A THEORY FOR THE WAVE-INDUCED MOTION OF FINITE MONOMOLECULAR FILMS*

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Abstract. The fluctuating velocity field of monomolecular films of arbitrary configuration is investigated when gravity waves propagate on the air-water interface. The surface-active material is assumed to have visco-elastic properties and to be insoluble. Boundary-layer techniques are employed, and a Dirichlet boundary value problem, involving Helmholtz' equation for the divergence of the velocity field, is obtained for the film. Circular and rectangular films are considered explicitly, whilst an approximate method is given for slender films of arbitrary orientation. Application is made to viscous wave-damping.

1. Introduction. Effects of monomolecular surface films on waves at an air-water interface have been considered experimentally, both in the laboratory, [1, 11], and in the open ocean, [8, 12], many investigations being concerned with dissipative properties. In wave-tanks, the water surface is usually completely covered by the film. Conversely, oceanic surface films (or "slicks"), of natural or artificial origin, may sometimes be regarded as *finite*, depending on the wavelengths involved. For long-crested progressive waves, much theoretical work on films of infinite extent exists, as exemplified in [6, 7, 10, 11]. Miles [13] investigated temporal wave-damping for water in a container, the upper surface being a monolayer.

The situation when the film only partially covers the available water surface has been considered recently in [3], where a two-dimensional model is given for gravity waves, normally incident on a slick of finite width b. Conditions near the "edge" of the film were treated by analogy with [4], which concerns the special case of an inextensible, horizontally-immobile, semi-infinite film ($b = \infty$).

In the present work, the model of [3] is extended to slicks of finite area and arbitrary configuration, subjected to long-crested progressive waves. We make the following assumptions.

(a1) The monomolecular film has linear visco-elastic properties.

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(a2) The surface-active material is fully spread and insoluble.

(a3) The incident waves are gravity waves, and capillarity is negligible except near the film perimeter, cf. [4].

(a4) Boundary-layer theory is applicable to both the vortical layers at the air-water interface and the film.

Due to (a3, a4), inviscid theory yields an approximately constant wave-number k, satisfying the dispersion relation

$$\sigma^2 \simeq gk \tanh kh, \tag{1.1}$$

where σ is the prescribed wave-frequency, g is gravitational acceleration, and h is the uniform mean depth of water. If ν is the kinematic viscosity of water, the rotational layer has vertical length-scale $\delta = \varepsilon k^{-1} = (2\nu/\sigma)^{1/2}$, so that (a4) requires

$$\delta \ll \min(k^{-1}, h, L), \tag{1.2}$$

where L is a characteristic length-scale for the film. In particular,

$$\varepsilon = k\delta \ll 1. \tag{1.3}$$

Mainly, we seek a method for the determination of the oscillatory response of surfactant elements to the incident waves, and this is accomplished *via* a boundary value problem for Helmholtz' equation. In passing, some justification is given to a basic assumption of [13]. As an application of the theory, the direct influence of the film on the spatial attenuation of the wave amplitude is examined, and a numerical example is presented.

2. Preliminary details. We consider a laterally-unbounded region of water, having uniform equilibrium depth h, density ρ and viscosity $\mu = \rho\nu$. The air-water interface is clean, apart from the presence of a monomolecular film of finite area A. The oscillatory velocity field αq is sought within the boundary layer adjacent to A, when long-crested gravity waves are incident on the slick, α being the maximum wave-slope. We choose Cartesian co-ordinates, x, y and z, with z measured vertically upwards from the mean interfacial level. The incident wave is specified by the interfacial position

$$z = z_i = a\exp[i(\sigma t - k_v x)], \tag{2.1}$$

where a is the amplitude, σ the real angular frequency, and k_v the complex wave-number. (Only real parts of complex expressions are physically significant.) The *dominant* part of k_v is real, and, for gravity waves, remains constant within the slick, (cf. [4]). In the following, we retain only this dominant, inviscid part, $k_v \simeq k$, take $\alpha = ak \ll 1$, and write $\lambda = 2\pi/k$ for the wavelength; the relative error here is $O(\varepsilon)$. In the Cartesian system, the velocity and position vector $\underset{\sim}{r}$ are given by

$$\underset{\sim}{q} = u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}, \qquad \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}, \tag{2.2}$$

where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors, and $|\underset{\sim}{q}| = O(\sigma/k)$. The horizontal velocity of the film is denoted by αv, where

$$\underset{\sim}{v} = v_x(x, y, t)\underset{\sim}{i} + v_y(x, y, t)\underset{\sim}{j}. \tag{2.3}$$

3. The monomolecular film. The linearized constitutive relation satisfied by the surface stress $\underset{\sim}{\tau}_f$ in the film is taken to be similar to that given by Miles [13] and Slattery [14, pp. 155 and 158], *viz.*

$$\underset{\sim}{\tau}_f = \underset{\sim}{\nabla} T_1 + \eta_d \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{v}) + \eta_s \nabla^2 \underset{\sim}{v}, \tag{3.1}$$

wherein T, the interfacial tension, has an expansion of the form $T_0 + \alpha T_1 + \ldots$, and η_d and η_s are the surface dilatational and shear viscosities. Miles considered a film *completely* covering the upper surface of fluid in a container having *vertical* sides, and made the assumption that the horizontal components of the local film velocity, and fluid velocity just beneath the oscillatory layer, are parallel. In fact, Miles assumed even more, in that the ratio of the magnitudes, and the difference in phases, of these components are actually *constants*. As a consequence, the vertical component of vorticity for the film,

$$\Omega = \underset{\sim}{k} \cdot \operatorname{curl} \underset{\sim}{v}, \tag{3.2}$$

is zero, whence the corresponding component, $\underset{\sim}{k} \cdot \operatorname{curl} \underset{\sim}{q}$, vanishes throughout the fluid. In the present work, Miles' assumption is not made. Indeed, it is not, in general, valid within the framework of our idealized model of the finite slick. Further, we do *not*, in general, have the simplification that Ω vanishes.

As in [10], we take $T = T(\Gamma)$, where Γ is the surfactant concentration; for insoluble films, the linearized conservation equation gives

$$\partial \Gamma_1 / \partial t + \Gamma_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0, \tag{3.3}$$

or

$$\partial T_1 / \partial t - \chi \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0, \tag{3.4}$$

where $\chi = -\Gamma_0 (dT/d\Gamma)_{\Gamma=\Gamma_0}$. By Newton's second law of motion, the sum of the surface stress in the film, and the shear stresses due to the air and water, must be zero. We assume that the atmospheric forces acting on the film are due to the wave motion alone. Thus,

$$\underset{\sim}{\tau}_f - \mu(\partial \underset{\sim}{q}_H/\partial z + \underset{\sim}{\nabla}_H w) + \mu_a(\partial \underset{\sim}{q}_{aH}/\partial z + \underset{\sim}{\nabla}_H w_a) = 0 \qquad (z = 0), \tag{3.5}$$

where the subscript H denotes the horizontal component of a vector quantity, and the subscript a refers to a quantity evaluated for the air. On using equations (3.1) and (3.4) to eliminate $\underset{\sim}{\tau}_f$ and T_1, we obtain

$$\left[\chi + (\eta_d + \eta_s)\frac{\partial}{\partial t}\right] \underset{\sim}{\nabla} D + \eta_s \underset{\sim}{k} \times \underset{\sim}{\nabla} \frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial t}\left[\mu\left(\frac{\partial \underset{\sim}{q}_H}{\partial z} + \underset{\sim}{\nabla}_H w\right) - \mu_a\left(\frac{\partial \underset{\sim}{q}_{aH}}{\partial z} + \underset{\sim}{\nabla}_H w_a\right)\right], \tag{3.6}$$

where

$$D = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \tag{3.7}$$

denotes the *divergence* of the horizontal component of velocity of the slick.

