QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIII, NUMBER 2
JULY 1985, PAGES 189-197

NONOSCILLATION IN A DELAY-LOGISTIC EQUATION*

By

K. GOPALSAMY

The Flinders University of South Australia

Sufficient conditions are derived for the existence of nonoscillatory solutions of equations of the form

$$\frac{dx(t)}{dt} = x(t)\left\{b - \sum_{j=1}^{n} a_j x(t - \tau_j)\right\}$$

and

$$\frac{dx}{dt} = x(t)\left\{b - a\int_0^\infty k(s)x(t - s)\,ds\right\}.$$

and the global asymptotic stability of their positive steady states.

1. Introduction. Consider a nonlinear (delay-logistic) equation of the form

$$\frac{dx(t)}{dt} = x(t)\left\{b - \sum_{j=1}^{n} a_j x(t - \tau_j)\right\}; \qquad t > 0; \tag{1.1}$$

where b, a_j, τ_j $(j = 1, 2, \ldots, n)$ are positive constants. Equations of the form (1.1) occur in several apparently unrelated areas and has been investigated by numerous authors. It is easy to see that (1.1) has a positive steady state $x^* = b/(\sum_{j=1}^{n} a_j)$ and when $\sum_{j=1}^{n} \tau_j = 0$, solutions of (1.1) corresponding to initial conditions of the form $x^* \neq x(0) > 0$ are such that

$$x(t) - x^* \neq 0 \quad \text{for } t \geqslant 0 \quad \text{and} \quad x(t) - x^* \to 0 \quad \text{as } t \to \infty. \tag{1.2}$$

When each τ_j in (1.1) is not zero but sufficiently small, one will then intuitively expect at least some solutions of (1.1) corresponding to initial conditions of the form

$$x(s) = \varphi(s) \geqslant 0, \quad \varphi \in C[-\tau, 0], \quad \varphi(0) > 0; \qquad \tau = \max_{1 \leqslant j \leqslant n} \tau_j \tag{1.3}$$

will have a behaviour of the type in (1.2). We will first derive sufficient conditions for (1.1) to have solutions satisfying (1.2) and then indicate a generalisation of our analysis to equations with "unbounded delays".

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* Received February 1, 1984.

We will adopt the following definition: a real valued function x defined on an interval of the form $[c, \infty)$ for some real constant c is said to be nonoscillatory if and only if $x(t)$ has at most a finite number of zeros on $[c, \infty)$.

2. Discrete delays. It is convenient if we first consider the behaviour of solutions of (1.1) and (1.3) in a neighbourhood of x^* by means of a linearized analysis. If we let

$$x(t) \equiv x^* + z(t), \qquad t > 0, \tag{2.1}$$

in (1.1) then the linear variational system of (1.1) corresponding to x^* is given by

$$\frac{dz(t)}{dt} = -x^* \sum_{j=1}^{n} a_j z(t - \tau_j) \tag{2.2}$$

for which we have the following.

LEMMA 2.1. Assume that the positive constants b, τ_j, a_j $(j = 1, 2, \ldots, n)$ are such that

$$e\left(\sum_{j=1}^{n} a_j\right) x^* \tau \leqslant 1; \qquad x^* = b \Big/ \left(\sum_{j=1}^{n} a_j\right); \qquad \tau = \max_{1 \leqslant j \leqslant n} \tau_j; \tag{2.3}$$

then there exists a nonoscillatory solution of (2.2) in the form

$$z(t) = \alpha e^{-\mu t}; \qquad t \geqslant -\tau, \tag{2.4}$$

where α and μ are some positive constants.

