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STAGNATION FLOW ON THE SURFACE OF A QUIESCENT FLUID– AN EXACT SOLUTION OF THE NAVIER-STOKES EQUATIONS*

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Abstract. A lighter fluid impinges downward on a heavier, otherwise quiescent fluid. The region near the stagnation point is investigated. The Navier-Stokes equations yield similarity solutions for both upper and lower regions. The convective heat transfer is also determined.

Introduction. One of the most basic exact solutions of the Navier-Stokes equations is the stagnation point flow. Hiemenz [1] found the similarity solution for the two-dimensional stagnation point flow against a solid plate and Homann [2] studied the axisymmetric case. We ask, what would be the solution if the solid boundary is replaced by a free surface of a different, heavier fluid? This situation can be visualized when we blow softly against the surface of a cup of hot coffee.

In order to obtain exact solutions, we require the interface to be flat. This criterium can be quantified as follows. Let the upper light fluid be denoted by the subscript 1 and the lower heavier fluid be denoted by the subscript 2. Let (x, y_1) denote either Cartesian or cylindrical coordinates with $x = 0$ be the symmetry plane (2 dimensional case) or the symmetry axis (axisymmetric case). The potential, stagnation point flow of the lighter fluid is described by

$$u_1 = ax, \qquad v_1 = -may_1 \tag{1}$$

where u, v are velocity components in the x, y directions respectively, a is a constant with dimension (time)$^{-1}$ and

$$m = \begin{cases} 1, & \text{two dimensional,} \\ 2, & \text{axisymmetrical.} \end{cases} \tag{2}$$

The potential pressure distribution on $y_1 = 0$ is

$$p_1 = p_0 - \frac{1}{2}\rho_1 a^2 x^2 \tag{3}$$

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where p_0 is the stagnation pressure and ρ is the density. Using fluid statics, the deflection of the free surface δ is estimated to be

$$\delta = \frac{p_0 - p_1}{(\rho_2 - \rho_1)g} = \frac{\rho_1 a^2 x^2}{2(\rho_2 - \rho_1)g}. \tag{4}$$

Thus the criterium for a nearly flat interface is

$$\frac{\delta}{x} = \frac{a^2 x}{2(\rho_2/\rho_1 - 1)g} \ll 1. \tag{5}$$

This can be realized by small x (the region near the stagnation point) or small a (soft blowing) or large density difference ($\rho_2 \gg \rho_1$) or large gravitational acceleration g. Notice that the condition of small x is already implicit in all stagnation point flows. Also if surface tension effects are included, the estimated δ would be even smaller.

Formulation. Figure 1 shows the coordinate axes. For the upper lighter fluid let

$$u_1 = axf'(\eta), \quad v_1 = -m\sqrt{\nu_1 a}\, f(\eta), \quad \eta \equiv \frac{y_1}{\sqrt{\nu_1/a}} \tag{6}$$

where ν is the kinematic viscosity. The Navier-Stokes equations reduce to

$$f'''(\eta) + mff'' - (f')^2 + 1 = 0 \tag{7}$$

Equation (7) is the same as that of Hiemenz [1] and Homann [2]. However, the boundary conditions are different.

$$f(0) = 0, \qquad f'(0) = \beta, \tag{8}$$

$$f'(\infty) = 1. \tag{9}$$

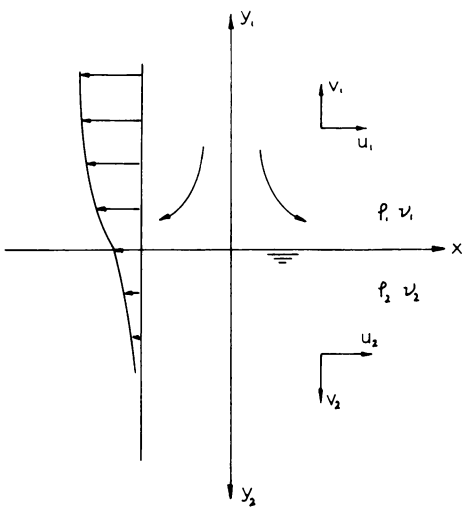

Fig. 1. The coordinate system.