At the film perimeter, C, the remaining conditions required for determination of the linearized molecular velocities are shown in Appendix A to be given by

$$\left.\begin{aligned} T_1 + (\eta_d - \eta_s)D + 2\eta_s \partial v_n/\partial n = 0 \\ \eta_s(\Omega - 2\partial v_s/\partial n) = 0 \end{aligned}\right\} \quad (\underline{r}_H \in C), \quad \begin{aligned} (3.8a) \\ (3.8b) \end{aligned}$$

where (s, n) are orthogonal curvilinear co-ordinates for the slick, with s denoting arc-length measured clockwise around C (viewed from above), and n being measured normally inwards to C. The quantities v_s, v_n denote the components of the velocity vector $\underline{v}$ in terms of this co-ordinate system. By (3.4), we note that, for slicks of uniform width, and infinite extent in the y-direction, the conditions (3.8a,b) reduce to the requirement that T_1 vanishes at the edges of the film, when the latter are normal to the direction of propagation of long-crested waves. This is in agreement with [3]. We emphasize, however, that, whilst the boundary conditions (3.8a,b) are taken to be physically appropriate for our model, the intrinsic composition of "edges" of natural or artificial slicks is, in general, very difficult to quantify.

The oscillatory boundary layers adjacent to the whole interface have thickness $O(\delta)$. By analogy with [3, 4], they are viewed in terms of the following sub-division, which, apart from "weak" slicks (cf. §4), proves sufficient for our purposes.

(a) Adjacent to the clean part of the interface, we have region S.

(b) In the neighbourhood of the film perimeter C, there is a "transition region" TR, wherein δ is also a horizontal scale normal to C. Region TR connects S to

(c) a cylindrical region S', containing the remaining part A' (outside that in TR) of the slick.

For weak slicks, a further length-scale, Δ say, normal to C, which is much smaller than the wavelength λ, may be $\ll$, $\simeq$ or $\gg$ δ, depending on the intensity of concentration of the surfactant. Such slicks are considered in Appendix C.

As discussed in Appendix B, the conditions (3.8a,b) may, to the *present order of accuracy*, be effectively applied at the "perimeter", C' say, of A'. This property is of extreme significance in that it enables us to *simplify* the calculation of the dominant correction (due to the presence of the slick) of the velocity field $\underline{q}_H^{(0)}$ of §4. Specifically, we are able to *avoid* the technically-complicated calculation of $\underline{q}_H$ in TR, as is apparent from Appendix B. For weak slicks, the property holds only when $\lambda \gg \Delta \gg \delta$, but, for very weak slicks such that $\Delta \leqslant O(\delta)$, $\underline{q}_H^{(0)}$ is readily calculated within A' (see Appendix C).

4. Boundary value problem for the slick. Henceforth, the formulation is in terms of non-dimensional variables

$$t' = \sigma t, \quad \underline{r}' = k\underline{r}, \quad \underline{q}' = (\sigma/k)\underline{q}, \quad \underline{v}' = (\sigma/k)\underline{v}, \quad T' = k^3 T/\sigma^2 \rho,$$

but the "dash" is omitted for simplicity. Excluding weak slicks for the moment, the *dominant part*, $O(1)$, of the horizontal velocity within region S' satisfies the boundary-layer equation

$$\partial \underline{q}_H/\partial t = \partial \underline{q}_H^{(\infty)}/\partial t + \partial^2 \underline{q}_H/\partial Z^2 \qquad (Z = z/\varepsilon), \tag{4.1}$$

whence

$$\underline{q}_H = \underline{q}_H^{(\infty)} + \left(\underline{q}_H^{(0)} - \underline{q}_H^{(\infty)}\right)\exp\left[(1+i)Z/2^{1/2}\right]. \tag{4.2}$$

In these equations, Z is a stretched variable, $\underline{q}_H^{(\infty)}$ denotes the inviscid, irrotational velocity just beneath the vortical region S', specifically, as $Z \to -\infty$, and $\underline{q}_H^{(0)}$ is the horizontal velocity at the monomolecular slick. Since $|\underline{q}_H| = O(1)$, the continuity equation gives $w = w^{(0)} + O(\varepsilon)$ within S', whence irrotationality of the velocity field beyond the oscillatory layer leads to

$$\underline{\nabla}_H w^{(0)} = \underline{\nabla}_H w^{(\infty)} = \left(\partial \underline{q}_H/\partial z\right)^{(\infty)}. \tag{4.3}$$

Upon introduction of the parameters

$$\{\xi, \zeta_d, \zeta_s\} = k^2(\rho\mu\sigma/2)^{-1/2}\{\chi/\sigma, \eta_d, \eta_s\}, \tag{4.4}$$

which represent surface compressional modulus and surface viscosity, employment of the time-periodicity of the linearized velocity field, and use of equations (4.2, 4.3), equation (3.6) becomes

$$\begin{aligned}(\xi + i\zeta)\underline{\nabla}D + i\zeta_s\underline{k} \times \underline{\nabla}\Omega = (1-i)\left[\Delta\underline{q}_H + (\rho_a\mu_a/\rho\mu)^{1/2}\Delta\underline{q}_{a_H}\right] \\ + i2^{1/2}\varepsilon\underline{\nabla}_H\left[w^{(\infty)} - (\mu_a/\mu)w_a^{(\infty)}\right],\end{aligned} \tag{4.5}$$

where $\zeta = \zeta_d + \zeta_s$ and $\Delta\underline{q}_H = \underline{q}_H^{(\infty)} - \underline{q}_H^{(0)}$ denotes the change in horizontal velocity across the interfacial boundary-layer region S', adjacent to the film. In the present work, the viscous and capillary length-scales are assumed to be comparable, whence the elastic parameter ξ becomes a measure of the ratio $d[\log(T/T_0)]/d[\log(\Gamma/\Gamma_0)]$.

We note that

$$\underline{q}_H^{(0)} = \underline{q}_{aH}^{(0)} = \underline{v}, \qquad w^{(0)} = w^{(\infty)} = w_a^{(\infty)}, \tag{4.6}$$

by the requirement of continuity of velocity at the slick. For gravity waves, $10^{-5} \leqslant \varepsilon \leqslant 10^{-2}$ (approximately), $\mu_a/\mu \simeq 1.35 \times 10^{-2}$, $(\rho_a\mu_a/\rho\mu)^{1/2} = (\varepsilon/\varepsilon_a)(\mu_a/\mu) \simeq 3.67 \times 10^{-3}$. Moreover, whilst we expect $|\underline{q}_{aH}^{(\infty)} - \underline{q}_{aH}^{(0)}| = O(1)$, $\forall\xi, \zeta$, which holds even in the absence of a slick, the difference in horizontal velocity, $\underline{q}_H^{(\infty)} - \underline{q}_H^{(0)}$, across the rotational layer in the *water* is $O(1)$ only for slicks of sufficient strength that $|\xi + i\zeta| \geqslant O(1)$. In such cases, equation (4.5) may be written as

$$(\xi + i\zeta)\underline{\nabla}D + i\zeta_s\underline{k} \times \underline{\nabla}\Omega = (1-i)\Delta\underline{q}_H, \tag{4.7}$$

to a good degree of accuracy; "weak" slicks, having small visco-elastic parameters such that $|\xi + i\zeta| \ll 1$, are considered in Appendix C, and are relatively *ineffective* in that they do not modify the velocity field at lowest order, $O(1)$. From equation (4.7), D and Ω satisfy the following non-homogeneous and homogeneous forms, respectively, of Helmholtz' equation,

$$\left.\begin{aligned}\nabla^2 D + m^2 D = m^2 d^{(\infty)} \quad (4.8a)\\ \nabla^2\Omega + p^2\Omega = 0 \quad (4.8b)\end{aligned}\right\} \quad (\underline{r}_H \in A'),$$

where

$$m^2 = (1-i)/(\xi + i\zeta), \quad p^2 = (1-i)/i\zeta_s, \quad d^{(\infty)} = \underline{\nabla}_H \cdot \underline{q}_H^{(\infty)} = O(1), \tag{4.9}$$

and the irrotationality of $\underline{q}^{(\infty)}$ has been used. From equation (B6) of Appendix B, together with principles of asymptotic matching, the boundary conditions for equations (4.8a,b) are

$$\left.\begin{aligned}(\xi + i\zeta)D &= 2i\zeta_s(\partial v_s/\partial s - \kappa v_n) \\ \Omega = 2\partial v_s/\partial n &= -2(\partial v_n/\partial s + \kappa v_s)\end{aligned}\right\} \quad (\underline{r}_H \in C'), \qquad \begin{aligned}&(4.10\text{a})\\&(4.10\text{b})\end{aligned}$$

where $\kappa(s)$ denotes the curvature of the film perimeter C, and the terms on the right-hand sides are, in general, unknown at this stage. Thus, equations (4.8a,b) and (4.10a,b) represent a coupled pair of boundary value problems for the $O(1)$ velocity components v_s, v_n in the interior, A', of the film, wherein both D and Ω are $O(1)$ when the parameters ξ, ζ_d and ζ_s are $O(1)$.

