

NOTE ON 120 DEGREE AXIAL COORDINATES*

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Abstract. Recently the author introduced [1] a symmetrical system of distance coordinates for studying the planar motion of a three-body system of point masses in its internal degrees of freedom, and applied them to the small vibration problem. Subsequently he applied them [2] to the gravitational three-body problem. The purpose of this note is to give a geometrical identification of the auxiliary angles θ_1, θ_2, and θ_3, related to the system, and to prove the identity which they satisfy.

Note. Figure 1 shows the masses m_1, m_2, and m_3 at the vertices of a triangle with sides y_1, y_2, and y_3, which subtend angles of 120 degrees at a moving origin 0. The coordinates x_1, x_2, and x_3 are taken as the signed distances from 0 to the masses. Only the positive halves of the $0x_1, 0x_2$, and $0x_3$ coordinate axes shown in Fig. 1. The coordinate system is completed by the angle of rotation ϕ of these axes about the normal to the plane of motion. The transformation from y_1, y_2, y_3 to x_1, x_2, x_3 was found by Bleick [1] to be

$$x_1\sqrt{3} = -y_1\cos\theta_1 + y_2\cos\theta_2 + y_3\cos\theta_3, \tag{1}$$

$$x_2\sqrt{3} = y_1\cos\theta_1 - y_2\cos\theta_2 + y_3\cos\theta_3, \tag{2}$$

$$x_3\sqrt{3} = y_1\cos\theta_1 + y_2\cos\theta_2 - y_3\cos\theta_3, \tag{3}$$

where

$$\cos\theta_1 = \pm\left[1 - \left(y_2^2 - y_3^2\right)^2 f/8y_1^2\right]^{1/2}, \tag{4}$$

$$f = \left\{y_1^2 + y_2^2 + y_3^2 + \left[6\left(y_1^2y_2^2 + y_2^2y_3^2 + y_3^2y_1^2\right) - 3\left(y_1^4 + y_2^4 + y_3^4\right)\right]^{1/2}\right\}/R, \tag{5}$$

$$R = y_1^4 + y_2^4 + y_3^4 - y_1^2y_2^2 - y_2^2y_3^2 - y_3^2y_1^2, \tag{6}$$

and similarly for $\cos\theta_2$ and $\cos\theta_3$.

Solve Eqs. (1)–(3) to find

$$y_3\cos\theta_3 = (x_1 + x_2)\sqrt{3}/2, \tag{7}$$

$$y_1\cos\theta_1 = (x_2 + x_3)\sqrt{3}/2, \tag{8}$$

$$y_2\cos\theta_2 = (x_3 + x_1)\sqrt{3}/2. \tag{9}$$

* Received November 15, 1983.

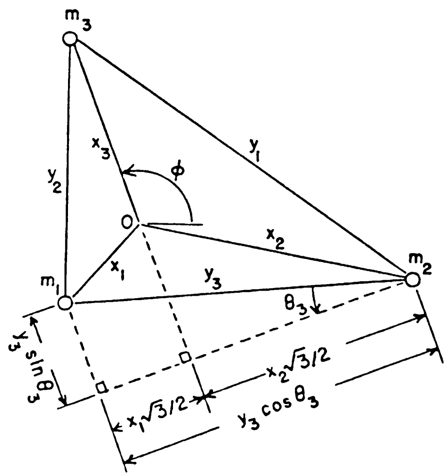

FIG. 1 120 *degree axial coordinates* x_1, x_2, *and* x_3.

The geometrical identification of θ_3 in Eq. (7) is achieved in Fig. 1 by resolving m_1m_2 into components perpendicular and parallel to m_30, and similarly for θ_1 and θ_2. The $y_3 \sin\theta_3$ orthogonal projection of m_1m_2 in the m_30 direction is the same as the projection of $m_10 + 0m_2$ in that direction. But the projection of m_10 in the m_30 direction is the negative of the projection of $0m_1$ in the m_20 direction. Likewise the projection of $0m_2$, in the m_30 direction is the negative of the projection of m_20 in the m_10 direction. From the extension of these considerations to the triangle sides y_1 and y_2 there follows the identity

$$y_1 \sin\theta_1 + y_2 \sin\theta_2 + y_2 \sin\theta_3 = 0. \tag{10}$$

REFERENCES

[1] Bleick, W. E., *New coordinates for three-body problems*, International Journal of Quantum Chemistry, Vol. 22, Quantum Chemistry Symposium No. 16, 241–245, 1982.

[2] Bleick, W. E., *Gravitational three-body problem in* 120 *deg axial coordinates*, Journal of Guidance, Control and Dynamics, **6** 124–128 (1983)