VELOCITY AND VORTICITY CORRELATIONS*

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Abstract. It is shown that one point velocity correlations may be expressed as a volume integral of the product of the singular part of Green's function and a function, $h_{jM}(x, X)$, which satisfies Poisson's equation and vanishes for points on the boundary. An explicit expression is found for $h_{jM}(x, X)$. These results provide a computational method for determining the velocity correlations.

Introduction. Vorticity correlations are discussed in [1] in connection with homogeneous shear flows. Expressions relating vorticity correlations to velocity correlations are given in [2]. The inverse problem, the determination of velocity correlations from vorticity correlations, arises in various areas of turbulence research including turbulence closures, [3], [4].

In the following, it is shown that one point velocity correlations may be expressed as a function of the volume integral of the product, $g(\mathbf{x},\mathbf{X})h_{jM}(\mathbf{x},\mathbf{X})$, where $g(\mathbf{x},\mathbf{X})$ is the singular part of a Green's function, $h_{jM}(\mathbf{x},\mathbf{X})$ satisfies $\nabla_x^2 h_{jM}(\mathbf{x},\mathbf{X}) = e_{JRS}w_{MR,s}(\mathbf{X},\mathbf{x})$, $w_{MR}(\mathbf{X},\mathbf{x})$ is the two point vorticity correlation and $h_{jM}(\mathbf{x},\mathbf{X}) = 0$ for points, $\mathbf{x}$, on the boundary. These results extend and sharpen the methods given in [3] and [4] for determining velocity correlations from vorticity correlations by proving that, for the no slip boundary condition and finite regions, only the singular part of the Green's function need be considered and further by showing that the function h_{jM}, in the kernel of the integral expression involved, satisfies Poisson's equation. This result, together with boundary and other conditions proven for h_{jM}, provides a useful method for computing h_{jM} and thus for determining the velocity correlation, u_{ij} from w_{kl}.

The theory for the determination of the velocity vector as a function of the vorticity vector is briefly reviewed. It is shown that the theory is applicable to the determination of the velocity fluctuation vector as a function of the vorticity fluctuation vector. For the no slip boundary condition and finite regions this leads to an integral expression, the kernel of which is the product of the singular part of the Green's function and the vorticity

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vector. Products of the integral expression with itself are used to derive an iterated integral giving velocity correlations as functions of vorticity correlations. The kernel of this expression contains the function, h_{jM}, which is shown to satisfy Poisson's equation with homogeneous boundary conditions. An explicit expression for $h_{jM}(\mathbf{x}, \mathbf{X})$ is derived and it is shown that $h_{jM,j} = 0$ and $h_{jM,M} = 0$.

Velocity and vorticity fluctuations. Let $U_i = \langle U_i \rangle + u_i$ where $U_i \equiv$ components of the velocity vector, $\langle U_i \rangle \equiv$ mean velocity vector and $u_i \equiv$ velocity fluctuation vector. For incompressible fluids the continuity equation implies that

$$u_{i,i} = 0 \tag{1}$$

where $(\ )_{,i} \equiv \partial(\ )/\partial x_i$ and repeated indices are summed, [1]. From the conditions satisfied by U_i on a stationary boundary, [5, sec. 69], with unit normal n_i, it follows that on the boundary, in general, $u_i n_i = 0$ and for viscous fluids, assuming no slip, $u_i = 0$.

The vorticity vector $W_i = e_{ijk} U_{k,j}$ where $e_{ijk} \equiv$ the permutation symbol. It follows that $\langle W_i \rangle = e_{ijk} \langle U_{k,j} \rangle$ and

$$w_i = e_{ijk} u_{k,j} \tag{2}$$

where $w_i \equiv$ vorticity fluctuation vector, [1]. In [5 sec. 106] it is shown that $W_i n_i = 0$ for any stationary surface over which the no slip condition is satisfied. According to [5], the earliest proof of the result appears in [6]. Since $\langle w_i \rangle = 0$ and $\langle \langle W_i \rangle \rangle = \langle W_i \rangle$ it follows from the foregoing that

$$w_i n_i = 0 \tag{3}$$

for any stationary surface assuming no slip. It follows from (2) that

$$w_{i,i} = 0. \tag{4}$$

The conditions which have been shown to be satisfied by the velocity and vorticity fluctuations are also satisfied by the velocity, U_i, and vorticity, W_i.