4.1. *Weak shear viscosity.* In what follows, we consider only the case when effects of shear viscosity are small (cf. [15, p. I-236]), so that the film vorticity Ω is appreciable *only* near the perimeter C. Thus, we take

$$\zeta_s \ll 1, \tag{4.1.1}$$

which is considered to be adequate for gravity waves, although, in [13, p. 465], the more restrictive condition $\zeta \ll 1$ is assumed. As discussed in Appendix C, there are *three* cases, according as

$$\text{(i) } 1 \gg \zeta_s^{1/2} \gg \varepsilon, \quad \text{(ii) } \zeta_s^{1/2} = O(\varepsilon), \quad \text{(iii) } \zeta_s^{1/2} \ll \varepsilon. \tag{4.1.2}$$

When (i) applies, it is seen from equation (4.8b) that the solution for Ω ($\underline{r}_H \in A'$) assumes a boundary-layer character. Specifically, there is a further boundary-layer region, A'_s say, of scale $O(\zeta_s^{1/2})$ normal to C, which is essentially located within A' and adjacent to C', and whose width is much greater than that, $O(\varepsilon)$, of A_{TR}. In case (ii), this boundary-layer region *coincides* with A_{TR}, whilst, for case (iii), it is embedded within A_{TR}.

From equation (3.7) and boundary conditions (3.8a,b) at the film perimeter C, it is indicated in Appendix B that, in each of (i), (ii) and (iii),

(a) the leading terms, $O(1)$, in v_s, v_n do not vary normally across A_{TR} (that is, such terms are independent of the magnified variable $N = n/\varepsilon$),

(b) the leading non-zero term in D is $O(\max\{\zeta_s, \varepsilon\})$ in A_{TR}, but that

(c) the leading non-zero term in Ω is $\leqslant O(1)$, and, if it is $O(1)$, is independent of N in A_{TR} in case (i), but varies rapidly with n in A_{TR} for cases (ii) and (iii), (since, to $O(1)$, $\Omega = 0$ at the inner edge, C', of A_{TR}, by asymptotic matching principles applied to the implication $\Omega = 0$ ($\underline{r}_H \in A'$) of equation (4.8b)).

Thus, when (4.1.1) holds, the conditions (4.10a,b) may effectively be replaced by

$$\left.\begin{aligned}D &= 0 \quad [\text{cases (i), (ii), (iii)}], \\ \Omega &= 0 \quad [\text{cases (ii), (iii)}],\end{aligned}\right\} \quad (\underline{r}_H \in C'), \qquad \begin{aligned}&(4.1.3\text{a})\\&(4.1.3\text{b})\end{aligned}$$

$$\Omega = 0 \quad [\text{case (i)}], \qquad (\underline{r}_H \in C'_s), \tag{4.1.3c}$$

where C'_s denotes the inner "edge" of the boundary-layer region A'_s. Helmholtz' equation (4.8a), with the slight modification that m^2 now represents $(1 - i)/(\xi + i\zeta_d)$ for purposes of consistency, remains valid for D in each case (i), (ii), (iii). This applies, in case (i), even to region A'_s, wherein equation (4.2), on which (4.8a) is based, holds, because the horizontal scale, $O(\zeta_s^{1/2})$, greatly exceeds the vertical scale, $O(\varepsilon)$.

Case (*i*). Within the boundary-layer region A'_s, we define the stretched variable

$$N_s = n/\zeta_s^{1/2} \qquad (\underline{r}_H \in A'_s), \tag{4.1.4}$$

whence equations (4.8a,b) become

$$\left.\begin{aligned} \partial^2 D/\partial N_s^2 &= 0 \\ \partial^2\Omega/\partial N_s^2 - (1+i)\Omega &= 0 \end{aligned}\right\} \qquad (\underline{r}_H \in A'_s). \qquad \begin{aligned}(4.1.5a)\\(4.1.5b)\end{aligned}$$

The solutions appropriate to conditions (4.1.3a,c), and to boundedness of D as $N_s \to \infty$, are

$$\left.\begin{aligned} D &= 0 \\ \Omega &= \phi(s)\exp\left[it - 2^{1/4}e^{i\pi/8}N_s\right] \end{aligned}\right\} \qquad (\underline{r}_H \in A'_s), \qquad \begin{aligned}(4.1.6a)\\(4.1.6b)\end{aligned}$$

where $\phi(s)$ is to be determined *via* asymptotic matching with the solution for Ω in A_{TR}. Consequently, within the major interior part, $A' - A'_s$, of the slick, D satisfies the following boundary value problem:

$$\left.\begin{aligned} \nabla^2 D + m^2 D = m^2 d^{(\infty)} \qquad &(n > 0) \\ D = 0 \qquad &(n = 0) \end{aligned}\right\}. \tag{4.1.7}$$

The condition on $n = 0$ here is deduced from equation (4.1.6a) *via* principles of asymptotic matching. Similarly, within $A' - A'_s$,

$$\Omega = 0 \qquad (n > 0), \tag{4.1.8}$$

by equation (4.8b). Thus, if, for any particular film, D can be obtained from equation (4.1.7), the corresponding velocity components v_s, v_n are found from equation (4.7) to be given by

$$\underline{v} = \underline{q}_H^{(\infty)} - m^{-2}\underline{\nabla} D \qquad (\underline{r}_H \in A' - A'_s), \tag{4.1.9}$$

which shows that, apart from a constant factor, D plays the role of a *velocity potential* for the film velocity $\underline{v}$ ($\underline{r}_H \in A' - A'_s$) relative to the horizontal component of velocity just beyond the oscillatory layer in the water. The expression for $\underline{v}$ at the "perimeter" C'_s of $A' - A'_s$, as given by equation (4.1.9), evaluated on $n = 0$, yields the dominant parts, $O(1)$, independent of N_s and N, of the film velocity $\underline{v}$ in the boundary-layer regions A'_s and A_{TR}, (cf. (a), above), and complete our primary objective in case (i) of equation (4.1.2.). We note once more that

$$v_s = q_s^{(\infty)} = -u^{(\infty)}\underline{n}\cdot\underline{j} \qquad (\underline{r}_H \in A'_s + A_{TR}), \tag{4.1.10}$$

that is, the dominant part of the film velocity *parallel to C* is equal to the corresponding component of $\underline{q}_H^{(\infty)}$.

From the expressions for v_s, v_n in A_{TR}, the boundary condition (3.8b), and item (c) above, we obtain

$$\Omega = -2(\partial v_n/\partial s + \kappa v_s)_{n=0} \qquad (\underline{r}_H \in A_{TR}), \tag{4.1.11}$$

provided that the right-hand side is non-zero. Then asymptotic matching at the "edge" C' of A_{TR} yields

$$\phi(s)e^{it} = -2(\partial v_n/\partial s + \kappa v_s)_{n=0}, \tag{4.1.12}$$

and hence completes the solution for Ω ($\underline{r}_H \in A'_s$) in equation (4.1.6b).

