates in this section may be generalized considerably. We assumed that $\underline{u}$ vanishes on $\partial_0 R(\varepsilon)$ to simplify its extension to $\tilde{R}(\varepsilon)$. One verifies, with a little more effort, that Propositions 4.1 and 4.2 remain valid without this condition (modulo a rigid motion, in the case of (4.10)). The argument presented here also works in the case $a < 1$; it applies, moreover, even if $\tilde{R}(\varepsilon)$ is only a Lipschitz domain; and the periodicity of the domain is not essential.

The case $a > 1$ is more subtle; we do not know nontrivial conditions on h which assure (4.10) or (4.14) for that scaling. The methods of [1] and [4] may be relevant in that case. □

5. An averaging lemma. Our attention remains restricted to the case $a = 1$. Q denotes the rescaled period cell (3.7); θ is the mean thickness $\theta = 2\mathscr{M}[h]$; and $\partial_\pm Q$ is the "non-periodic" part of ∂Q,

$$\partial_\pm Q = \partial Q \cap \{ \underline{\eta} : |\eta_\alpha| < L_\alpha/2, \alpha = 1, 2 \}.$$

For any pair of functions $G(\underset{\sim}{x}; \underline{\eta})$ and $g(\underset{\sim}{x}; \underline{\eta})$ which are periodic in $\underset{\sim}{\eta}$, we define

$$\mathscr{F}(G, g)(\underset{\sim}{x}) = \frac{1}{L_1 L_2} \left\{ \int_Q G \, d\underline{\eta} + \int_{\partial_\pm Q} g \, d\sigma \right\} \tag{5.1}$$

($d\sigma$ denotes surface measure). Our goal is the following result.

PROPOSITION 5.1. *Suppose that G and g have derivatives in $\underset{\sim}{x}$ of order $\leqslant 2$ which are $C^{0,\alpha}$ and $C^{1,\alpha}$ in $\underline{\eta}$ uniformly in $\underset{\sim}{x}$, respectively. Then for any $\underline{u} \in X_\varepsilon$*

$$\left| \int_{R(\varepsilon)} G(\underset{\sim}{x}; \underline{x}/\varepsilon) u_3 \, d\underline{x} + \varepsilon \int_{\partial_\pm R(\varepsilon)} g(\underset{\sim}{x}; \underline{x}/\varepsilon) u_3 \, d\sigma \right.$$
$$\left. - \theta^{-1} \int_{R(\varepsilon)} \mathscr{F}(G, g)(\underset{\sim}{x}) u_3 \, d\underline{x} \right| \leqslant C \varepsilon^{3/2} \| e(\underline{u}) \|_{2,\varepsilon}. \tag{5.2}$$

The constant C depends on G, g, and h, but not on ε.

The essence of Proposition 5.1 is the following: if $\mathcal{F}(G, g) = 0$, and if $\underline{w}^\varepsilon$ solves

$$-\partial_j\big(\sigma_{ij}(\underline{w}^\varepsilon)\big) = \begin{cases} 0, & i = 1,2 \\ \varepsilon^2 G, & i = 3 \end{cases} \quad \text{in } R(\varepsilon),$$

$$\sigma_{ij}(\underline{w}^\varepsilon)\nu_j^\varepsilon = \begin{cases} 0, & i = 1,2 \\ \varepsilon^3 g, & i = 3 \end{cases} \quad \text{on } \partial_\pm R(\varepsilon),$$

$$\underline{w}^\varepsilon = 0 \quad \text{on } \partial_0 R(\varepsilon),$$

then

$$\|e(\underline{w}^\varepsilon)\|_{2,\varepsilon} \leqslant C\varepsilon^{7/2}. \tag{5.3}$$

Indeed, if G and g are even in η_3, one proves (5.3) by substituting $\underline{w}^\varepsilon$ in (5.2) and integrating by parts. If G and g are odd then (5.3) is the same as (2.12).

Before beginning the proof, we introduce some more notation. Given a pair G, g with $\mathcal{F}(G, g) = 0$, we say "$\underline{\phi}$ solves the cell problem associated to G and g" if

$$\int_Q \Sigma_{ij}(\underline{\phi})E_{ij}(\underline{\psi})\,d\underline{\eta} = \int_Q G\psi_3\,d\underline{\eta} + \int_{\partial_\pm Q} g\psi_3\,d\sigma \tag{5.4}$$

for every $\underline{\psi} \in H^1(Q)$ which is $\underset{\sim}{\eta}$-periodic.