The variables β, yet to be determined, represents lateral motion of the interface. β ranges from zero (solid boundary) to one (stress-free boundary). When β is one, the solution for the upper flow is potential:

$$f = \eta. \tag{10}$$

For the lower, heavier fluid we set

$$u_2 = a\beta x h'(\zeta), \quad v_2 = -m\sqrt{\nu_2 a\beta}\, h, \quad \zeta \equiv \frac{y_2}{\sqrt{\nu_2/a\beta}}. \tag{11}$$

Since the flow decays to zero as $y_2 \to \infty$, the Navier-Stokes equations reduce to

$$h'''(\zeta) + mhh'' - (h')^2 = 0. \tag{12}$$

Noting that the velocities must be equal at the interface, the boundary conditions are

$$h(0) = 0, \qquad h'(0) = 1, \tag{13}$$

$$h'(\infty) = 0. \tag{14}$$

The function h is independent of β. The solution for the two-dimensional case ($m = 1$) is

$$h = 1 - e^{-\zeta}. \tag{15}$$

This closed form solution was found by Stuart [3] in connection with acoustic streaming problems and later by Crane [4] in modelling the flow due to a two-dimensional stretching plate. The tangential stresses are then matched at the interface

$$\rho_1\nu_1 \frac{\partial u_1}{\partial y_1}(0) = -\rho_2\nu_2 \frac{\partial u_2}{\partial y_2}(0). \tag{16}$$

This yields

$$\frac{f''(0)}{-\beta^{3/2}h''(0)} = \frac{\rho_2}{\rho_1}\left(\frac{\nu_2}{\nu_1}\right)^{1/2} \equiv K. \tag{17}$$

Equation (17) is used to determine β.

The solution for $h(\zeta)$, $m = 2$. This problem is the axisymmetric analog of Crane's [4] solution. Suppose a membrane is radially stretched in a viscous fluid. The governing equations are Eqs. (12)–(14). For $m = 2$ the solution cannot be expressed in closed form. Asymptotic properties are obtained by setting

$$h = b + \varphi(\zeta) \tag{18}$$

where b is a constant and $\varphi(\zeta)$ is small. Then Eq. (12) linearizes to

$$\varphi''' + 2b\varphi'' = 0 \tag{19}$$

or

$$\varphi = b_0 + b_1\zeta + b_2 e^{-2b\zeta}. \tag{20}$$

In order for φ to decay, $b_0 = b_1 = 0$ and $b > 0$.

Perhaps the simplest method to solve Eqs. (12)–(14) is by numerical integration. We guess $h''(0)$ and integrate Eqs. (12), (13) as an initial value problem by the fifth order

Runge-Kutta-Fehlberg algorithm. A solution is found when $h(\zeta)$ decay exponentially to zero. We find

$$h''(0) = -1.1737247, \tag{21}$$

$$h(\infty) = b = 0.7514252 \tag{22}$$

This universal solution is plotted in Fig. 2, Eq. (15) is also plotted for comparison.

Perturbation solution forl $f(\eta)$ when $\beta \approx 1$. This is the case when the lower, heavier fluid is almost inviscid. Since $\nu_2 \to 0$ Eq. (17) gives $f''(0) \approx 0$ and the solution is $f \approx \eta$. Let $\varepsilon \approx 1 - \beta$ and we perturb as follows

$$f = \eta + \varepsilon F(\eta) + O(\varepsilon^2) \tag{23}$$

Equations (7)–(9) yield

$$F''' + m\eta F'' - 2F' = 0, \tag{24}$$

$$F(0) = 0, \quad F'(0) = -1, \quad F'(\infty) = 0. \tag{25}$$

For $m = 1$, the solution is

$$F(\eta) = \frac{1}{C_1}\int_0^{\eta}(s^2 + 1)\int_{\infty}^{s} e^{-t^2/2}(t^2 + 1)^{-2}\, dt\, ds, \tag{26}$$

$$C_1 \equiv \int_0^{\infty} e^{-s^2/2}(s^2 + 1)^{-2}\, ds = 0.6266571. \tag{27}$$