Cases (*ii*), (*iii*). When $\zeta_s^{1/2} \leqslant O(\varepsilon)$, D satisfies the boundary value problem (4.1.7) in the interior region A' of the slick, wherein equation (4.8b) yields $\Omega = 0$ and $\underset{\sim}{v}$ is given by equation (4.1.9). Then, as in case (i), the velocities v_s, v_n are known in A_{TR}, once D ($\underset{\sim}{r}_H \in A'$) has been determined. The value of Ω at the film perimeter C is given by

$$\Omega = -2(\partial v_n/\partial s + \kappa v_s)_{n=0} \qquad (\underset{\sim}{r}_H \in C), \tag{4.1.13}$$

but, otherwise, Ω remains undetermined in A_{TR} (cf. (c), above).

Work of Miles [13]. When the fluid has fixed lateral boundaries with vertical generators, Miles assumed that

$$\Omega = 0 \qquad (\underset{\sim}{r}_H \in A'), \tag{4.1.14}$$

which satisfies equation (4.8b). Instead of the condition (4.10b), he took

$$\underset{\sim}{n} \cdot \underset{\sim}{\nabla} D = 0 \qquad (\underset{\sim}{r}_H \in C'), \tag{4.1.15}$$

where $\underset{\sim}{n}$ is a unit horizontal vector normal to the perimeter C. This follows from (4.7), since $v_n = O(\varepsilon) = q_n^{(\infty)}$ on C'. Outside the oscillatory layers at the boundaries, the linear velocity field is irrotational, and $\underset{\sim}{q}_H^{(\infty)}$ satisfies a homogeneous Helmholtz equation. In consequence, it is readily found that a simple solution of the form $\underset{\sim}{v} = B\underset{\sim}{q}_H^{(\infty)}$ is possible, where B is a complex constant. This clarifies Miles' assumption of such a form, which, however, is not generally valid for containers of *variable* depth. In the present context, adoption of this expression for $\underset{\sim}{v}$ would violate the condition (4.10b).

Progressive waves. With a view to application of the boundary value problem (4.1.7) to specific geometries of films with weak parametric shear viscosity ζ_s, we first write

$$\left.\begin{aligned} (u^{(\infty)}, v^{(\infty)}) &= (P, 0)e^{i(t-x)} \qquad (P = \coth h), \\ d^{(\infty)} &= -iPe^{i(t-x)} \end{aligned}\right\} \tag{4.1.16}$$

for propagating waves. Then, on setting

$$D = Pf(\underset{\sim}{r}_H)e^{it} + [im^2/(1-m^2)]\,Pe^{i(t-x)}, \tag{4.1.17}$$

the function f is related to the finite dimensions of the film, and, for $\zeta_s \ll 1$, satisfies the boundary value problem

$$\nabla^2 f + m^2 f = 0 \qquad (n > 0), \tag{4.1.18}$$

$$f = -[im^2/(1-m^2)]e^{-ix}(n = 0). \tag{4.1.19}$$

The film velocity is given by

$$\underset{\sim}{v} = -m^{-2}Pe^{it}\underset{\sim}{\nabla} f - [m^2P/(1-m^2)]\,\underset{\sim}{i}e^{i(t-x)}. \tag{4.1.20}$$

In the important limiting case $|\xi + i\zeta| \to \infty$, as when the molecular elements are closely packed, $m \to 0$ and it is seen from equations (4.1.17, 4.1.18, 4.1.19) that $D \to 0$, so that the slick is inextensible—its area is conserved, but its shape may change with time. This extreme is now considered in some detail.

5. Inextensible slicks. In the limit $\xi^2 + \zeta^2 \to \infty$, both D, which is the rate of extension of a surfactant element per unit area, and f tend to zero. However, from equation (4.1.20), the horizontal velocity of the surfactant molecules is given by

$$\underset{\sim}{v}_I = \lim_{\xi^2+\zeta^2\to\infty}\{\underset{\sim}{v}\} = -Pe^{it}\lim_{m\to 0}\{m^{-2}\underset{\sim}{\nabla} f\}, \tag{5.1}$$

so that stagnation of the film does not necessarily occur. In fact, as shown for infinitely-long films in [3], horizontal immobility is exceptional.

If we set

$$\lim_{m\to 0}\{f/m^2\} = -iF, \tag{5.2}$$

equations (4.1.18, 4.1.19) show that F is a harmonic function which satisfies the Dirichlet problem

$$\left.\begin{aligned} \nabla^2 F &= 0 \quad (n > 0),\\ F &= e^{-ix} \quad (n = 0).\end{aligned}\right\} \tag{5.3}$$

Moreover, as already indicated, the horizontal component of velocity of the slick is non-divergent,

$$D_I = \lim_{\xi^2+\zeta^2\to\infty}\{D\} = \lim_{\xi^2+\zeta^2\to\infty}\{\underline{\nabla}\cdot\underline{v}\} \propto e^{it}\nabla^2 F = 0. \tag{5.4}$$

Some specific examples are now considered.

(1) *Two-dimensional theory*. When the slick has uniform width b and length $l\to\infty$, as measured in the x- and y-directions, respectively, F is a function of x only, and it is readily found that the horizontal velocity field is *spatially uniform*,

$$\underline{v}_I = iPe^{it}\underline{\nabla}F = i(P/b)(e^{-ib}-1)\underline{i}e^{it}, \tag{5.5}$$

which is in agreement with [3], and partially confirms the boundary value problem (5.3) for the inextensible slick of finite area. When $b\to\infty$, (5.5) shows that the slick is horizontally-immobile, as deduced by Lamb [9] and others, and this static condition also occurs if $b = 2n\pi$ ($n = 1,2,3,\ldots$). For $b \neq 2n\pi$, the slick oscillates to and fro, and has *constant* width due to the non-divergent property.

(2) *Circular slicks*. We use polar co-ordinates such that $x + iy = Re^{i\phi}$, with origin at the centre of a circular slick of radius $\overline{R}$. By use of the relation

$$e^{-ix} = J_0(R) + 2\sum_{n=1}^{\infty}(-i)^n J_n(R)\cos n\phi \tag{5.6}$$

involving Bessel functions J_n of the first kind, the solution of the boundary value problem (5.3) is readily obtained:

$$F = J_0(\overline{R}) + 2\sum_{n=1}^{\infty}(-i)^n J_n(\overline{R})(R/\overline{R})^n\cos n\phi. \tag{5.7}$$

Some particular properties may be mentioned.

(i) When the radius is small compared with the wavelength, $\overline{R} \ll 1$ and

$$v_{Ix} = \left[1 - \frac{1}{2}iR\cos\phi + O(\overline{R}^2)\right]Pe^{it}, \tag{5.8}$$

$$v_{Iy} = \left[\frac{1}{2}iR\sin\phi + O(\overline{R}^2)\right]Pe^{it}. \tag{5.9}$$

The motion of the slick is essentially that of a speck, or flake, on a pure air-water interface.

(ii) At the other extreme when $\bar{R} \gg 1$, it is found from the asymptotic form of $J_n(\bar{R})$ as $\bar{R} \to \infty$ that the slick is stationary:

$$v_{Ix} \to 0, \quad v_{Iy} \to 0 \quad \text{as } \bar{R} \to \infty, \tag{5.10}$$

in agreement with (1), above, when $b \to \infty$.