Velocity fluctuations from vorticity fluctuations. The problem of the determination of the velocity vector, U_i, in terms of the vorticity vector, W_i, has been reviewed in [8], Chap. VI, with extensive references to the work of Lichtenstein, [6], p. 101, Poincaré, [9], Villat, [10], Chap. II and Crudeli, [11]. It is shown in [8, 9, 10] and in a formal development in [12], 18.20, that if $W_i = e_{ijk} U_{k,j}$, continuity is satisfied, $U_{i,i} = 0$, and the boundary condition $U_i n_i = 0$ is satisfied on stationary surface, then

$$U_i = e_{ijk} A_{k,j} \tag{5}$$

where

$$A_i(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{X}) W_I(\mathbf{X})\, dV + e_{iJK} \int_S g(\mathbf{x}, \mathbf{X}) n_K U_J(\mathbf{X})\, dS. \tag{6}$$

$g(\mathbf{x}, \mathbf{X})$ is the singular part of Green's function for Poisson's equation defined in a volume V with surface S, n_J is the unit outward normal to the surface S and $g(\mathbf{x}, \mathbf{X}) = 1/(4\pi r(\mathbf{x}, \mathbf{X}))$ [13], for three dimensional space. If the no slip condition is imposed over the surface S, then $U_j(\mathbf{X}) = 0$ on S and (6) becomes

$$A_i(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{X}) W_I(\mathbf{X})\, dV. \tag{7}$$

Since the velocity and vorticity fluctuations, u_i and w_i, satisfy the same conditions as the velocity and vorticity, U_i and W_i, it follows that for a viscous fluid, with no slip,

$$u_i = e_{ijk} a_{k,j} \tag{8}$$

where

$$a_i(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{X}) w_I(\mathbf{X})\, dV. \tag{9}$$

The more comprehensive theories discussed in [8] may be shown to reduce to (5) and (7) for the no slip case. The validity of (5) and (7) may also be deduced from the Helmholtz representation theorem, [14], sec. 89. In [15] the validity of Helmholtz's representation theorem is established for infinite domains if $\mathbf{U} = \mathbf{C} + O(r^{-\delta})$ as $r \to \infty$ where $\mathbf{C}$ and $\delta > 0$ are constants which is a weaker requirement that that of [14].

Velocity and vorticity correlations. The velocity correlation, $u_{ij}(\mathbf{x}) \equiv \langle u_i(\mathbf{x}) u_j(\mathbf{x})\rangle$, may be found from the vorticity correlation, $w_{MR}(\mathbf{X}, \mathbf{Y}) \equiv \langle w_M(\mathbf{X}) w_R(\mathbf{Y})\rangle$, by averaging the product of $u_i(\mathbf{x})$, (8, 9), with itself to give

$$u_{ij}(\mathbf{x}) = e_{IMN} e_{JRS} \int_V g,_N(\mathbf{x}, \mathbf{X}) \left(\int_V g,_S(\mathbf{x}, \mathbf{Y}) w_{MR}(\mathbf{X}, \mathbf{Y})\, dV(\mathbf{Y}) \right) dV(\mathbf{X}) \tag{10}$$

where, in general, $Q,_N(\mathbf{x}, \mathbf{Z}) \equiv \partial Q/\partial Z_N$, $Q,_n(\mathbf{x}, \mathbf{Z}) \equiv \partial Q/\partial x_n$ and it follows from the definition of $g(\mathbf{x}, \mathbf{X})$ that $g,_n = -g,_N$. Let

$$f_{MR}(\mathbf{x}, \mathbf{X}) \equiv \int_V g(\mathbf{x}, \mathbf{Y}) w_{MR}(\mathbf{X}, \mathbf{Y})\, dV(\mathbf{Y}),$$

$$f_{MRS}(\mathbf{x}, \mathbf{X}) \equiv \int_V g,_S(\mathbf{x}, \mathbf{Y}) w_{MR}(\mathbf{X}, \mathbf{Y})\, dV(\mathbf{Y})$$

