QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIII, NUMBER 3
OCTOBER 1985, PAGES 325-336

ASYMPTOTIC ANALYSIS OF STOKES FLOW IN A TORTUOUS VESSEL*

By

R. S. CHADWICK

National Institutes of Health

Abstract. Steady slow viscous flow is considered inside a vessel with circular cross section. The centerline curvature is specified as a function of arc length. The Stokes equations are written in orthogonal curvilinear coordinates. The primary small parameter is the slenderness ratio ε, which is the ratio of vessel radius to vessel length or wavelength. The product of centerline curvature and vessel length is assumed to be of order unity. A transverse drift appears at $O(\varepsilon^2)$ that is proportional to the rate of change of curvature. Contours of axial velocity show a primary peak shifted toward the inside wall and a secondary peak grows toward the outside wall as curvature is increased. The flux ratio or relative hydrodynamic conductance is calculated to $O(\varepsilon^4)$ and includes the effect of variable curvature. The present calculations tend to indicate that the sinusoidal mode of buckled micro-vessel could offer substantially more resistance to flow than the helical buckled mode.

I. Introduction. In another related work, Chadwick [1] has determined the Stokes flow inside a torus. The solution was represented by an eigenfunction expansion in toroidal coordinates. The resistance was calculated as a function of the ratio of tube radius to coil radius, and it was found not to be more than a few percent different than that of a straight tube. The motivation for both that work and the present study is to see if the buckled modes that sometimes occur in the microcirculation offer a significantly different resistance to flow than straight vessels. The torus was used as an approximation to a tightly wound helix. A sinusoidally shaped tortuosity also occurs at times. In this case the centerline curvature is not constant, and that leads to complicating effects not present in the torus. The analytical technique used here is to introduce curvilinear coordinates that in effect straighten the tube but complicate the component form of the Stokes equations. Scaling the equations introduces a single small parameter ε, which is both the ratio of tube radius to length and the product of a reference centerline curvature and tube radius. In

*Received May 1, 1984. The author has received valuable advice and encouragement from Leonard Schwartz, Milton Van Dyke, and Julian Cole.

general the latter is an independent small parameter $\varepsilon\kappa$, but here we keep ε of order unity. The problem is then one of internal slender body theory for the Stokes equations. Van Dyke [2] has recently considered the two dimensional version of the present problem for the Navier-Stokes equations, also allowing for variable channel width. In that problem Reynolds number also appears as a parameter since the inertial terms are included. In the present work the Reynolds number does not appear. We have decided to use the Stokes equations as a starting point rather than the Navier-Stokes equations because the application is for the microcirculation where the Reynolds number Re is substantially less than unity (10^{-3} in capillaries and 10^{-2} in small arterioles). There have been a number of order asymptotic expansions for flow in curved tubes, that are calculated, like the present work, as small deviations from Poiseuille flow.

It would be useful to summarize how some of the authors treat the three parameters, ε, $\varepsilon\kappa$, and Re. The present work uses the limiting process $\varepsilon \to 0$, with $\kappa = O(1)$ and $\mathrm{Re} = 0$. Dean [3] considered steady, fully developed flow in a tube with constant centerline curvature, based on special approximating equations. That work used the limiting process $\varepsilon\kappa\,\mathrm{Re}^2 \to 0$, with $\mathrm{Re} \to \infty$, and $\varepsilon\kappa \to 0$. The product $\varepsilon\kappa\,\mathrm{Re}^2$ is now known as the Dean number. The parameter ε plays no role in the constant curvature, fully developed case. Van Dyke [4] has extended this expansion to high order using the computer, and he reviews other related work on this problem. Wang [5] calculated the first correction term for flow in a varicose vein having sinusoidal centerline curvature, with $\varepsilon \to 0$, $\tilde{\kappa} = O(1)$ and $\mathrm{Re} = O(1)$. Murata, et al. [6] also considered variable curvature with the double expansion $\varepsilon \to 0$, $\kappa \to 0$, and $\mathrm{Re} = O(1)$.