(iii) The slick is symmetrical about the x-axis, on which $v_{Iy} = 0$. Elsewhere, the transverse, or y-, component of velocity is, in general, non-zero. At the centre of the slick,

$$(v_{Ix})_{R=0} = 2P\left[J_1(\bar{R})/\bar{R}\right]e^{it}, \tag{5.11}$$

which is readily shown to be equal to the average value of v_{Ix} over both the perimeter C (by potential theory) and the area A; that is,

$$(v_{Ix})_{R=0} = \frac{1}{2\pi}\int_0^{2\pi} v_{Ix}\,d\phi = \frac{1}{\pi\bar{R}^2}\int_0^{\bar{R}}\int_0^{2\pi} v_{Ix}\,Rd\,Rd\,\phi. \tag{5.12}$$

The motion at the centre of the circular slick is therefore of some significance as a spatial average, and, in Fig. 1, the function

$$|(v_{Ix})_{R=0}|/P = 2|J_1(\bar{R})|/\bar{R}, \tag{5.13}$$

representing the ratio of the maximum velocity at the centre to that for a pure interface, is plotted against $\bar{R}$. By use of the identities

$$\frac{1}{2}RJ_0(R) = \sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)J_{2n-1}(R), \quad \frac{1}{2}RJ_1(R) = \sum_{n=1}^{\infty}(-1)^{n-1}2nJ_{2n}(R),$$

[16, p. 18], we obtain

$$(v_{Ix})_{R=\bar{R}} = P\left[\pm J_0(\bar{R}) - iJ_1(\bar{R})\right]e^{it} \qquad \left(\phi = \frac{\pi}{2} \mp \frac{\pi}{2}\right).$$

The corresponding ratio is also shown in Fig. 1 for these boundary points, and decreases monotonically with $\bar{R}$, in contrast with that at the mid-point. Conversely, for these inextensible slicks, the centre is virtually motionless when the radius exceeds a wavelength or so, but there remains significant displacement at the perimeter.

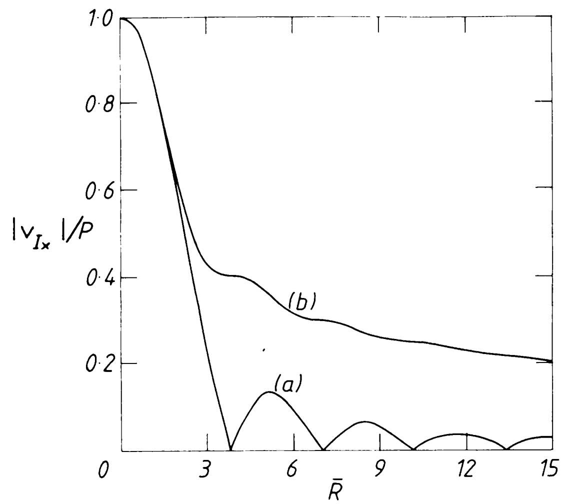

FIG. 1

(3) *Rectangular slicks.* We consider a rectangular slick, of width b and length l, bounded by the intersection of planes $x = 0$, $x = b$ and $y = \pm \frac{1}{2}l$ with the horizontal plane $z = 0$. The solution of the boundary value problem (5.3) is

$$F = \left[1 + \frac{x(e^{-ib} - 1)}{b} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{b} \cosh\frac{n\pi y}{b}\right], \tag{5.14}$$

where

$$B_n = \frac{2b^2}{n\pi} \frac{\{1 - e^{-ib}(-1)^n\}}{n^2\pi^2 - b^2} \operatorname{sech}\frac{n\pi l}{2b} \quad (b \neq n\pi), \tag{5.15}$$

$$B_n = -i \operatorname{sech} \frac{1}{2} l \quad (b = n\pi), \tag{5.16}$$

and the horizontal velocity components of the slick are given by

$$v_{Ix} = \frac{iP}{b}\left[e^{-ib} - 1 + \sum_{n=1}^{\infty} n\pi B_n \cos\frac{n\pi x}{b} \cosh\frac{n\pi y}{b}\right] e^{it}, \tag{5.17}$$

$$v_{Iy} = \frac{iP}{b}\left[\sum_{n=1}^{\infty} n\pi B_n \sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}\right] e^{it}. \tag{5.18}$$

(i) When the width of the slick is much smaller than the wavelength, $b \ll 1$ and

$$\left.\begin{aligned} v_{Ix} &= P[1 + O(b)]\, e^{it} \\ v_{Iy} &= O(b) P e^{it} \end{aligned}\right\}, \tag{5.19}$$

for *any* value of the ratio b/l. The presence of such a slick has virtually no effect on the motion of the water within the oscillatory boundary layer.

(ii) At the intersection of the lines of symmetry of the slick,

$$(v_{Ix})_{x=1/2b,y=0} = \frac{P \sin\frac{1}{2}b}{\frac{1}{2}b}\left[1 - 2b^2 \sum_{m=1}^{\infty} \frac{(-1)^m \operatorname{sech}(m\pi l/b)}{4m^2\pi^2 - b^2}\right] e^{i(t-1/2b)}$$

$$(b \neq 2m\pi), \tag{5.20}$$

$$= (-1)^m P e^{it} \operatorname{sech} \frac{1}{2} l \quad (b = 2m\pi), \tag{5.21}$$

whereas the average value of v_{Ix} over A is given by

$$\frac{1}{bl}\int_0^b \int_{-1/2l}^{1/2l} v_{Ix}\, dx\, dy = i(P/b)(e^{-ib} - 1)e^{it}, \tag{5.22}$$

which is independent of l. The transverse component of velocity vanishes on the front ($x = 0$) and rear ($x = b$) edges, and on $y = 0$; $v_{Ix} = u^{(\infty)} = Pe^{i(t-x)}$ on $y = \pm \frac{1}{2}l$. If $b = 2m\pi$ ($m = 1, 2, \ldots$),

$$v_{Ix} = (-1)^m P e^{it} \cosh y \operatorname{sech} \frac{1}{2} l \quad \left(x = \frac{1}{2}b\right); \tag{5.23}$$

if $b = (2m + 1)\pi$,

$$v_{Iy} = (-1)^m P e^{it} \sinh y \operatorname{sech} \frac{1}{2} l \quad \left(x = \frac{1}{2}b\right). \tag{5.24}$$

In (5.23), v_{Ix} is in phase with the velocity $u^{(\infty)} = (-1)^m Pe^{it}$ at $x = \frac{1}{2}b$; v_{Iy} in (5.24) is out of phase by $\frac{1}{2}\pi$ with the corresponding velocity $u^{(\infty)} = (-1)^m iPe^{it}$. The ratios $v_{Ix}/u^{(\infty)}$ and $|v_{Iy}/u^{(\infty)}|$ are plotted against y for this median line of inextensible rectangular slicks in Fig. 2.

(4) *Approximate theory*. Consider a long, narrow slick having a smooth perimeter C, but, otherwise, of quite general shape. The orientation with respect to the direction of wave propagation is arbitrary, and slow variations therein (due, for example, to drifting) are ignored. The configuration is shown in Fig. 3, where the X-axis makes a positive angle θ ($< \frac{1}{2}\pi$) with the x-axis. On writing

$$F = e^{-iY\sin\theta} G(X, Y), \tag{5.25}$$

the boundary value problem for G is

$$\left.\begin{aligned} \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2} - 2i\sin\theta\frac{\partial G}{\partial Y} - G\sin^2\theta = 0 \quad & (n > 0), \\ G = e^{-iX}\cos\theta \quad & (n = 0). \end{aligned}\right\} \tag{5.26}$$

Let the equation of C be $X = b_{\pm}(Y)$ for $X \gtrless 0$, and assume that $|db_{\pm}/dY| \ll 1$. In consequence, $|\partial G/\partial Y|$ should generally be much smaller than $|\partial G/\partial X|$, whence an