The value of C_1 is obtained by numerical quadrature. Thus

$$F''(0) = \frac{1}{C_1} = 1.595769. \tag{28}$$

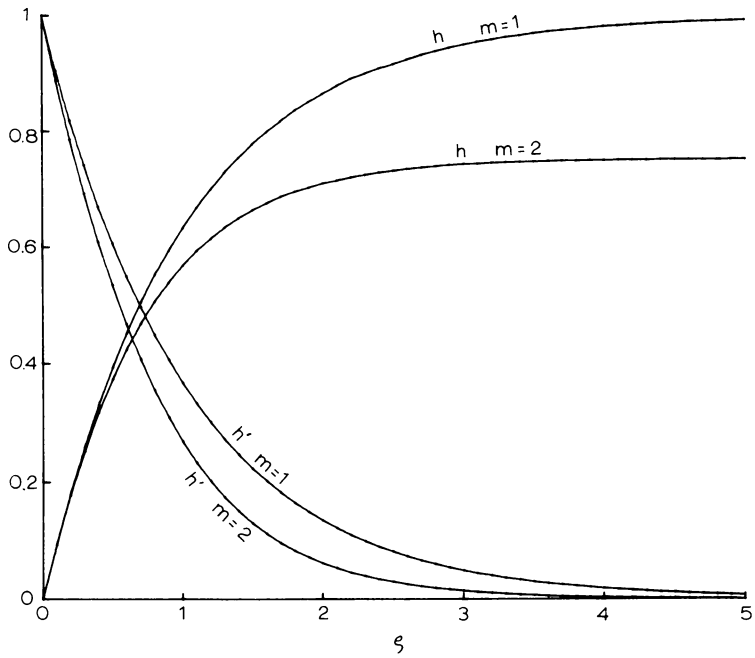

Fig. 2. The universal function $h(\zeta)$.

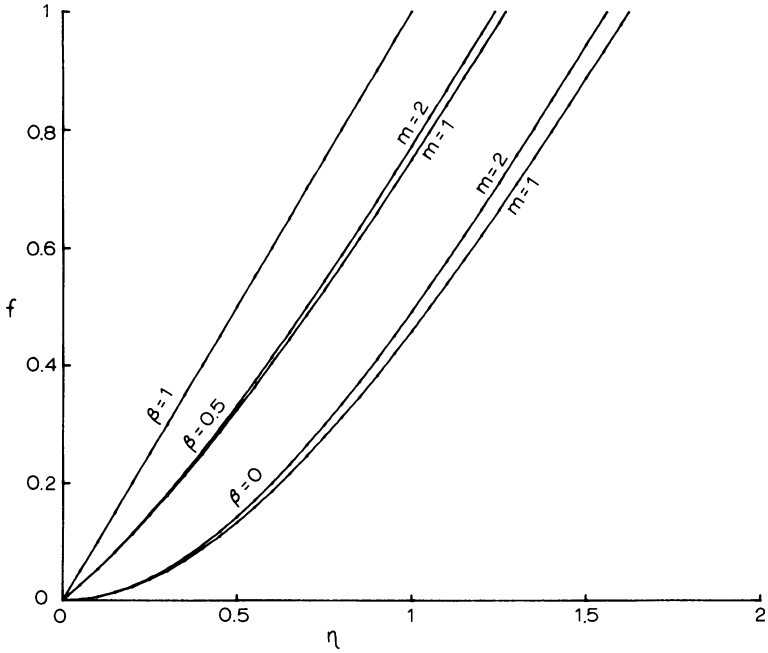

Fig. 3. The function $f(\eta)$.