and

$$f_{MR,s}(\mathbf{x}, \mathbf{X}) \equiv \partial f_{MR}/\partial x_s.$$

Since $g,_s = -g,_S$ and from the definitions of f_{MRS} and $f_{MR,s}$ it follows that $f_{MRS} = -f_{MR,s}$. Noting that $g(\mathbf{x}, \mathbf{X})$ is a Green's function for the Poisson equation in V, then from the definition of $f_{MR}(\mathbf{x}, \mathbf{X})$ it follows that $\nabla_x^2 f_{MR}(\mathbf{x}, \mathbf{X}) = -w_{MR}(\mathbf{X}, \mathbf{x})$, [13]. Taking $\partial/\partial x_s$ and since $f_{MR,s} = -f_{MRS}$ the Poisson equation becomes

$$\nabla_x^2 f_{MRS}(\mathbf{x}, \mathbf{X}) = w_{MR,s}(\mathbf{X}, \mathbf{x}) \tag{11}$$

where $\nabla_{\mathbf{x}}^2 \equiv$ Laplacian operator with respect to the $\mathbf{x}$, x_i, coordinates. Define $h_{jM}(\mathbf{x}, \mathbf{X}) \equiv e_{JRS} f_{MRS}(\mathbf{x}, \mathbf{X})$. Then from (10)

$$u_{ij}(\mathbf{x}) = e_{iMN} \int_V g,_N(\mathbf{x}, \mathbf{X}) h_{jM}(\mathbf{x}, \mathbf{X})\, dV(\mathbf{X}). \tag{12}$$

From (11) and the definition of h_{jM} it follows that

$$\nabla_x^2 h_{jM}(\mathbf{x}, \mathbf{X}) = e_{JRS} w_{MR,s}(\mathbf{X}, \mathbf{x}) \tag{13}$$

where S and s are considered to be repeated indices and therefore summed. Let S be the surface of the simply connected domain V. Then if $\mathbf{x}$ are points on S it follows that $h_{jM}(\mathbf{x},\mathbf{X}) = 0$. This result may be proved as follows: The definition of f_{MRS} together with $w_R(\mathbf{Y}) = e_{RLP}\partial u_P(\mathbf{Y})/\partial Y_L$ then substituted into h_{jM} give

$$h_{jM}(\mathbf{x},\mathbf{X}) = \left\langle w_m(\mathbf{X})\int_V e_{JRS}e_{RLP}g,_S(\mathbf{x},\mathbf{X})u_{P,L}(\mathbf{Y})\,dV(\mathbf{Y})\right\rangle. \tag{14}$$

Substituting $e_{JRS}e_{RLP} = \delta_{JP}\delta_{SL} - \delta_{JL}\delta_{SP}$, [16], into (14) and summing on repeated indices gives

$$h_{jM}(\mathbf{x},\mathbf{X}) = \left\langle w_m(\mathbf{X})\left(\int_V g,_P(\mathbf{x},\mathbf{Y})u_{J,P}(\mathbf{Y})\,dV(\mathbf{Y}) - \int_V g,_P(\mathbf{x},\mathbf{Y})u_{P,J}(\mathbf{Y})\,dV(\mathbf{Y})\right)\right\rangle. \tag{15}$$

Green's theorem, [13], for functions f_1 and f_2 may be written as

$$\int_V f_{1,P}f_{2,P}\,dV = \int_S f_2\frac{\partial f_1}{\partial n}\,dS - \int_V f_2\nabla^2 f_1\,dV \tag{16}$$

where $\mathbf{n}$ is an outward pointing unit vector normal to the surface S. Letting $f_1 = g$ and $f_2 = u_j$ in (16) gives

$$\int_V g,_P(\mathbf{x},\mathbf{Y})u_{J,P}(\mathbf{Y})\,dV(\mathbf{Y}) = \int_S u_J(\mathbf{Y})\frac{\partial g}{\partial n}(\mathbf{x},\mathbf{Y})\,dS - u_j(\mathbf{x}) \tag{17}$$

since $\int_V u_J(\mathbf{Y})\nabla^2 g(\mathbf{x},\mathbf{Y})\,dV(\mathbf{Y}) = u_J(\mathbf{x})$, [13]. The surface integral, in (17), vanishes since $u_J(\mathbf{x}) = 0$ on the stationary surface, S, because of the no slip condition. If the point $\mathbf{x}$ approaches the fluid side of boundary S then $u_J(\mathbf{x})$ approaches zero because of the no slip condition. Thus the first integral on the right side of (15) is zero for points $\mathbf{x}$ on S. Continuity, (1), implies that $(gu_{P,J}),_P = g,_P u_{P,J}$. Gauss's theorem implies that $\int_V T_{IJ,I}\,dV = \int_S T_{IJ}n_I\,dS$. Letting $T_{IJ,I} = (gu_{I,J}),_I = g,_I u_{I,J}$; in Gauss's theorem gives