Recall that $E(\underline{\phi})$ and $\Sigma(\underline{\phi})$ are defined by (3.5) and (3.6). One verifies easily that (5.4) has an $\underset{\sim}{\eta}$-periodic solution, unique up to a translation. Since we have assumed that $\tilde{R}(\varepsilon)$ is a $C^{2,\tilde{\alpha}}$ domain,

$$\|\Sigma_{ij}(\underline{\phi})\|_{C^{1,\alpha}} \leqslant C\big(\|G\|_{C^{0,\alpha}} + \|g\|_{C^{1,\alpha}}\big). \tag{5.5}$$

All norms are on the rescaled period cell Q or the boundary $\partial_\pm Q$. The constant C depends only on Q and not on $\underset{\sim}{x}$, which occurs in (5.4)–(5.5) as a parameter. If we define

$$\tau_{ij} = \Sigma_{ij}(\underline{\phi})(\underset{\sim}{x}; \underline{x}/\varepsilon), \tag{5.6}$$

then (5.5) leads to

$$\|\tau_{ij}\|_{2,\varepsilon} \leqslant C\varepsilon^{1/2}\sup_{\underset{\sim}{x}}\big(\|G\|_{C^{0,\alpha}} + \|g\|_{C^{1,\alpha}}\big). \tag{5.7}$$

Proof of Proposition 5.1. It suffices to consider the case $\mathcal{F}(G, g) = 0$; the general case will follow by considering $G' = G - \theta^{-1}\mathcal{F}(G, g)$, $g' = g$.

Let $\underline{\phi}$ solve the cell problem (5.4) associated to G and g; let τ be as in (5.6); and let $\underline{u} \in X_\varepsilon$. By Green's formula,

$$\begin{aligned}\int_{R(\varepsilon)} Gu_3\,d\underline{x} &+ \varepsilon\int_{\partial_\pm R(\varepsilon)} gu_3\,d\sigma \\ &= \varepsilon\int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta}\Sigma_{i\beta}u_i\,d\underline{x} + \varepsilon\int_{R(\varepsilon)} \tau_{ij}e_{ij}(\underline{u})\,d\underline{x},\end{aligned} \tag{5.8}$$

where $(\partial/\partial x_\beta)\Sigma_{i\beta} = (\partial/\partial x_\beta)\Sigma_{i\beta}(\underline{\phi})$ is evaluated at $\underline{\eta} = \underline{x}/\varepsilon$ *after* differentiation. Notice that $(\partial/\partial x_\gamma)\Sigma_{i\beta}$ is the stress of the cell problem associated to $\partial G/\partial x_\gamma$ and $\partial g/\partial x_\gamma$; since the $\underset{\sim}{x}$-derivatives of G and g are assumed $C^{0,\alpha}$ and $C^{1,\alpha}$ in $\underline{\eta}$, uniformly in $\underset{\sim}{x}$,

$$\left\| \frac{\partial}{\partial x_\beta} \Sigma_{i\beta} \right\|_{2,\varepsilon} \leqslant C\varepsilon^{1/2}. \tag{5.9}$$

We estimate the various terms in (5.8) separately. First,

$$\left| \int_{R(\varepsilon)} \tau_{ij} e_{ij}(\underline{u})\, d\underline{x} \right| \leqslant C\varepsilon^{1/2} \|e(\underline{u})\|_{2,\varepsilon} \tag{5.10}$$

by (5.7) and Hölder's inequality. Next,

$$\begin{aligned} \left| \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Sigma_{\alpha\beta} \cdot (u_\alpha + x_3 \partial_\alpha u_3)\, d\underline{x} \right| &\leqslant C\varepsilon^{1/2} \|u_\alpha + x_3 \partial_\alpha u_3\|_{2,\varepsilon} \\ &\leqslant C\varepsilon^{3/2} \|e(\underline{u})\|_{2,\varepsilon} \end{aligned} \tag{5.11}$$