Proof. Supplying (2.4) in (2.2) we note that z in (2.4) will be a solution of (2.2) if and only if μ is a positive root of the equation

$$\mu = x^* \sum_{j=1}^{n} a_j e^{\mu \tau_j}. \tag{2.5}$$

If we let $F(\mu) = \mu - x^* \sum_{j=1}^{n} a_j e^{\mu \tau_j}$ we note that

$$F(0) = -x^* \left(\sum_{j=1}^{n} a_j\right) < 0$$

$$F\left(\frac{1}{\tau}\right) = \frac{1}{\tau} - x^* \left(\sum_{j=1}^{n} a_j\right) e^{(\tau_j/\tau)}$$

$$\geqslant \left[1 - x^* \tau e \left(\sum_{j=1}^{n} a_j\right)\right] \Big/ \tau \geqslant 0$$

from which it will follow that there exists a real number $\mu > 0$ such that $F(\mu) = 0$. For such μ, z defined in (2.4) is a solution of (2.2) where α is any arbitrary real constant and such z has no zeros on $[0, \infty)$.

We can now formulate our nonoscillation result.

THEOREM 2.1. Assume that (2.3) holds. Then there exists a solution x of (1.1) and (1.3) such that $x(t) - x^*$ has no zeros on $[0, \infty)$.

Proof. Since solutions of (1.1) and (1.3) remain nonnegative we can let

$$X(t) = \log[x(t)/x^*] \tag{2.6}$$

and derive that

$$\frac{dX(t)}{dt} = -x^* \sum_{j=1}^{n} a_j[e^{X(t-\tau_j)} - 1]. \tag{2.7}$$

Define a sequence $\{X_n(t); t \geqslant -\tau, n = 0, 1, 2, \dots\}$ of functions as follows:

$$X_0(t) \equiv \alpha e^{-\mu t}, \qquad t \in [-\tau, \infty); \alpha > 0 \text{ is a constant;}$$

$$X_{n+1}(t) = \begin{cases} X_0(t) t \in [-\tau, 0] \\ X_0(0) + x^* \sum_{j=1}^{n} \int_t^{\infty} [e^{X_n(s-\tau_j)} - 1]\, ds, \end{cases} \tag{2.8}$$

$$t > 0, n = 0, 1, 2, \dots .$$

It will follow from the definition of $\{X_n\}$ that

$$X_1(t) - X_0(t) = \begin{cases} 0 \text{ for } t \in [-\tau, 0] \\ x^* \sum_{j=1}^{n} \int_t^{\infty} [e^{X_0(s-\tau_j)} - 1]\, ds, & t > 0, \end{cases}$$

and hence

$$X_1(t) - X_0(t) \geqslant 0 \quad \text{for } t \geqslant -\tau. \tag{2.9}$$

Similarly it will follow from (2.9) that

$$X_2(t) - X_1(t) \geqslant 0 \quad \text{for } t \geqslant -\tau$$

and thus

$$0 \leqslant X_0(t) \leqslant X_1(t) \leqslant \dots \leqslant X_n(t) \leqslant X_{n+1}(t) \leqslant \dots \quad \text{for } t \geqslant -\tau. \tag{2.10}$$

We claim that the sequence $\{X_n(t)\}$ is pointwise bounded on $[-\tau, \infty)$. Suppose our claim is not valid; then there will exist an integer $m \geqslant 1$ and a real number $t^* > 0$ such that

$$X_m(t) \to \infty \quad \text{as } t \to t^* \quad \text{and} \quad X_j(t) < \infty, \qquad j = 1, 2, \dots, m-1; t \geqslant 0.$$

This will then imply that given any arbitrary positive number say N there exists a $t^{**} < t^*$ such that

$$X_m(t^{**}) = N \text{ and } \left.\frac{dX_m}{dt}\right]_{t=t^{**}} > 0. \tag{2.11}$$