$$\int_V g,_P u_{P,J}\,dV = \int_S g(\mathbf{x},\mathbf{Y})u_{P,J}(\mathbf{Y})n_P\,dS(\mathbf{Y}) \tag{18}$$

where $\mathbf{n}$ is the outward pointing unit vector which is normal to S. Define a normal coordinate system, [17], sec. 2.6, such that $Y_I = f_I(\alpha_1,\alpha_2,\alpha_3)$ and $Y_I = f_I(\alpha_1,\alpha_2,0)$ defines the surface S. α_1 and α_2 are surface coordinates in S and α_3 a normal coordinate to S. In this coordinate system the normal to S has components $n_1 = 0$, $n_2 = 0$, $n_3 = 1$. The metric for the normal coordinate system is known to satisfy $g^{3\rho} = 0$, $g_{3\rho} = 0$, $\rho = 1,2$ and $g_{33} \neq 0$, [17]. Since u_P and n_P are vectors then $u_{P,J}n_P$ is a vector and may be expressed in the normal coordinates as $u_i|_k n^i = (\partial Y_J/\partial\alpha_k)u_{P,J}n_P$ by the tensor transformation law for a vector where $u_i|_k \equiv$ covarient derivative of u_i with respect to α_k. Since $n_1 = 0$, $n_2 = 0$ and $n_3 = 1$ it follows that $n^i = g^{ij}n_j = g^{i3} = 0$ if $i = 1,2$. Then $n^1 = 0$, $n^2 = 0$, $n^3 \neq 0$ and $u_i|_k n^i = u_3|_k n^3$ where $u_3|_k = \partial u_3/\partial\alpha_k - \{{}_3{}^n{}_k\}u_n$. On $S\mathbf{u} = 0$ which implies that $u_n = 0$ and therefore $\{{}_3{}^n{}_k\}u_n = 0$ for all k. On S, $u_3(\alpha_1,\alpha_2,0) = 0$ and therefore $\partial u_3/\partial\alpha_1 = 0$ and $\partial u_3/\partial\alpha_2 = 0$ on S. In the curvilinear normal coordinates (1) becomes $g^{ij}u_i|_j = 0$ which reduces to $u_3|_3 = 0$ on S. Therefore $u_3|_k = 0$ for all k, $u_i|_k n^i = 0$,

$u_{P,J} n_P = 0$ on S, the right side of (18) vanishes and the second integral on the right side of (15) vanishes. It follows that $h_{jM}(\mathbf{x}, \mathbf{X}) = 0$ for points $\mathbf{x}$ on S. Thus $h_{jM}(\mathbf{x}, \mathbf{X})$ satisfies Poisson's equation, (13), with homogeneous boundary conditions in the variable $\mathbf{x}$.

Substituting (17) and (18) into (15) with no slip boundary conditions gives

$$h_{jM}(\mathbf{x}, \mathbf{X}) = -\langle w_M(\mathbf{X}) u_J(\mathbf{x}) \rangle. \tag{19}$$

It follows from (19) and (1) that

$$h_{jM,j}(\mathbf{x}, \mathbf{X}) = 0 \tag{20}$$

and from (19) and (4) that

$$h_{jM,M}(\mathbf{x}, \mathbf{X}) = 0. \tag{21}$$

It may be shown, through a direct calculation, that (19) is consistant with (13). It follows from (19) and (3) that

$$h_{jM}(\mathbf{x}, \mathbf{X}) n_M = 0 \tag{22}$$

for points, $\mathbf{X}$, on any stationary surface.

Conclusions. It has been shown for finite regions, assuming a no slip condition, that the velocity fluctuation vector is expressible as a function of the volume integral of the product of the singular part of the Green's function with the vorticity fluctuation vector. This result has been applied to integral expressions for velocity correlations as functions of vorticity correlations given in [3] and [4]. The kernel of the resulting expressions has been shown to satisfy Poisson's equation with homogeneous boundary conditions. An explicit expression has been derived for the kernel and it has been shown to satisfy a variety of field equations and boundary conditions. These results provide a practical computational method for determining the velocity correlation corresponding to a given field of vorticity correlations.

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