by (5.9), Hölder's inequality, and Proposition 4.2. Finally,

$$\begin{aligned} \left| \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Sigma_{\alpha\beta} x_3 \partial_\alpha u_3\, d\underline{x} \right| &\leqslant C\varepsilon^{3/2} \|\nabla \underline{u}\|_{2,\varepsilon} \\ &\leqslant C\varepsilon^{1/2} \|e(\underline{u})\|_{2,\varepsilon} \end{aligned} \tag{5.12}$$

by (5.9), Hölder's inequality, and Proposition 4.1. Combining (5.10)–(5.12), we conclude that the right side of (5.8) equals

$$\varepsilon \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Sigma_{3\beta} u_3\, d\underline{x} + \mathcal{O}\left(\varepsilon^{3/2} \|e(\underline{u})\|_{2,\varepsilon}\right). \tag{5.13}$$

At this point we need the following identity, which will be proved later:

$$\int_Q \frac{\partial}{\partial x_\beta} \Sigma_{3\beta}\, d\underline{\eta} = 0. \tag{5.14}$$

This means that $\mathscr{F}(\partial \Sigma_{3\beta}/\partial x_\beta, 0) = 0$. Repeating the above argument with the cell problem associated to $\partial \Sigma_{3\beta}/\partial x_\beta$ and 0, we conclude that

$$\begin{aligned} &\varepsilon \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Sigma_{3\beta} u_3\, d\underline{x} \\ &\quad = \varepsilon^2 \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Xi_{3\beta} u_3\, d\underline{x} + \mathcal{O}\left(\varepsilon^{5/2} \|e(\underline{u})\|_{2,\varepsilon}\right), \end{aligned} \tag{5.15}$$

where Ξ are the stresses associated to the new cell problem. (We use here our hypothesis on the second derivatives of G and g.) By Hölder's inequality, Proposition 4.1, and the analogue of (5.9),

$$\left| \int_{R(\varepsilon)} \frac{\partial}{\partial x_\beta} \Xi_{3\beta} u_3\, d\underline{x} \right| \leqslant C\varepsilon^{-1/2} \|e(\underline{u})\|_{2,\varepsilon}. \tag{5.16}$$

Combining (5.8), (5.13), (5.15), and (5.16), we obtain (5.2).

It remains to prove (5.14). Substituting $\underline{\psi} = (\eta_3, 0, 0)$ into (5.4) gives

$$\int_Q \Sigma_{31}(\underline{\phi})\, d\underline{\eta} = \int_Q \Sigma_{ij}(\underline{\psi}) E_{ij}(\underline{\psi})\, d\underline{\eta} = 0.$$

Since Q is independent of $\underset{\sim}{x}$, it follows that

$$\int_Q \frac{\partial}{\partial x_1}\Sigma_{31}\, d\underline{\eta} = \frac{\partial}{\partial x_1}\int_Q \Sigma_{31}\, d\underline{\eta} = 0.$$

The corresponding assertion for $(\partial/\partial x_2)\Sigma_{32}$ follows using $\underline{\psi} = (0, \eta_3, 0)$, and summation leads to (5.14). □

6. *Convergence*. Let w solve

$$\partial_{\alpha\beta}\left(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w\right) = \mathscr{F}(F, f) \quad \text{in } \Omega,$$

$$w = \partial_n w = 0 \quad \text{on } \partial\Omega,$$

where $M_{\alpha\beta\gamma\delta}$ is defined by (3.10); let $\underline{u}^*$ be as in (3.11); and let $\underline{u}^\varepsilon$ solve the three dimensional elasticity problem (2.7)–(2.9). We shall prove that $\underline{u}^* - \underline{u}^\varepsilon$ converges to zero in energy, and that w is really the limiting vertical displacement. In addition to the regularity hypotheses on Ω and $\tilde{R}(\varepsilon)$, formulated in Sec. 2, we must assume that

All $\underset{\sim}{x}$-derivatives of order $\leqslant 2$ of F and f are $C^{0,\alpha}$ and $C^{1,\alpha}$ in $\underline{\eta}$, respectively (uniformly in $\underset{\sim}{x}$).