But we have from the definition of $\{X_n\}$ that

$$\left.\frac{dX_m}{dt}\right]_{t=t^{**}} = -x^* \sum_{j=1}^{n} a_j[e^{X_{m-1}(t^{**}-\tau_j)} - 1] \leqslant 0$$

and this contradicts (2.11). Thus the pointwise boundedness of the sequence $\{X_n(t); t \geqslant -\tau; n = 0, 1, 2, \dots\}$ follows and the following limit exists in a pointwise sense;

$$\lim_{t \to \infty} X_n(t) = X^*(t); \qquad t \geqslant -\tau. \tag{2.12}$$

Since $X_n(t) \geqslant 0$ for $n = 0, 1, 2, \ldots$ and $t \geqslant -\tau$, it will follow that $X^*(t) \geqslant 0$ for $t \geqslant -\tau$. By the Lebesgue's convergence theorem we have from (2.8) that

$$X^*(t) = X_0(0) + x^* \sum_{j=1}^{n} a_j \int_t^\infty [e^{X^*(s-\tau_j)} - 1]\, ds, \qquad t > 0. \tag{2.13}$$

It will follow from (2.12) and (2.13) that

$$\frac{dX^*}{dt} = -x^* \sum_{j=1}^{n} a_j [e^{X^*(t-\tau_j)} - 1]; \qquad t > 0; \tag{2.14}$$

$$X^*(t) = X_0(t), \qquad t \in [-\tau, 0],$$

showing that X^* is a nonnegative solution of (2.7) on $[-\tau, \infty)$ such that $X^*(t) \geqslant X_0(t)$ on $[-\tau, \infty)$ which implies that $x(t) - x^*$ has no zeros on $[-\tau, \infty)$ and the proof is complete.

It is interesting to examine how the foregoing analysis can be extended for integrodifferential equations of the form

$$\frac{dy(t)}{dt} = y(t)\left\{b - a\int_0^\infty k(s)y(t-s)\, ds\right\} \tag{2.15}$$

supplemented with initial conditions of the type

$$y(s) = \varphi(s) \geqslant 0; \quad s \in (-\infty, 0]; \ \varphi \text{ is continuous on } (-\infty, 0]; \tag{2.16}$$

$$\varphi(0) > 0; \qquad \sup_{s \leqslant 0} \varphi(s) < \infty.$$

We assume the following for (2.15)–(2.16);

(i) a, b are positive constants;

(ii) the delay kernel k is piecewise (locally) continuous on $[0, \infty)$ such that $k(s) \geqslant 0$ for $s \in [0, \infty)$ and

$$\int_0^\infty k(s)\, ds = 1; \quad \int_0^\infty k(s)s\, ds = \alpha < \infty; \quad \int_0^\infty k(s)e^{(s/\alpha)}\, ds \leqslant e. \tag{2.17}$$

The system (2.15)–(2.17) has a steady state $y^* = a/b$ and under (2.16) all solutions of (2.15) will remain nonnegative. As before we let

$$Y(t) = \log[y(t)/y^*]; \ t > 0$$

and derive that

$$\frac{dY(t)}{dt} = -ay^* \int_0^\infty k(s)[e^{Y(t-)} - 1]\, ds. \tag{2.18}$$

The following lemma is analogous to that of Lemma 2.1.

LEMMA 2.2. Assume that a, b are positive constants and the delay kernel k satisfies the above assumptions. Furthermore suppose

$$e\alpha a y^* \leqslant 1 \tag{2.19}$$

Then there exists a solution of the linear integrodifferential equation

$$\frac{du}{dt} = -ay^* \int_0^\infty k(s)u(t-s)\, ds; \qquad t > 0. \tag{2.20}$$

such that $u(t) \neq 0$ for $t \geqslant 0$.

Proof. It is enough to prove that the characteristic equation associated with (2.20) given by

$$\lambda = -ay^* \int_0^\infty k(s) e^{-\lambda s}\, ds \tag{2.21}$$

has a real root. For instance if we define $H(\lambda)$ so that

$$H(\lambda) = \lambda + ay^* \int_0^\infty k(s) e^{-\lambda s}\, ds$$

then

$$H(0) = ay^* > 0,$$
$$H\left(-\frac{1}{\alpha}\right) = -\frac{1}{\alpha} + ay^* \int_0^\infty k(s) e^{(s/\alpha)}\, ds \leqslant (-1 + ay^* \alpha e)/\alpha \leqslant 0,$$

from which it will follow that (2.21) has at least one negative root corresponding to which (2.20) will have a solution of constant sign on $[0, \infty)$.