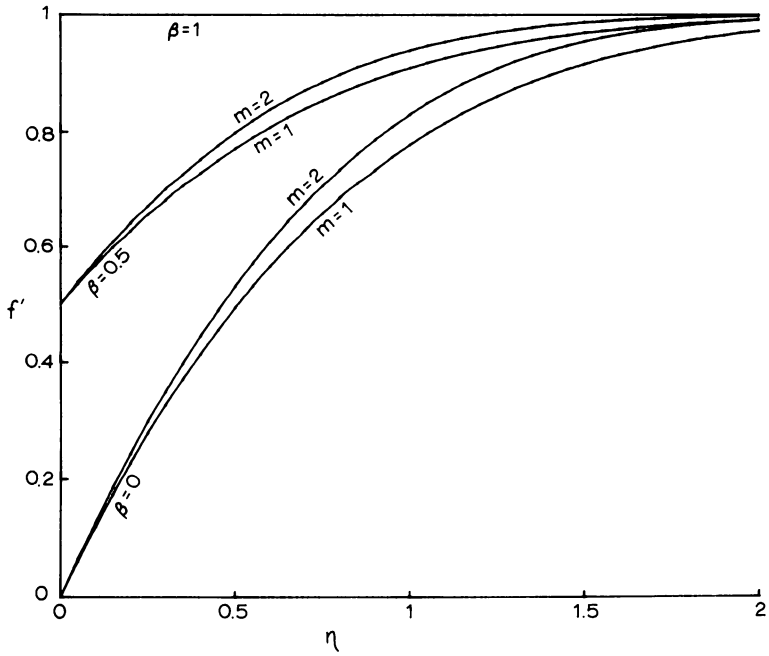

Fig. 4. The function $f'(\eta)$.

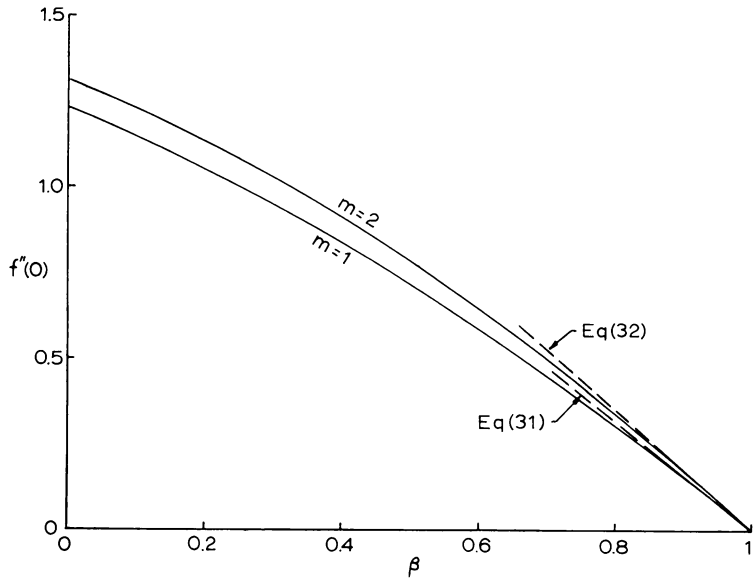

Fig. 5. The initial values $f''(0)$. Dashed lines are analytic approximations.