By [6], $M_{\alpha\beta\gamma\delta}$ is positive definite; it follows that

$$w \text{ has bounded } \underset{\sim}{x}\text{-derivatives of order } \leqslant 6. \tag{6.2}$$

Let $\zeta(t) \in C^1(0, \infty)$ with $\zeta(t) = 0$ for $t \leqslant 1/2$ and $\zeta(t) = 1$ for $t \geqslant 1$; we define

$$\hat{\zeta}_\varepsilon(\underset{\sim}{x}) = \mathbf{z}\left(\varepsilon^{-1}\operatorname{dist}(\underset{\sim}{x}, \partial\Omega)\right) \text{ for } \underset{\sim}{x} \in \Omega,$$

and

$$\underline{u}^\# = (-x_3\partial_1 w, -x_3\partial_2 w, w) + \left(\left(0, 0, \tfrac{1}{2}(x_3)^2\frac{B_{\alpha\beta33}}{B_{3333}}\right) + \varepsilon^2\underline{\phi}^{\alpha\beta}(\underset{\sim}{x}/\varepsilon)\right)\hat{\zeta}_\varepsilon \cdot \partial_{\alpha\beta}w, \tag{6.3}$$

where $\underline{\phi}^{\alpha\beta}$ is as in (3.9). Notice that $\underline{u}^\# \in X_\varepsilon$.

Since $\tilde{R}(\varepsilon)$ is assumed to be a $C^{2,\alpha}$ domain, a standard regularity result shows that $\underline{\phi}^{\gamma\delta} \in C^{2,\alpha}(\overline{Q})$. In particular the functions

$$\Sigma_{ij}\left(\underline{\phi}^{\gamma\delta}\right) \quad \text{and} \quad \frac{\partial}{\partial\eta_k}\Sigma_{ij}\left(\underline{\phi}^{\gamma\delta}\right)$$

are $C^{1,\alpha}$ and $C^{0,\alpha}$, respectively. We shall use this fact repeatedly, sometimes without direct mention, in what follows.

Lemma 6.1. The functions $\underline{u}^\#$, $\underline{u}^*$ satisfy

$$\left\|e\left(\underline{u}^* - \underline{u}^\#\right)\right\|_{2,\varepsilon} \leqslant C\varepsilon^2. \tag{6.4}$$

Proof. Since $\underline{\phi}^{\alpha\beta}$ and $E_{ij}(\underline{\phi}^{\alpha\beta})$ are bounded functions,

$$\left\|\,|\nabla\hat{\zeta}_\varepsilon| \cdot \underline{\phi}^{\alpha\beta}(\underset{\sim}{x}/\varepsilon)\right\|_{2,\varepsilon} \leqslant C,$$

$$\left\|(1 - \hat{\zeta}_\varepsilon)\underline{\phi}^{\alpha\beta}(\underset{\sim}{x}/\varepsilon)\right\|_{2,\varepsilon} \leqslant C\varepsilon,$$

$$\left\|(1 - \hat{\zeta}_\varepsilon)E\left(\underline{\phi}^{\alpha\beta}\right)(\underset{\sim}{x}/\varepsilon)\right\|_{2,\varepsilon} \leqslant C\varepsilon.$$

The estimate (6.4) follows easily, using (6.2), □

LEMMA 6.2 For each β, γ, δ

$$\int_Q \Sigma_{3\beta}\left(\underline{\phi}^{\gamma\delta}\right) d\underline{\eta} = 0. \tag{6.5}$$

Proof. One argues as in the proof of (5.14), using (3.9) instead of (5.4). □

LEMMA 6.3. Let $\underline{\nu}$ denote the outward unit vector normal to $\partial_{\pm} Q$. For each β, γ, δ, the functions

$$G(\underline{\eta}) = -\eta_3 \frac{\partial}{\partial \eta_\alpha} \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right), \qquad \underline{\eta} \in Q,$$

$$g(\underline{\eta}) = \left[-\eta_3^2 \tilde{B}_{\alpha\beta\gamma\delta} + \eta_3 \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right)\right] \nu_\alpha, \quad \underline{\eta} \in \partial_{\pm} Q,$$

satisfy

$$\mathscr{F}(G, g) = 0. \tag{6.6}$$

Proof. By Green's formula,

$$\int_Q \eta_3 \frac{\partial}{\partial \eta_\alpha} \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right) d\underline{\eta} = \int_{\partial_{\pm} Q} \left[-\eta_3^2 \tilde{B}_{\alpha\beta\gamma\delta} + \eta_3 \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right)\right] \nu_\alpha \, d\sigma;$$