The proof of the following result is similar to that of Theorem 2.1 and hence we will omit the details of proof.

THEOREM 2.2. Assume that the conditions of lemma 3.1 hold; then there exists a nonoscillatory solution of (2.15)–(2.16) in the sense that $y(t) - y^*$ has no zeros on $[0, \infty)$.

3. Global asymptotic stability. Let $x(t, \varphi)$ denote a solution of (1.1) corresponding to an initial condition φ of the form in (1.3). If all solutions of (1.1) have the property

$$\lim_{t \to \infty} x(t, \varphi) = x^* \tag{3.1}$$

for arbitrary φ as in (1.3) then the steady state x^* of (1.1) is said to be globally asymptotically stable or globally attractive. Note that only nonnegative initial conditions φ with $\varphi(0) > 0$ are considered. The following result offers an interesting generalisation of a result known for the case of a single delay.

THEOREM 3.1. Suppose the positive constants b, a_j, τ_j $(j = 1, 2, \ldots, n)$ in (1.1) are such that

$$x^* \left(\sum_{j=1}^n a_j \tau_j \right) < 1. \tag{3.2}$$

then x^* of (1.1) is globally attractive of all other solutions with continuous nonnegative initial conditions as in (1.3).

Proof. We let $x(t) \equiv x^* + y(t)$ in (1.1) and derive that

$$\frac{dy}{dt} = -\sum_{j=1}^n a_j x^* y(t - \tau_j) - \sum_{j=1}^n a_j y(t) y(t - \tau_j), \qquad t > 0, \tag{3.3}$$

which for $t > \tau$ can be written as

$$\frac{dy}{dt} = -y(t) \sum_{j=1}^n a_j x(t - \tau_j) + x^* \sum_{j=1}^n a_j \left\{ \int_{t - \tau_j}^t \left(\frac{dy}{du} \right) du \right\}, \qquad t > \tau. \tag{3.4}$$

From elementary considerations one can show that solutions of (1.1) and (1.3) remain bounded on $[-\tau, \infty)$ and hence solutions of (3.4) are defined for all $t \geqslant \tau$ and remain bounded. Define

$$m(t) = \sup_{s \in [t-\tau, t]} \left| \frac{dy}{ds} \right| \quad \text{and let} \quad c = \left(\sum_{j=1}^{n} a_j \tau_j \right) x^*.$$

It will follow from (3.4) that

$$m(t) \leqslant |y(t)| \left\{ \sum_{j=1}^{n} a_j x(t - \tau_j) \right\} + x^* \left(\sum_{j=1}^{n} a_j \tau_j \right) m(t)$$

implying that

$$m(t) \leqslant \frac{1}{1-c} |y(t)| \sum_{j=1}^{n} a_j x(t - \tau_j) < \infty, \qquad t \geqslant \tau. \tag{3.5}$$

(3.4) and (3.5) imply that dy/dt is bounded on $[\tau, \infty)$ and hence y is uniformly continuous on $[\tau, \infty)$. Let us consider a Lyapunov functional $v(t, y)$ defined by

$$\begin{aligned} v(t, y) = |y(t)| - c \int_{t^*}^{t} \left| \frac{dy}{ds} \right| ds \\ + x^* \sum_{i=1}^{n} a_j \int_{t-\tau_j}^{t} \left\{ \int_{s}^{t} \left| \frac{dy}{du} \right| \right\} ds. \end{aligned} \tag{3.6}$$

where $t^* \geqslant \tau$ is determined such that on (t^*, t), $\frac{dy}{du}$ has same sign. The boundedness of y on $[0, \infty)$ and (3.5) imply that $v(t, y)$ is bounded for $t \in [\tau, \infty)$. Calculating the right derivative $D^+ v$ of v along the solutions of (3.4) we have