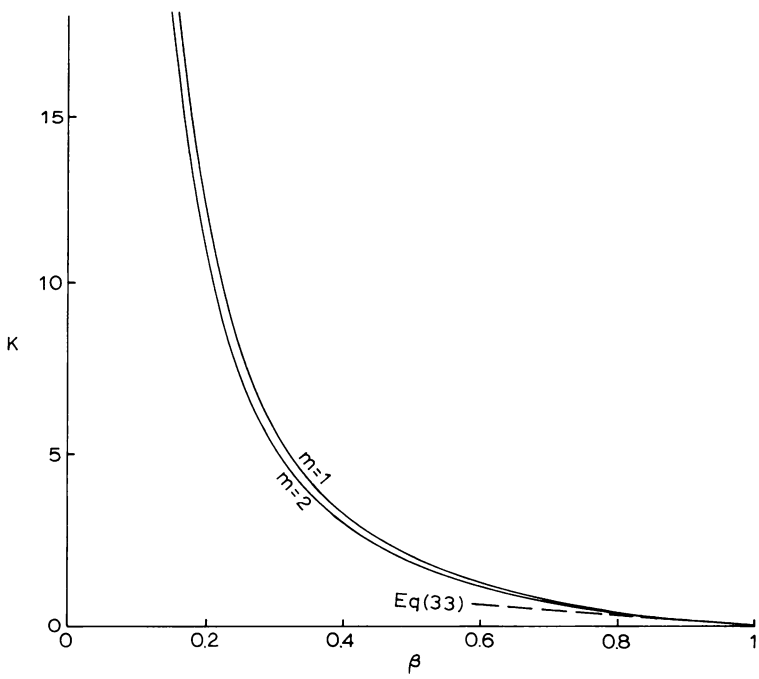

Fig. 6. Graph of $K = (\rho_2/\rho_1)(\nu_2/\nu_1)^{1/2}$ as a function of β.

For $m = 2$ the solution is

$$F = \int_0^\eta (s^2 - 1)e^{-s^2}\,ds - \eta^2 \int_\infty^\eta e^{-s^2}\,ds. \tag{29}$$

Thus

$$F''(0) = 2\int_0^\infty e^{-s^2}\,ds = \sqrt{\pi}\,. \tag{30}$$

Therefore the approximate initial values are

$$f''(0) = 1.595769(1 - \beta) + O(1 - \beta)^2, \qquad m = 1, \tag{31}$$

$$f''(0) = 1.772454(1 - \beta) + O(1 - \beta)^2, \qquad m = 2. \tag{32}$$

Numerical solution for $f(\eta)$. An asymptotic study shows that for large η and arbitrary β, the governing equation is similar to Eq. (24). Thus exponential decay occurs. For given β, Eqs. (7), (8) are integrated numerically with a guessed $f''(0)$, guided by Eqs. (31), (32). We find a step size of $\Delta\eta = 0.05$ guarantees accuracy to 10^{-6}. Some results are graphically shown in Figs. 3 and 4. When $\beta = 0$ we recover the stagnation point solutions of Hiemenz [1] and Homann [2]. Fig. 5 shows the variation of $f''(0)$ with β. Our approximate solutions compare well when $\beta \approx 1$. Table 1 shows the more accurate initial values. Since nowadays computers are fairly common, it is no longer necessary to tabulate functional values for all η here.

Given the density ratio and the viscosity ratio, one can determine the value of β from Eq. (17). The particular combination $K = -f''(0)\beta^{-3/2}/h''(0)$ is graphed in Fig. 6 for practical purposes. For $\beta \approx 1$

$$K = \begin{cases} 1.5958(1 - \beta) + O(1 - \beta)^2, & m = 1, \\ 1.5101(1 - \beta) + O(1 - \beta)^2, & m = 2. \end{cases} \tag{33}$$

The temperature profile. The problem considered in this paper is particularly important in forced convective heat transfer. Suppose the temperature at $y_1 \to \infty$ is $T_{1\infty}$ and the temperature at $y_2 \to \infty$ is $T_{2\infty}$. We would like to predict the heat transfer rate.

For the lighter fluid the energy equation yields

$$\chi_1'' + mP_1 f \chi_1' = 0 \tag{34}$$

where

$$\chi_1(\eta) \equiv \frac{T_1(\eta) - T_{1\infty}}{T_0 - T_{1\infty}}, \qquad P_1 \equiv \frac{\nu_1}{\kappa_1}. \tag{35}$$

Here κ is the thermal diffusivity and T_0 is the unknown constant temperature of the interface. The boundary conditions are

$$\chi_1(\infty) = 0, \qquad \chi_1(0) = 1. \tag{36}$$

The normalized temperature function χ_1 depends on m, β, and P_1.