This is equivalent to (6.6). □

LEMMA 6.4. For each α, β, γ, δ

$$\int_Q \eta_3 \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right) d\underline{\eta} = \int_Q \eta_3 \tilde{B}_{\alpha\beta\lambda\mu} E_{\lambda\mu}\left(\underline{\phi}^{\gamma\delta}\right) d\underline{\eta}. \tag{6.7}$$

Proof. Substituting $\underline{\psi} = (0, 0, \frac{1}{2}(\eta_3)^2)$ in (3.9), and noting that $\Sigma_{33}(\underline{\Gamma}^{\gamma\delta}) = 0$, we see that

$$\int_Q \eta_3 \Sigma_{33}\left(\underline{\phi}^{\gamma\delta}\right) d\underline{\eta} = 0; \tag{6.8}$$

on the other hand,

$$\tilde{B}_{\alpha\beta\lambda\mu} E_{\lambda\mu}\left(\underline{\phi}^{\gamma\delta}\right) = \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right) - \frac{B_{\alpha\beta 33}}{B_{3333}} \Sigma_{33}\left(\underline{\phi}^{\gamma\delta}\right); \tag{6.9}$$

a combination of (6.8) and (6.9) yields (6.7). □

One easily verifies that (3.9) is equivalent to

$$\frac{\partial}{\partial \eta_j} \Sigma_{ij}\left(\underline{\phi}^{\gamma\delta}\right) = 0 \qquad \text{in } Q,$$

$$\Sigma_{ij}\left(\underline{\phi}^{\gamma\delta}\right) \nu_j = \eta_3 \tilde{B}_{i\beta\gamma\delta} \nu_\beta \quad \text{on } \partial_{\pm} Q.$$

If τ is defined by

$$\tau_{ij} = \left[-x_3 \tilde{B}_{ij\gamma\delta} + \varepsilon \Sigma_{ij}\left(\underline{\phi}^{\gamma\delta}\right)(\underline{x}/\varepsilon)\right] \partial_{\gamma\delta} w, \tag{6.10}$$

one computes that

$$\partial_j \tau_{ij} = \left[-x_3 \tilde{B}_{i\beta\gamma\delta} + \varepsilon \Sigma_{i\beta}\left(\underline{\phi}^{\gamma\delta}\right)(\underline{x}/\varepsilon)\right] \partial_{\beta\gamma\delta} w \tag{6.11}$$

in $R(\varepsilon)$, and

$$\tau_{ij} \nu_j^{\varepsilon} = 0 \tag{6.12}$$

on $\partial_{\pm} R(\varepsilon)$.

PROPOSITION 6.1. For any $\underline{v} \in X_\varepsilon$,

$$\int_{R(\varepsilon)} \tau_{ij} e_{ij}(\underline{v})\, d\underline{x} = \varepsilon^2 \theta^{-1} \int_{R(\varepsilon)} \mathscr{F}(F, f) v_3\, d\underline{x} + \mathscr{O}\big(\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}\big). \tag{6.13}$$

Proof. By Green's formula and (6.12),

$$\int_Q \tau_{ij} e_{ij}(\underline{v})\, d\underline{x} = -\int_{R(\varepsilon)} \partial_j(\tau_{ij}) v_i\, d\underline{x}.$$

We apply Proposition 5.1, using (6.11), (6.5), and the fact that $\tilde{B}_{3\beta\gamma\delta} = 0$, to see that

$$\int_{R(\varepsilon)} \partial_j(\tau_{3j}) v_3\, d\underline{x} \leqslant C\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}.$$

Writing $v_\alpha = -x_3 \partial_\alpha v_3 + (v_\alpha + x_3 \partial_\alpha v_3)$, and applying Proposition 4.2, we obtain

$$\int_{R(\varepsilon)} \partial_j(\tau_{\alpha j}) v_\alpha\, d\underline{x} = -\int_{R(\varepsilon)} x_3 \partial_j(\tau_{\alpha j}) \partial_\alpha v_3\, d\underline{x} + \mathscr{O}\big(\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}\big).$$