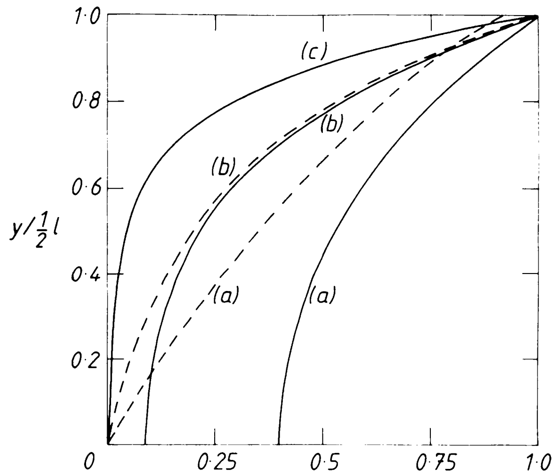

FIG. 2

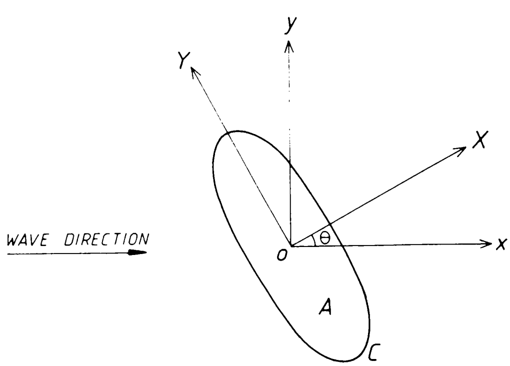

FIG. 3

approximate solution, based on the neglect of $\partial^r G/\partial Y^r$ $(r = 1, 2)$, is

$$G = \frac{e^{-ib_+\cos\theta}\sinh[(X - b_-)\sin\theta] + e^{-ib_-\cos\theta}\sinh[(b_+ - X)\sin\theta]}{\sinh[c(Y)\sin\theta]}, \tag{5.27}$$

where $c(Y) = b_+(Y) - b_-(Y)$. The approximation is justified provided that

$$\left| c(Y)\frac{db_\pm(Y)}{dY} \right| \ll 1 \quad \text{and} \quad \left| \frac{db_\pm(Y)}{dY} \right| \ll 1; \tag{5.28}$$

the latter condition implies that this analysis usually fails in the vicinity of the extremities of the slick. If $\theta = 0$, so that $Y = y$ and the slick is broadside on to the wave direction,

$$\left.\begin{aligned} v_{Ix} &\simeq [iP/c(Y)][e^{-ib_+(Y)} - e^{-ib_-(Y)}]e^{it}, \\ v_{Iy} &= O(|v_{Ix}\,dc/dY|), \end{aligned}\right\} \tag{5.29}$$

which is a simple extension of equation (5.5), and is analogous to "strip theory" in aerodynamical wing-theory. The motion of molecular elements is essentially in the wave direction. On the other hand, if $\theta = \frac{1}{2}\pi$, so that $Y = x$ and the slick is elongated in the direction of wave propagation,

$$\begin{aligned} (v_{Ix}, v_{Iy}) \simeq P\bigg(&\cosh\left[y - \frac{1}{2}(b_- + b_+)\right], \\ &i\sinh\left[y - \frac{1}{2}(b_- + b_+)\right]\bigg)e^{i(t-x)}\Big/\cosh\frac{1}{2}c(Y). \end{aligned} \tag{5.30}$$

Thus, horizontal components of molecular displacements are comparable, unless $|b_\pm(Y)| \ll 1$, in which case motion is predominantly in the wave direction. For general values of θ, it is readily found that, when $|b_\pm(Y)| \ll 1$,

$$G \simeq 1, \quad \partial G/\partial X \simeq -i\cos\theta, \quad \partial G/\partial Y \simeq -(dc/dY)/c(Y),$$

and that

$$v_{Ix} \simeq Pe^{i(t-x)}, \qquad v_{Iy} = O(|db_\pm/dY|)P. \tag{5.31}$$

(5) *Shape of the slick.* As mentioned previously, the *area* of the slick is conserved whilst the shape of the perimeter varies throughout the wave cycle. If the equation (averaged over a wave period) of the boundary is $\gamma(x, y) = 0$, the instantaneous equation is

$$\gamma(x, y) - \alpha P e^{it}(\underline{\nabla} F)_{\gamma=0} \cdot \underline{\nabla}\gamma = 0. \tag{5.32}$$

Thus, for a circular slick, the equation of C is

$$R - 2\alpha P e^{it}\sum_{n=1}^{\infty}(-1)^n J_n(\overline{R})\frac{n}{\overline{R}}\cos n\phi = \overline{R}, \tag{5.33}$$

which, when $\overline{R} \ll 1$, reduces to

$$R + \alpha P e^{it}\left(i\cos\phi + \frac{1}{2}\overline{R}\cos 2\phi\right) \simeq \overline{R}, \tag{5.34}$$

provided that $\overline{R} \gg \alpha\coth h$, so that second-order effects (in α) are negligible. If the term in $\cos 2\phi$ is ignored in equation (5.34), the perimeter is a circle, of radius $\overline{R}$ and centre

$x = \alpha P \sin t$, $y = 0$; retention of this term indicates that the boundary is an ellipse, having this same centre, with axes of length $\bar{R}(2 - \alpha P \cos t)$ and $\bar{R}(2 + \alpha P \cos t)$ in the x- and y-directions, respectively.

6. Extensible slicks. For general values of the elastic and viscous parameters ξ and ζ_d, we must solve the boundary value problem of equations (4.1.18, 4.1.19) involving the Helmholtz equation for f. In this case, both shape and area of the slick vary during wave motion. We confine attention to circular slicks, for which it is found that

$$f = -\left[im^2P/(1 - m^2)\right] \cdot \left[\frac{J_0(\bar{R})}{J_0(m\bar{R})} J_0(mR) + 2\sum_{n=1}^{\infty} (-1)^n \frac{J_n(\bar{R})}{J_n(m\bar{R})} J_n(mR) \cos n\phi\right]. \tag{6.1}$$

The velocity of surfactant elements is obtained from equation (4.1.10), and, at the centre of the slick,

$$(v_x)_{R=0} = \left[mP/(1 - m^2)\right]\left[\frac{J_1(\bar{R})}{J_1(m\bar{R})} - m\right] e^{it}. \tag{6.2}$$

If $\bar{R} \ll 1$ and $|m|\bar{R} \ll 1$,

$$(v_x)_{R=0} \simeq P\left(1 - \frac{1}{8}\bar{R}^2 + \frac{1 - 2m^2}{192}\bar{R}^4 + \dots\right) e^{it}, \tag{6.3}$$

and deviates only slightly from the inextensible expression.

The perimeter C is represented by

$$R = \bar{R} + \alpha\left[im^{-2}\frac{\partial f}{\partial R} + \left\{im^2P/(1 - m^2)\right\}\cos\phi\, e^{-ix}\right]_{R=\bar{R}} e^{it}, \tag{6.4}$$

which, when $\bar{R} \ll 1$ and $|m|\bar{R} \ll 1$, gives

$$R = \bar{R} - \alpha P\left[i\cos\phi + \frac{1}{2}\bar{R}\cos 2\phi - \frac{1}{8}i\bar{R}^2(\cos\phi + \cos 3\phi)\right] e^{it}, \tag{6.5}$$

so that effects of finite visco-elastic properties, which enter at $O(\bar{R}^3)$, are small, and the slick behaves very much as in the inextensible limit.

The area $A(t)$ of the slick is readily found:

$$A(t) = \pi\bar{R}^2 + \left[2\pi\alpha P m\bar{R}/(1 - m^2)\right]\left[mJ_1(\bar{R}) - \frac{J_0(\bar{R})J_1(m\bar{R})}{J_0(m\bar{R})}\right] e^{it}, \tag{6.6}$$

whence, asymptotic expansions of Bessel functions give

$$A(t) = \begin{cases} \pi\bar{R}^2\left[1 + \frac{1}{8}\alpha P m^2\bar{R}^2 e^{it}\right] \qquad (\bar{R} \ll 1, |m|\bar{R} \ll 1), \\ \pi\bar{R}^2\left[1 + \alpha P\left(\frac{2}{\pi\bar{R}}\right)^{3/2} \frac{\pi m}{1 - m^2}\left\{m\cos\left(\bar{R} - \frac{3}{4}\pi\right) + i\cos\left(\bar{R} - \frac{1}{4}\pi\right)\right\} e^{it}\right] \\ (\bar{R} \gg 1, |m|\bar{R} \gg 1). \end{cases}$$

6.1. *Viscous dissipation*. We recall that the propagating waves are assumed to be strictly periodic in time (§2), and seek an estimate of the direct effect, in the form of viscous damping, of the surface film on their spatial attenuation.

