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NONOSCILLATORY DIFFERENTIAL EQUATIONS WITH RETARDED AND ADVANCED ARGUMENTS*

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Abstract. Sufficient conditions are derived for a vector-matrix system of the form

$$\frac{d^nX(t)}{dt^n}+(-1)^{n-1}[P(t)X(t-\tau_1(t))+Q(t)X(t+\tau_2(t))]=0$$

to be nonoscillatory.

1. Introduction. We will first derive a set of sufficient conditions for scalar differential equations of the form

$$\frac{d^nx(t)}{dt^n}+(-1)^{n-1}[a(t)x(t-\tau_1(t))+b(t)x(t+\tau_2(t))]=0 \tag{1.1}$$

to be nonoscillatory. Although some authors (Kusano [3], Anderson [1]) have discussed oscillatory nature of (1.1), the literature concerned with nonoscillation of equations of the form (1.1) is scarce. We will assume the following for (1.1);

(A_1) a, b, τ_1, τ_2 are bounded continuous functions defined on $\mathbb{R}=(-\infty,\infty)$ such that for $t\in\mathbb{R}$.

$$\begin{aligned}&0<a(t)\leqslant\alpha; &&0\leqslant t-\tau_1(t)\\&0\leqslant b(t)\leqslant\beta; &&0\leqslant\tau_2(t)\\&0<\tau_1(t)\leqslant\sigma;\end{aligned}$$

where α, β, σ are positive constants.

(A_2) The positive constants α, β, σ are such that

$$(\alpha+\beta)e^n\sigma^n/n^n\leqslant 1. \tag{1.2}$$

The following elementary observation is useful in proving our nonoscillation result. Consider a function g: $[0,\infty)\to\mathbb{R}$ defined by

$$g(\mu)=\mu^n-(\alpha+\beta)e^{\mu\sigma}.$$

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Since we have

$$\begin{aligned} g(0) &= -(\alpha+\beta) < 0 \\ g(n/\sigma) &= (n/\sigma)^n - (\alpha+\beta)e^n \\ &= (n/\sigma)^n[1-(\alpha+\beta)e^n\sigma^n/n^n] \geqslant 0 \quad \text{(by (1.2))}, \end{aligned}$$

it will follow that $g(\mu) = 0$ has a positive root say μ^* such that

$$(\mu^*)^n = (\alpha+\beta)e^{\sigma\mu^*}. \tag{1.3}$$

2. Nonoscillatory scalar systems. As it is customary we will say that (1.1) is oscillatory if and only if all solutions of (1.1) have zeros on every interval of the form $[\alpha, \infty)$ for arbitrary real constants α and (1.1) will be called nonoscillatory if there exists at least one solution of (1.1) having no zeros on an interval of the form $[\beta, \infty)$ for some real constant β. We can now establish the following:

THEOREM 2.1. Suppose a, b, τ_1, τ_2 in (1.1) satisfy the hypothesis (A_1) and (A_2); then (1.1) is nonoscillatory.

Proof. As one will see the proof is surprisingly simple. We consider a sequence $\{x_m(t);\, t \geqslant -\sigma;\, m = 0,1,2,\dots\}$ defined as follows:

$$x_0(t) = \exp[-\mu^* t]; \qquad t \geqslant -\sigma; \tag{2.1}$$

$$x_{m+1}(t) = \begin{cases} \exp[-\mu^* t]; & t \in [-\sigma, 0]; \\ \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}[a(s)x_m(s-\tau_1(s)) + b(s)x_m(s+\tau_2(s))]\,ds, & t > 0. \end{cases} \tag{2.2}$$

It is immediate from (2.1)–(2.2) that

$$\begin{aligned} x_1(t) &\leqslant \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}[\alpha\exp[-\mu^*(s-\sigma)] + \beta\exp[-\mu^*(s+\tau_2(s))]]\,ds \\ &\leqslant (\alpha+\beta)(\exp[\mu^*\sigma])\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\exp[-\mu^* s]\,ds \\ &\leqslant (\alpha+\beta)\{\exp[\mu^*\sigma]/(\mu^*)^n\}\exp[-\mu^* t] \\ &\leqslant e^{-\mu^* t} \quad \text{(by the choice of } \mu^*) \\ &\leqslant x_0(t) \quad \text{for } t > 0, \end{aligned}$$

and hence

$$x_1(t) - x_0(t) \leqslant 0 \quad \text{for } t \geqslant -\sigma. \tag{2.3}$$

From (2.2) and (2.3) one can similarly obtain

$$x_2(t) - x_1(t) \leqslant 0 \quad \text{for } t \geqslant -\sigma, \tag{2.4}$$

and repeating the above procedure we derive

$$0 \leqslant x_{m+1}(t) \leqslant x_m(t) \leqslant \cdots \leqslant x_1(t) \leqslant x_0(t) \quad \text{for } t \geqslant -\sigma. \tag{2.5}$$

The pointwise limit of the sequence $\{x_m(t)\}$ as $m \to \infty$ exists for $t \geqslant -\sigma$ and so we can let

$$\lim_{m\to\infty} x_n(t) = x^*(t), \qquad t \geqslant -\sigma. \tag{2.6}$$

It will now follow by Lebesgue's dominated convergence theorem that

$$x^*(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[a(s)x^*(s-\tau_1(s)) + b(s)x^*(s+\tau_2(s))\right] ds, \qquad t > 0, \tag{2.7}$$

showing that x^* is a solution of (1.1) for $t > 0$. Since x^* is the limit of a sequence of nonnegative functions, x^* itself is nonnegative. Now it will follow from (2.7) that $x^*(t) > 0$ on $[-\sigma, \infty)$ for if $x^*(t) > 0$ on $[-\sigma, \tilde{t})$ and $x^*(\tilde{t}) = 0$ then (2.7) will lead to a contradiction. The result follows.