A combination of these results yields

$$\int_{R(\varepsilon)} \tau_{ij} e_{ij}(\underline{v})\, d\underline{x} = \int_{R(\varepsilon)} x_3 \partial_j(\tau_{\alpha j}) \partial_\alpha v_3\, d\underline{x} + \mathscr{O}\big(\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}\big). \tag{6.14}$$

We use Green's formula again:

$$\int_{R(\varepsilon)} x_3 \partial_j(\tau_{\alpha j}) \partial_\alpha v_3\, d\underline{x} = -\int_{R(\varepsilon)} x_3 \partial_{\alpha j}(\tau_{\alpha j}) v_3\, d\underline{x} + \int_{\partial_\pm R(\varepsilon)} x_3 \partial_j(\tau_{\alpha j}) \nu_\alpha^\varepsilon v_3\, ds. \tag{6.15}$$

Now,

$$x_3 \partial_j(\tau_{\alpha j}) \nu_\alpha^\varepsilon = \varepsilon^2 (\mathrm{I} + \mathrm{II}),$$

with

$$\mathrm{I} = -\eta_3^2 \tilde{B}_{\alpha\beta\gamma\delta} \nu_\alpha \big|_{\underline{\eta} = \underline{x}/\varepsilon} \partial_{\beta\gamma\delta} w,$$
$$\mathrm{II} = \eta_3 \Sigma_{\alpha\beta}\big(\underline{\phi}^{\gamma\delta}\big) \nu_\alpha \big|_{\underline{\eta} = \underline{x}/\varepsilon} \partial_{\beta\gamma\delta} w,$$

and

$$-x_3 \partial_{\alpha j}(\tau_{\alpha j}) = \varepsilon \mathrm{III} + \varepsilon^2 \mathrm{IV},$$

with

$$\mathrm{III} = -\eta_3 \frac{\partial}{\partial \eta_\alpha} \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right)\big|_{\underline{\eta}=\underline{x}/\varepsilon} \partial_{\beta\gamma\delta} w,$$

$$\mathrm{IV} = \left(\eta_3^2 \tilde{B}_{\alpha\beta\gamma\delta} - \eta_3 \Sigma_{\alpha\beta}\left(\underline{\phi}^{\gamma\delta}\right)\right)\big|_{\underline{\eta}=\underline{x}/\varepsilon} \partial_{\alpha\beta\gamma\delta} w.$$

By Lemma 6.3 and Proposition 5.1,

$$\varepsilon \int_{R(\varepsilon)} (\mathrm{III})\, v_3\, d\underline{x} + \varepsilon^2 \int_{\partial_\pm R(\varepsilon)} (\mathrm{I} + \mathrm{II})\, v_3\, ds = \mathscr{O}\left(\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}\right) \tag{6.16}$$

similarly, by Lemma 6.4, Proposition 5.1 and (3.10),

$$\begin{aligned} \varepsilon^2 \int_{R(\varepsilon)} (\mathrm{IV})\, v_3\, d\underline{x} \\ = \varepsilon^2 \theta^{-1} \int_{R(\varepsilon)} M_{\alpha\beta\gamma\delta} \partial_{\alpha\beta\gamma\delta} w v_3\, d\underline{x} + \mathscr{O}\left(\varepsilon^{7/2} \|e(\underline{v})\|_{2,\varepsilon}\right). \end{aligned} \tag{6.17}$$

Since $M_{\alpha\beta\gamma\delta}$ is constant and w satisfies (3.1), (6.15)–(6.17) imply

$$\begin{aligned} \int_{R(\varepsilon)} x_3 \partial_j(\tau_{\alpha j}) \partial_\alpha v_3\, d\underline{x} \\ = \varepsilon^2 \theta^{-1} \int_{R(\varepsilon)} \mathscr{F}(F, f)\, v_3\, d\underline{x} + \mathscr{O}\left(\varepsilon^{5/2} \|e(\underline{v})\|_{2,\varepsilon}\right). \end{aligned} \tag{6.18}$$

A combination of (6.14) and (6.18) yields (6.13). □

We are ready to prove the main result of this paper.