Thus, consider the energy equation for the fluid within a fixed vertical cylinder V, extending over the whole depth and having a small rectangular cross-section with sides of length δx and δy parallel to the x- and y-axes, respectively. Since the energy of this fluid is constant in time, an approximation for the local wave amplitude, $A(x, y)$ say, is obtained as follows. The dominant part of the energy flux, averaged over a wave period, out of the vertical sides of V is given by

$$\delta y \int_{-h}^{0} [-p_w u]_x^{x+\delta x}\, dz \simeq \frac{2h + \sinh 2h}{8 \sinh^2 h}\left[A^2(x, y) - A^2(x + \delta x, y)\right] \delta y, \qquad (6.1.1)$$

where p_w denotes the fluid pressure (beyond the oscillatory boundary layers) due to the wave motion. Also, the dominant part of $\overline{D}$, the mean rate of viscous dissipation of energy of the fluid in V, arises from the vortical layers at the bottom and at the free surface (provided that V pierces the slick). From the bottom layer, it is found that

$$\overline{D}_b \simeq 2^{-3/2} \varepsilon \delta x \delta y \left|\underline{q}_b^{(\infty)}\right|^2 = 2^{-3/2} \varepsilon \delta x \delta y A^2 \operatorname{cosech}^2 h \qquad (6.1.2)$$

(cf. equation (6.1.3) and set $\underline{v} = \underline{0}$), and, from the surface layer,

$$\overline{D}_s \simeq 2^{-3/2} \varepsilon \delta x \delta y \mathscr{R}\left\{ |\underline{q}^{(\infty)} - \underline{v}|^2 + (1 + i)\left(\underline{q}^{(\infty)} - \underline{v}\right) \cdot \underline{v}^* \right\} \qquad (6.1.3)$$

(cf. equation (4.12a) of [13]), where $\underline{v}^*$ is the complex conjugate of $\underline{v}$. However

$$\mathscr{R}\left\{(1 + i)\left(\underline{q}^{(\infty)} - \underline{v}\right) \cdot \left(\underline{q}^{(\infty)} - \underline{v}\right)^*\right\} = \mathscr{R}\left|\underline{q}^{(\infty)} - \underline{v}\right|^2,$$

whence (6.1.3) can be written as a linear expression in $\underline{v}$, *viz*,

$$\overline{D}_s = 2^{-3/2} \varepsilon \delta x \delta y \mathscr{R}\left\{(1 + i)\left(\underline{q}^{(\infty)} - \underline{v}\right) \cdot \underline{q}^{(\infty)*}\right\}, \qquad (6.1.4)$$

from which equations (4.1.9), (4.1.16) and (4.1.17) give

$$\overline{D}_s = 2^{-3/2} \varepsilon P^2 A^2 \delta x \delta y \mathscr{R}\left\{(1 + i)\left[m^{-2} e^{ix} \partial f/\partial x + (1 - m^2)^{-1}\right]\right\}. \qquad (6.1.5)$$

Then, from equations (6.1.1), (6.1.2) and (6.1.5), the local wave-attenuation rate within the interior of the film is given by

$$\frac{1}{A}\frac{\partial A}{\partial x} \simeq -2^{1/2} \varepsilon \frac{1 + \mathscr{R}\left\{(1 + i)\left[m^{-2} e^{ix} \partial f/\partial x + (1 - m^2)^{-1}\right]\right\} \cosh^2 h}{2h + \sinh 2h}. \qquad (6.1.6)$$

For inextensible slicks, $m \to 0$, and we have

$$\frac{1}{A}\frac{\partial A}{\partial x} \simeq -2^{1/2} \varepsilon \frac{1 + \cosh^2 h + \mathscr{R}\left\{(1 - i) e^{ix} \partial F/\partial x\right\} \cosh^2 h}{2h + \sinh 2h}. \qquad (6.1.7)$$

In the two-dimensional theory, the solution of equations (4.1.18) and (4.1.19) is

$$f = -\left[im^2/(1 - m^2)\right]\left[e^{-ib} \sin mx - \sin m(x - b)\right] \operatorname{cosec} mb, \qquad (6.1.8)$$

whence integration of (6.1.6) yields

$$[\log A]_{x=0}^{x=b} = -2^{1/2} \varepsilon\left[\gamma_b + \mathscr{R}(\gamma_{s1} + \gamma_{s2})\right]/(2h + \sinh 2h), \qquad (6.1.9)$$

where

$$\gamma_b = 1,$$
$$\gamma_{s1} = b\big[(1+i)/(1-m^2)\big]\cosh^2 h,$$
$$\gamma_{s2} = 2m\Big[(1+i)/(1-m^2)^2\Big](\cos b - \cos mb)\operatorname{cosec} mb \cosh^2 h.$$

Equation (6.1.9) represents an estimate of the reduction in amplitude experienced by the wave in passing across the entire slick. The quantities γ_b and $\gamma_s = \gamma_{s1} + \gamma_{s2}$ are associated with damping in the bottom and surface boundary layers, respectively; γ_{s1} corresponds to an unbounded slick ($b \to \infty$), so that γ_{s2} represents the effect of finite width. In the inextensible case,

$$\gamma_b = 1,\ \mathscr{R}\gamma_{s1} = b\cosh^2 h,\ \mathscr{R}\gamma_{s2} = 2b^{-1}(\cos b - 1)\cosh^2 h,$$

so that the contribution due to finite width (hence, to horizontal, spatially-uniform motion of the slick) tends to *reduce* wave damping, unless $b = 2n\pi$ ($n = 1, 2, 3, \ldots$), although this effect of finiteness is significant only when b is less than about 3, or, in other words, the wavelength exceeds about twice the width of the slick. Thus, for example, in deep-water conditions, a single inextensible slick is, in general, more effective in damping waves than a number of smaller slicks of the *same total width*.

Numerical example. In the two-dimensional case, we illustrate the dissipative influence of an inextensible film, of width 200m, in water of depth 16m and having kinematic viscosity $\nu = 0.01$ cm^2 sec^{-1}. The percentage loss of the square of the wave amplitude due to passage of the wave across the film, *viz.*

$$1 - (A^2)_{x=b}/(A^2)_{x=0},$$

which is a measure of the wave energy deficit, is found from equation (6.1.9) to be about 1.1% and 26.4% for deep-water waves of length 29.5m (period 4.35 sec) and 4.3m (period 1.67 sec), respectively. The chosen data corresponds very closely to some of the measurements reported in [8; cf. Fig. 4] and made in the North Sea. Direct comparison is not possible for a variety of reasons, e.g. observations include additional damping effects due to modification of wind input and of wave-wave interactions, directionality of the wave field, non-uniformity of the film. Nevertheless, it seems worth recording that the measurements yield values of about 92% and 79% for the ratio of the wave energy in the slick area B1 to that in the "nonslick" area C, (see Fig. 3a of [8]), corresponding to "wave attenuations" of 8% and 21% for the above longer and shorter waves, respectively.

7. Concluding remarks. Boundary-layer techniques and, in the case of weak shear viscosities $\zeta_s \ll 1$ for the film, the vanishing (to lowest order) of the horizontal divergence D at the "perimeter" C', have facilitated the determination of the oscillatory velocity $\underline{v}$ of an idealized insoluble slick of intermediate visco-elastic parameters ξ, ζ_d. For a specific configuration and known $m(\xi, \zeta_d)$, knowledge of $\underline{v}$ enables calculation of various film characteristics, particularly (a) elliptical orbits of the horizontal molecular motion, and (b) energy dissipated within the oscillatory layer adjacent to A. From (b), estimates may be found for the total viscous damping when a wave propagates through a region containing either one large slick, as indicated in §6.1, or (say) a non-uniform distribution of small or medium slicks (cf. [8, p. 436]).

Other extensions of the present work may prove possible, e.g. to soluble films or to second-order drift, and consideration of a model for films of small, but finite, thickness might be viable.