3. Nonoscillatory vector-matrix systems. Let us now consider the vector-matrix system

$$\frac{d^n X(t)}{dt^n} + (-1)^{n-1}\left[P(t)X(t-\tau_1(t)) + Q(t)X(t+\tau_2(t))\right] = 0, \qquad t > 0, \tag{3.1}$$

with the following assumptions:

(A_3) τ_1, τ_2 are bounded continuous scalar functions as in (A_1).

(A_4) $P(t)$, $Q(t)$ are $m \times m$ matrices with nonnegative elements such that at least one element of $P(t)$ is positive and in an element wise ordering we have

$$0 \leqslant P(t) + Q(t) \leqslant M \quad \text{for } t \geqslant 0 \tag{3.2}$$

where M is a constant $m \times m$ matrix with positive elements.

We will need the following preparation; it is well known (Perron's theorem) that M will have a positive eigenvalue say α^* corresponding to which M will have an eigenvector say Z with positive elements. Consider now a "majorant" of (3.1) in the form

$$\frac{d^n Y(t)}{dt^n} + (-1)^{n-1} M Y(t-\sigma) = 0; \qquad t > 0. \tag{3.3}$$

The characteristic equation associated with (3.3) is given by

$$\det\left[\lambda^n I + (-1)^{n-1} M e^{-\lambda\sigma}\right] = 0 \tag{3.4}$$

or equivalently

$$\det[\mu^n I - M e^{\mu\sigma}] = 0, \quad \text{with } \mu = -\lambda.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the eigenvalues of M we have

$$\det[\mu^n I - M e^{\mu\sigma}] = 0 \Leftrightarrow \prod_{j=1}^{m}\left[\mu^n - \alpha_j e^{\mu\sigma}\right] = 0.$$

If $\alpha_s = \alpha^*$ for some $s \in (1, 2, \ldots, m)$ we can consider

$$\mu^n - \alpha^* e^{\mu\sigma} = 0 \tag{3.5}$$

in looking for real roots of (3.4). Let us now assume that

$$\alpha^*(\sigma)^n e^n/n^n \leqslant 1. \tag{3.6}$$

It will then follow that (3.5) has a positive root say μ^* corresponding to which (3.3) will have a solution given by

$$Y(t) = Z(\exp[-\mu^* t]); \qquad t \geqslant -\sigma \tag{3.7}$$

(Z being a positive eigenvector associated with the positive eigenvalue μ^* of M). With this preparation we can now formulate the following for (3.1).

THEOREM 3.1. Assume that P, Q, τ_1, τ_2 satisfy the hypotheses (A_3) and (A_4). Furthermore assume that (3.6) holds. Then (3.1) is nonoscillatory.

Proof. Proof is quite similar to that of the scalar case and we provide a brief outline only. Define a sequence $\{X^K(t);\ t \geqslant -\sigma;\ K = 0, 1, 2, \ldots\}$ as follows:

$$X^{(0)}(t) = Z(\exp[-\mu^* t]); \qquad t \geqslant -\sigma; \tag{3.8}$$

$$X^{K+1}(t) = \begin{cases} Z(\exp[-\mu^* t]); & t \in [-\sigma, 0]; \\ \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[P(s)X^{(K)}(s-\tau_1(s)) + Q(s)X^{(K)}(s+\tau_2(s))\right] ds; & t > 0. \end{cases} \tag{3.9}$$

With a componentwise comparison, it will follow as in the scalar case (on using (3.6)),

$$0 \leqslant X^{(K)}(t) \leqslant X^{(K-1)}(t) \leqslant \cdots \leqslant X^{(1)}(t) \leqslant X^{(0)}(t), \qquad t > -\sigma, \tag{3.10}$$

and the rest of the proof is exactly similar to that in Theorem (2.1) and we will omit further details.

We conclude with a remark that we have shown elsewhere [2] that conditions of the type in (1.2) and (3.6) are in fact necessary also for equations of the form (1.1) with (3.1) with constant coefficients to be nonoscillatory.

REFERENCES

[1] C. H. Anderson, *Asymptotic oscillation results for solutions of first-order nonlinear differential-difference equations of advanced type*, J. Math. Anal. Appl. **24**, 430–439 (1968).

[2] K. Gopalsamy, *Oscillations in linear systems of differential-difference equations*, Bull. Austral. Math. Soc. **29**, 377–387 (1984)

[3] T. Kusano, *On even order functional differential equations with advanced and retarded arguments*, J. Diff. Equns. **45**, 75–84 (1984)