THEOREM 6.1. The Ansatz $\underline{u}^*$, defined by (3.11), and the displacement $\underline{u}^\varepsilon$, defined by (2.7)–(2.9), satisfy

$$\|e(\underline{u}^* - \underline{u}^\varepsilon)\|_{2,\varepsilon} \leqslant C\varepsilon^2. \tag{6.19}$$

Proof. We shall prove

$$\left\|e\left(\underline{u}^\# - \underline{u}^\varepsilon\right)\right\|_{2,\varepsilon} \leqslant C\varepsilon^2, \tag{6.20}$$

with $\underline{u}^\#$ as in (6.3). The estimate (6.19) is an immediate consequence, using Lemma 6.1.

To prove (6.20), we first observe that

$$\left\|\sigma\left(\underline{u}^\#\right) - \tau\right\|_{2,\varepsilon} \leqslant C\varepsilon^2 \tag{6.21}$$

where τ is defined by (6.10). Indeed, a simple computation gives that

$$\|\sigma(\underline{u}^*) - \tau\|_{2,\varepsilon} \leqslant C\varepsilon^{5/2},$$

while by Lemma 6.1,

$$\left\|\sigma(\underline{u}^*) - \sigma\left(\underline{u}^\#\right)\right\|_{2,\varepsilon} \leqslant C\varepsilon^2;$$

(6.21) follows by means of the triangle inequality.

By Proposition 6.1 and (6.21),

$$\int_{R(\varepsilon)} \sigma_{ij}(\underline{u}^{\#}) e_{ij}(\underline{v})\, d\underline{x} = \varepsilon^2 \theta^{-1} \int_{R(\varepsilon)} \mathscr{F} \cdot v_3\, d\underline{x} + \mathscr{O}\left(\varepsilon^2 \|e(\underline{v})\|_{2,\varepsilon}\right) \tag{6.22}$$

for any $\underline{v} \in X_\varepsilon$. Also,

$$\int_{R(\varepsilon)} \sigma_{ij}(\underline{u}^{\varepsilon}) e_{ij}(\underline{v})\, d\underline{x} = \varepsilon^2 \int_{R(\varepsilon)} F v_3\, d\underline{x} + \varepsilon^3 \int_{\partial_{\pm} R(\varepsilon)} f v_3\, d\sigma$$

$$= \varepsilon^2 \theta^{-1} \int_{R(\varepsilon)} \mathscr{F} v_3\, d\underline{x} + \mathscr{O}\left(\varepsilon^{7/2} \|e(\underline{v})\|_{2,\varepsilon}\right) \tag{6.23}$$

by (2.7)–(2.9), Green's formula and Proposition 5.1. Taking $\underline{v} = \underline{u}^{\#} - \underline{u}^{\varepsilon}$, and subtracting (6.23) from (6.22), we conclude that

$$\int_{R(\varepsilon)} \sigma_{ij}(\underline{u}^{\#} - \underline{u}^{\varepsilon}) e_{ij}(\underline{u}^{\#} - \underline{u}^{\varepsilon})\, d\underline{x} \leqslant C\varepsilon^2 \|e(\underline{u}^{\#} - \underline{u}^{\varepsilon})\|_{2,\varepsilon},$$

from which (6.20) follows. □

Remark 6.1. Had we specified the ε-dependent boundary condition

$$\underline{u}^{\varepsilon}|_{\partial R_0(\varepsilon)} = \underline{u}^{*}|_{\partial R_0(\varepsilon)}$$

instead of (2.9) then the introduction of $\underline{u}^{\#}$ would not have been necessary. The above argument yields $\|e(\underline{u}^{*} - \underline{u}^{\varepsilon})\|_{2,\varepsilon} \leqslant C\varepsilon^{5/2}$ when $\underline{u}^{\varepsilon}$ is defined this way. □

One verifies readily that $\|e(\underline{u}^{*})\|_{2,\varepsilon} \sim \varepsilon^{3/2}$ whenever $\mathscr{F} \neq 0$. It follows, using (6.19), that

$$C^{-1}\varepsilon^{3/2} \leqslant \|e(\underline{u}^{\varepsilon})\|_{2,\varepsilon} \leqslant C\varepsilon^{3/2}, \tag{6.24}$$

with C depending on $\mathscr{F}$ but not on ε. A combination of (6.19) and (6.24) yields the relative error estimate

$$\|e(\underline{u}^{*} - \underline{u}^{\varepsilon})\|_{2,\varepsilon} / \|e(\underline{u}^{\varepsilon})\|_{2,\varepsilon} \leqslant C\varepsilon^{1/2}.$$