$$\begin{aligned} D^+ v(t, y) = \left(\frac{dy}{dt} \right) [\operatorname{sign}(y(t))] - c|\dot{y}(t)| \\ + x^* \sum_{j=1}^{n} a_j \left[-\int_{t-\tau_j}^{t} \left| \frac{dy}{du} \right| du + \tau_j |\dot{y}(t)| \right], \end{aligned} \tag{3.7}$$

$$t \geqslant \tau; \ (\dot{y} = dy/dt).$$

Simplifying (3.7) using (3.4) we derive that

$$D^+ v(t, y) \leqslant -|y(t)| \sum_{j=1}^{n} a_j x(t - \tau_j), \qquad t \geqslant \tau. \tag{3.8}$$

Since we have from (3.6),

$$v(t, y) \geqslant |y(t)|(1 - c), \qquad t \geqslant \tau,$$

and integration of (3.8) leads to

$$v(t, y(t)) - v(\tau, y(\tau)) \leqslant -\int_{\tau}^{t} |y(s)| \left(\sum_{j=1}^{n} a_j x(s - \tau_j) \right) ds$$

and hence

$$(1 - c)|y(t)| + \int_{\tau}^{t} |y(s)| \sum_{j=1}^{n} a_j x(s - \tau_j) \, ds \leqslant v(\tau, y(\tau)) < \infty. \tag{3.9}$$

A consequence of (3.9) is that $|y(t)|\sum_{j=1}^{n} a_j x(t-\tau_j)$ is integrable on $[\tau, \infty)$. Now the uniform continuity of $|y(t)|\sum_{j=1}^{n} a_j x(t-\tau_j)$ with its integrability on $[\tau, \infty)$ will imply that

$$\lim_{t\to\infty} |y(t)|\left\{\sum_{j=1}^{n} a_j x(t-\tau_j)\right\} = 0;$$

that is

$$\lim_{t\to\infty} |x(t) - x^*| \sum_{j=1}^{n} a_j x(t-\tau_j) = 0. \tag{3.10}$$

It is not difficult to see that (3.10) actually leads to the assertion of the theorem. For instance suppose the conclusion of the theorem is not valid; that is suppose $|x(t) - x^*|$ does not converge to zero as $t \to \infty$; then we have from (3.10) that $x(t-\tau_j) \to 0$ as $t \to \infty$ $(j = 1, 2, 3, \dots, n)$; this means that we can find a sequence $\{t_m\}$ as $m \to \infty$ such that for arbitrary $\varepsilon > 0$,

$$x(t_m) > 0, \quad \sum_{j=1}^{n} a_j x(t_m - \tau_j) < \varepsilon, \quad \left.\frac{dx}{dt}\right|_{t=t_m} < 0, \tag{3.11}$$

$$m = 1, 2, \dots.$$

But from the equation governing x, we have

$$\begin{aligned}\left.\frac{dx}{dt}\right|_{t=t_m} &= x(t_m)\left(b - \sum_{j=1}^{n} a_j x(t_m - \tau_j)\right)\\ &> x(t_m)[b-\varepsilon] > 0 \quad \text{if } \varepsilon < b.\end{aligned}$$

which contradicts (3.11). Thus it will follow from (3.10) that

$$\lim_{t\to\infty} |x(t) - x^*| = 0$$

and the proof is complete.

We will briefly indicate an analogue of Theorem 3.1 to systems of the form (2.15)–(2.16). Precisely we have the following.