For the heavier fluid we deduce

$$\chi_2'' + mP_2 h\chi_2' = 0, \tag{37}$$

$$\chi_2(\zeta) \equiv \frac{T_2(\zeta) - T_{2\infty}}{T_0 - T_{2\infty}}, \qquad P_2 \equiv \frac{\nu_2}{\kappa_2},$$

$$\chi_2(\infty) = 0, \qquad \chi_2(0) = 1. \tag{38}$$

χ_2 is a function of m and P_2 only. The interface temperature is determined by the constant heat flux

$$\kappa_1 \frac{\partial T_1}{\partial y_1}(0) = -\kappa_2 \frac{\partial T_2}{\partial y_2}(0). \tag{39}$$

TABLE 1

β	$f''(0)$ for $m = 1$	$f''(0)$ for $m = 2$
0	1.232588	1.311938
0.25	1.000054	1.081629
0.5	0.713295	0.780324
0.75	0.378421	0.417534
1	0	0

TABLE 2a. $\chi_1'(0)$ for $m = 1$

β \ P_1	1	10	100
0	−0.571	−1.340	−2.986
0.25	−0.634	−1.683	−4.600
0.5	−0.692	−1.990	−5.905
0.75	−0.747	−2.270	−7.010
1	−0.798	−2.523	−7.982

TABLE 2b. $\chi_1'(0)$ for $m = 2$

β \ P_1	1	10	100
0	−0.762	−1.752	−3.870
0.25	−0.867	−2.295	−6.346
0.5	−0.961	−2.768	−8.275
0.75	−1.048	−3.190	−9.883
1	−1.129	−3.569	−11.283

TABLE 3. $\chi_2'(0)$

m \ P_1	1	10	100
1	−0.582	−2.308	−7.765
2	−0.852	−3.308	−11.032

This gives

$$T_0 = T_{2\infty}\left(\frac{1}{1+A}\right) + T_{1\infty}\left(\frac{1}{1+A^{-1}}\right), \tag{40}$$

where

$$A \equiv \beta^{-1/2}\sqrt{\frac{\nu_2}{\nu_1}}\,\frac{\kappa_1}{\kappa_2}\,\frac{\partial x_1}{\partial \eta}(0)\Big/\frac{\partial \chi_2}{\partial \zeta}(0). \tag{41}$$

Although we can formally express the solution to Eqs. (34), (36) in terms of integrals, it is more practical to obtain a numerical solution. Using one parameter shooting we find the initial values listed in Table 2 and Table 3. In general, the functions $\chi_1(\eta)$ and $\chi_2(\zeta)$ decay monotonically to zero. The thermal boundary layer is thinner for larger Prandtl number P. As a reference, $P \sim 0.7$ for air and $P \sim 7$ for water. One can also include the energy of possible phase change in Eq. (39).

Discussion. Exact solutions of the Navier-Stokes equations are rare. This paper presents the exact solution of an important boundary value problem. Given the properties of the two fluids and the strength of the stagnation flow, one can determine the entire velocity and temperature field.

The relative motion of two immiscible fluids was also investigated by Lock [5]. He considered the two-dimensional case where the upper fluid is moving with constant velocity parallel to the interface. As in the present case, the lower fluid is being dragged along by interfacial shear. Lock's results, however, is a boundary layer solution and not an exact solution of the Navier-Stokes.

On the other hand, if the upper fluid is a thin layer of light, spreading material from a point source, such as the continuous spreading of split oil on water, then an exact solution is possible. One can show that the interface velocity is inversely proportional to the distance from the axis. The lower fluid then admits a closed-form solution in spherical coordinates [6].

References

[1] K. Hiemenz, *Die Grenzschicht an einem in den gleichförmingen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder*, Dinglers J. **326**, 321–410 (1911)

[2] F. Homann, *Der Einfluss grosser Zähigkeit bei der Strömung um den Zylinder und um die Kugel*, Z. angew. Math. Mech. **16**, 153–164 (1936)

[3] J. T. Stuart, *Double boundary layers in oscillatory viscous flow*, J. Fluid Mech. **24**, 673–687 (1966)

[4] L. J. Crane, *Flow past a stretching plate*, Zeit. Angew. Math. Phys. **21**, 645–647 (1970)

[5] R. C. Lock, *The velocity distribution in the laminar boundary layer between parallel streams*, Quart. J. Mech. Appl. Math. **4**, 42–63 (1951)

[6] C. Y. Wang, *Effect of spreading of material on the surface of a fluid–an exact solution*, Int. J. Nonlinear Mech. **6**, 255–262 (1971)