A similar argument, using (4.10), shows that

$$\frac{\|\underline{u}^{*} - \underline{u}^{\varepsilon}\|_{2,\varepsilon}}{\|\underline{u}^{\varepsilon}\|_{2,\varepsilon}} + \frac{\|\nabla \underline{u}^{*} - \nabla \underline{u}^{\varepsilon}\|_{2,\varepsilon}}{\|\nabla \underline{u}^{\varepsilon}\|_{2,\varepsilon}} \leqslant C\varepsilon^{1/2}.$$

One may also compare u_3^{ε} and w directly:

COROLLARY 6.1. If one defines

$$w^{\varepsilon}(\underset{\sim}{x}) = \frac{1}{2\varepsilon h(\underset{\sim}{x}/\varepsilon)} \int_{-\varepsilon h(\underset{\sim}{x}/\varepsilon)}^{+\varepsilon h(\underset{\sim}{x}/\varepsilon)} u_3^{\varepsilon}(\underline{x})\, dx_3$$

whenever $h(\underset{\sim}{x}/\varepsilon) \neq 0$, then

$$\left(\int_{\Omega} |w - w^{\varepsilon}|^2 h(\underset{\sim}{x}/\varepsilon)\, d\underset{\sim}{x}\right)^{1/2} \leqslant C\varepsilon^{1/2}. \tag{6.25}$$

Proof. We consider w, w^ε to be defined on $R(\varepsilon)$. By Poincaré's inequality

$$\int_{-\varepsilon h(\underset{\sim}{x}/\varepsilon)}^{\varepsilon h(\underset{\sim}{x}/\varepsilon)} |w^\varepsilon - u_3^\varepsilon|^2 \, dx_3 \leqslant C\varepsilon^2 \int_{-\varepsilon h(\underset{\sim}{x}/\varepsilon)}^{\varepsilon h(\underset{\sim}{x}/\varepsilon)} \left| \frac{\partial}{\partial x_3} u_3^\varepsilon \right|^2 dx_3$$

$$= C\varepsilon^2 \int_{-\varepsilon h(\underset{\sim}{x}/\varepsilon)}^{\varepsilon h(\underset{\sim}{x}/\varepsilon)} |e_{33}(\underline{u}^\varepsilon)|^2 \, dx_3;$$

integration over Ω yields

$$\int_{R(\varepsilon)} |u_3^\varepsilon - w^\varepsilon|^2 \, d\underline{x} \leqslant C\varepsilon^2 \int_{R(\varepsilon)} |e(\underline{u}^\varepsilon)|^2 \, d\underline{x} \leqslant C\varepsilon^5.$$

One computes directly from (6.3) that

$$\int_{R(\varepsilon)} |w - u_3^{\#}|^2 \, d\underline{x} \leqslant C\varepsilon^5.$$

Combining these two estimates with (4.10) and (6.20), we conclude that

$$\left(\int_{R(\varepsilon)} |w - w^\varepsilon|^2 \, d\underline{x} \right)^{1/2} \leqslant C\varepsilon^{5/2} + \|u_3^{\#} - u_3^\varepsilon\|_{2,\varepsilon} \leqslant C\varepsilon.$$

It follows that

$$\int_\Omega |w - w^\varepsilon|^2 h(\underset{\sim}{x}/\varepsilon) \, d\underset{\sim}{x} = \frac{1}{2\varepsilon} \int_{R(\varepsilon)} |w - w^\varepsilon|^2 \, d\underline{x} \leqslant C\varepsilon. \qquad \square$$

Remark 6.2. If $h(\underset{\sim}{\eta}) \geqslant c > 0$, *i.e.* if the plate has no holes, then (6.25) becomes

$$\left(\int_\Omega |w - w^\varepsilon|^2 \, d\underset{\sim}{x} \right)^{1/2} \leqslant C\varepsilon^{1/2}. \qquad \square$$

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