Appendix A. Boundary conditions at the edge of the film. It is readily found from [14, pp. 156–158; 15, pp. 230–235] that the linearized condition at the edge, C, of the film may be written as

$$[T_1 + (\eta_d - \eta_s)\operatorname{div} \underline{v}]\underline{n} + \eta_s[(\underline{n} \times \operatorname{curl} \underline{v}) + 2(\underline{n} \cdot \underline{\nabla})\underline{v}] = \underline{0} \qquad (\underline{r}_H \in C); \quad \text{(A1)}$$

see, also, [6, pp. 354–355] for the case with $\eta_d = 0$. Components of (A1) in directions normal and tangential to C give the following dimensional forms of boundary conditions:

$$\left.\begin{aligned} T_1 + (\eta_d - \eta_s)D + 2\eta_s \partial v_n/\partial n &= 0 \\ \eta_s(\Omega - 2\partial v_s/\partial n) &= 0 \end{aligned}\right\} \qquad (\underline{r}_H \in C), \qquad \begin{aligned}&\text{(A2a)}\\&\text{(A2b)}\end{aligned}$$

wherein $\Omega = \underline{k} \cdot \operatorname{curl} \underline{v}$,

$$D = \operatorname{div} \underline{v} = \partial v_s/\partial s - \kappa v_n + \partial v_n/\partial n \qquad (\underline{r}_H \in C),$$

and κ denotes the curvature of C, positive for centres of curvature having $n > 0$. Thus, equation (A2a) can be expressed non-dimensionally, *via* equation (4.4), as

$$\varepsilon^{-1}T_1 + (\zeta_d + \zeta_s)D = 2\zeta_s(\partial v_s/\partial s - \kappa v_n) \qquad (\underline{r}_H \in C). \qquad \text{(A3)}$$

Within the linearized framework for insoluble slicks, $iT_1 = \varepsilon\xi D$ by equation (3.4), whence (A3) reduces to

$$T_1 = 0 \qquad (\underline{r}_H \in C) \qquad \text{(A4)}$$

in each of the following cases:

1. $\zeta_s = 0$,

2. slicks, of uniform width and unlimited length, having edges parallel to straight wave-crests, allowing a *two*-dimensional theory, [3],

3. $\zeta_d \to \infty$, with ξ and ζ_s finite, whence it is readily shown from equation (3.6) that $T = 0 = D$ ($\underline{r}_H \in A$), so that this is an *inextensible* limiting case. [We note that $\xi \to \infty$, with ζ_d and ζ_s finite, yields $D = 0$ ($\underline{r}_H \in C$) and, in consequence, $D = 0$ ($\underline{r}_H \in A$), so that this is also an inextensible limit. However, by (A3), we then have $T_1 \neq 0$ ($\underline{r}_H \in C$), in general.]

Equation (A4), expressing the vanishing of the surface tension at the curved line of contact of water and contaminant, is equivalent to the *static* requirement, as mentioned in [2].

Appendix B. Transition region: velocity $\underline{v}(\underline{r}_H, t)$ for the film. When the film parameters ξ, ζ_d and ζ_s are $O(1)$ on the scale of ε, the shortest length-scale (in the plane $z = 0$) for the transition region is $O(\varepsilon)$, and is measured normal to C, provided that $\kappa^{-1} \gg \varepsilon$. Thus, if we introduce a stretched variable

$$N = n/\varepsilon \qquad (\underline{r}_H \in A_{TR}), \qquad \text{(B1)}$$

the dominant terms in equation (3.6) yield

$$\partial^2 \underline{v}/\partial N^2 = O(\varepsilon^2) \tag{B2}$$

within the transition region, A_{TR}, of the film. Consequently, the *major part*, $O(1)$, of $\underline{v}$ does not *vary rapidly* within A_{TR}, and

$$\underline{v} \simeq \lim_{N\to\infty} \underline{v} \qquad (\underline{r}_H \in A_{TR}) \tag{B3}$$

correct to $O(1)$. Therefore, both D and Ω are $O(1)$ in A_{TR}, whence (3.6) shows that their major parts are independent of N, that is

$$D \simeq \lim_{N\to\infty} D, \quad \Omega \simeq \lim_{N\to\infty} \Omega \quad (\underline{r}_H \in A_{TR}). \tag{B4}$$

From either (B2) or the expression $\Omega \simeq \partial v_s/\partial n - \partial v_n/\partial s - \kappa v_s$ in A_{TR}, we deduce that the major part of $\partial v_s/\partial n$ is $O(1)$ and independent of N, that is

$$\partial v_s/\partial n \simeq \lim_{N\to\infty} \partial v_s/\partial n \qquad (\underline{r}_H \in A_{TR}). \tag{B5}$$

Thus, to the present order of working, the boundary conditions (A2a,b) may be applied at the "perimeter" C' of the interior region A' of the slick,

$$\left.\begin{aligned}(\xi + i\zeta)D &= 2i\zeta_s(\partial v_s/\partial s - \kappa v_n)\\ \Omega &= 2\partial v_s/\partial n\end{aligned}\right\} \qquad (\underline{r}_H \in C'), \qquad \begin{aligned}&(\mathrm{B6a})\\ &(\mathrm{B6b})\end{aligned}$$

on account of equations (B3), (B4) and (B5).

By formulating the linearized boundary-layer version, in terms of stretched variables $N = n/\varepsilon$, $Z = z/\varepsilon$, of the three-dimensional boundary value problem for p and $\underline{q}$ in the transition region TR of the water, it is readily found (cf. [4]) that

$$\partial p/\partial N = 0 = \partial p/\partial Z. \tag{B7}$$

Hence, when $\zeta_s \ll 1$, the boundary value problem for $q_s - q_s^{(\infty)}$ is homogeneous in TR (cf. equation (B2)), and, at $O(1)$,

$$p = p^{(\infty)}, \quad q_s = q_s^{(\infty)} \quad (\underline{r} \in TR). \tag{B8}$$

In particular, the tangential component of the film velocity, v_s, is simply the tangential component of velocity of the water just beyond the oscillatory layer.

Appendix C. Weak slicks. When the visco-elastic parameters are so small that

$$|\xi + i\zeta| \ll 1, \qquad |m| \gg 1, \tag{C1}$$

the effect of the surfactant on the velocity field is relatively slight, except near the film perimeter C. In fact, in the water, the leading term in $\underline{q}_H$ within the *whole* oscillatory layer is simply $\underline{q}_H^{(\infty)}$. It is apparent that the value of $|\xi + i\zeta|$ relative to ε, $(\mu_a/\mu)\varepsilon$, $\varepsilon^2, \ldots$ is significant. Consequently, it is advantageous to write

$$\underline{q} = \underline{q}' + \underline{Q} \tag{C2}$$

for these weak slicks, where $\underline{q}'$ is the *known* velocity corresponding to the exact linear solution for waves on a clean interface, [5], and $\underline{Q}$ is the small addition due to the slick. Near C, equation (3.6) shows that $|\xi + i\zeta|^{1/2}$ is an important parameter, and that

$\underline{Q}_H^{(0)} = \underline{v} - \underline{q}'^{(0)}_H$ is expected to change rapidly over distances, normal to C, $O(|m|^{-1/2})$ and $O(\varepsilon)$. For illustrative purposes, suppose that $\zeta_s^{1/2} \ll \varepsilon$ (see §4.1). Then there are three cases, according as

$$\text{(i)}\quad 1 \gg |\xi + i\zeta|^{1/2} \gg \varepsilon, \qquad \text{(ii)}\quad |\xi + i\zeta|^{1/2} = O(\varepsilon),$$
$$\text{(iii)}\quad |\xi + i\zeta|^{1/2} \ll \varepsilon. \tag{C3}$$

When (i) applies, there is a further boundary-layer region, ΔA say, of scale $k\Delta = O(|\xi + i\zeta|^{1/2})$, which is essentially located within A' and adjacent to C', and whose width is much greater than that, $O(\varepsilon)$, of A_{TR}. In case (ii), this boundary-layer region coincides with A_{TR}, whilst, for the slicks of case (iii), it is embedded in A_{TR}. Cases (ii) and (iii) correspond to very weak slicks, and the dominant part of $\underline{v}$ in the more important case (i) can be determined *via* introduction of a boundary-layer variable $N_2 = |m|n$ in region ΔA.

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