THEOREM 3.2. Let a, b be positive constants and k: $[0, \infty) \to [0, \infty)$ be (locally) piecewise continuous on $[0, \infty)$ such that

$$\int_0^\infty k(s)\, ds = 1; \quad \int_0^\infty k(s) s\, ds = \alpha; \qquad a x^* \alpha < 1. \tag{3.12}$$

Then any solution of (2.15)–(2.16) has the property

$$\lim_{t\to\infty} x(t) = x^* = b/a. \tag{3.13}$$

Proof. Since the strategy of proof is exactly similar to that of Theorem 3.1 we will give a brief outline only. We let $x(t) = x^* + y(t)$ in (2.15) and derive that y is governed by

$$\begin{aligned}\frac{dy}{dt} &= -ay(t)\int_0^\infty k(s)x(t-s)\, ds\\ &\quad + ax^*\int_0^\infty k(s)[y(t) - y(t-s)]\, ds.\end{aligned} \tag{3.14}$$

(Since initial conditions φ are not necessarily in $C^1(-\infty, 0]$ but are in $C(-\infty, 0]$ we do not use derivatives under the intergal in (3.14).) A Lyapunov functional $v(t, y)$ suitable for (3.14) is given by

$$v(t, y) = |y(t)| - (ax^*\alpha)\int_{t^*}^{4}\left|\frac{dy}{ds}\right| ds$$
$$+ax^*\int_0^\infty k(s)\left[\int_{t-s}^{t} |y(t) - y(u)| su\right] ds, \qquad t > 0. \tag{3.15}$$

where $t^* \geqslant 0$ is determined as in Theorem 3.1 above. The boundedness of v for $t \geqslant 0$ will follow from that of solutions of (2.15)–(2.16) and the property of k in (3.12) relating to the finiteness of the mean of k. A calculation of the right derivative D^+v of v along the solutions of (3.14) will lead to

$$D^+v(t, y) \leqslant -a|y(t)|\int_0^\infty k(s)x(t-s)\, ds, \qquad t > 0, \tag{3.16}$$

and the remainder of the proof is similar to that of theorem 3.1 and we will omit further details.

4. Comments. We remark that the methods proposed here for studying nonoscillation and global stability of (1.1) and (2.15) can also be used for similar studies of

$$\frac{dx}{dt} = x(t)\left\{b - a_0x(t) - \sum_{j=1}^{n} a_jx(t-\tau_j)\right\} \tag{4.1}$$

where b, a_0, a_j, τ_j ($j = 1, 2, \ldots, n$) are positive constants and integro-differential equations of the form

$$\frac{dy}{dt} = y\left\{b - a_0y(t) - a_1y(t-\tau) - a_2\int_0^\infty k(s)y(t-s)\, ds\right\} \tag{4.2}$$

where b, a_0, a_1, a_2, τ are positive constants and k is a suitable kernel. The conditions derived by us here are sufficient conditions only; it is known (see [3]) that in the case of a single delay, our condition of Theorem 2.1 is also necessary for (1.1) to be nonoscillatory.

A number of authors (Kakutani and Markus [3], Hutchinson [1], Jones [2], Wright [4]) have studied in different forms the equation

$$\frac{dx}{dt} = \gamma x(t)[K - x(t-\tau)]/K. \tag{4.3}$$

It has been shown by Kakutani and Markus [3] and Wright [4] that when $\gamma\tau < 1$ in (4.3), the nonzero steady state of (4.3) is globally asymptotically stable; our theorem 3.1 offers a generalisation of this result. We conclude with the remark that there is an urgent need to derive sufficient conditions for the global asymptotic stability of a nonnegative steady state of systems of the form

$$\frac{dx_i}{dt} = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij})\right), \qquad i = 1, 2, \ldots, n \tag{4.4}$$

(b_i, a_{ij}, τ_{ij} being constants with $\tau_{ij} \geqslant 0$, $\tau_{ii} \neq 0$, $i, j = 1, 2, \ldots, n$) since systems of the form (4.4) are useful models of ecosystems and global stability will preclude the existence of periodic solutions believed to be common in time delayed systems. We have elsewhere performed a detailed study of (4.4) with $\tau_{ii} = 0$, $i = 1, 2, \ldots, n$.

Acknowledgement. The author wishes to thank a referee for suggesting the inclusion of Sec. 3 in this paper.

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