II. Formulation of the problem. We consider Stokes flow through a tube with a circular cross section and a curved centerline. The flow is driven by a prescribed pressure drop across the tube. The shape of the centerline is given intrinsically by a specified curvature $\tilde{\kappa}$ as a function of the arc length S along the centerline in the direction of primary flow. In general the centerline could also have torsion. Here the centerline will be assumed to lie in one plane, so the torsion is zero. Polar coordinates are introduced in the cross plane to form a right handed curvilinear system (R, θ, S) shown in Fig. 1. The ray $\theta = 0$ points in the same direction as the curvature vector

$$\vec{k} = d\vec{\tau}/dS$$

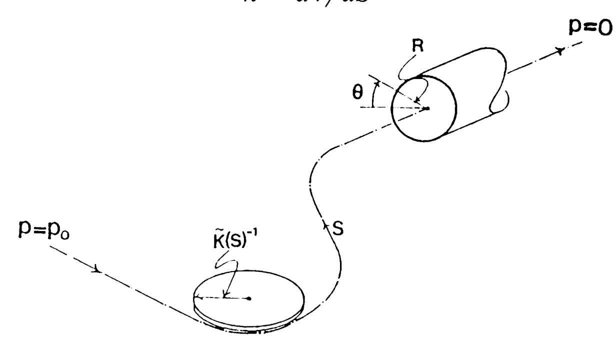


FIG. 1. Curvilinear coordinates system (R, θ, S) used in analysis.

where τ is the unit tangent vector to the centerline. In this coordinate system the square of a length element is

$$(dR)^2 + R^2(d\theta)^2 + (1 - \tilde{\kappa}(S)R\cos\theta)^2(dS)^2$$

from which we identify the metrical coefficients

$$\begin{aligned} h_1 &= 1, \\ h_2 &= R^{-1}, \\ h_3 &= (1 - \kappa(S)R\cos\theta)^{-1}. \end{aligned} \tag{1}$$

The vector form of the Stokes field equations is

$$\nabla P = -\mu\nabla \times \nabla \times \vec{V}, \tag{2}$$

$$\nabla \cdot \vec{V} = 0, \tag{3}$$

where $\vec{V}$ is the velocity, p is pressure, and μ is viscosity. We now scale the variables in the following way:

$$\begin{aligned} &r = R/a; \qquad \xi = S/L; \qquad \kappa = L\tilde{\kappa}; \\ &p = P/P_0; \qquad \vec{q} = \vec{V}/\left(P_0 a^2/L\mu\right) \end{aligned} \tag{4}$$

where a is the tube radius, L is the tube length or characteristic wavelength, and P_0 is the driving pressure (measured relative to the exit pressure taken to be zero). We now work with the dimensionless coordinates (r, θ, ξ) and dimensionless pressure p and velocity $\vec{q}$, which has components (u, v, w) in radial, theta, and axial directions. The divergence, gradient, and curl can be written in the (r, θ, ξ) coordinates using the method outlined by Love [7]. The dimensionless form of Eqs. (2), (3) then become

$$(rh^{-1}u)_r + (h^{-1}v)_\theta + \varepsilon r w_\xi = 0, \tag{5}$$

$$-rh^{-1}p_r = \varepsilon\left\{(rh)^{-1}\left[(rv)_r - u_\theta\right]\right\}_\theta + \varepsilon^2 r\left\{h(h^{-1}w)_r\right\}_\xi - \varepsilon^3 r\left\{hu_\xi\right\}_\xi, \tag{6}$$

$$-(rh)^{-1}p_\theta = -\varepsilon\left\{(rh)^{-1}\left[(rv)_r - u_\theta\right]\right\}_r + \varepsilon^2 r^{-1}\left\{h(h^{-1}w)_\theta\right\}_\xi - \varepsilon^3 r^{-1}\left\{hv_\xi\right\}_\xi, \tag{7}$$

$$\left\{rh(h^{-1}w)_r\right\}_r + r^{-1}\left\{h(h^{-1}w)_\theta\right\}_\theta - hrp_\xi = \varepsilon\left\{rhu_\xi\right\}_r + \varepsilon r^{-1}\left\{h(rv)_\xi\right\}_\theta. \tag{8}$$

We now use the notation

$$h_3 = h = (1 - \varepsilon\kappa r\cos\theta)^{-1}$$

with subscripts denoting partial differentiation. The parameter $\varepsilon = a/L$ is the basic small parameter in the problem. At this stage we also assume that $\kappa = O(1)$, i.e. the characteristic radius of curvature is like L in physical units. To complete the formulation, we impose boundary conditions on pressure,

$$p(r, \theta, 0) = 1; \qquad p(r, \theta, 1) = 0 \tag{9}$$

and conditions on velocity at the tube wall.

$$u(1, \theta, \xi) = v(1, \theta, \xi) = w(1, \theta, \xi) = 0. \tag{10}$$

III. Asymptotic solution. We consider the system of equations (5)–(10) in the narrow tube limit of $\varepsilon \to 0$. The following expansions of the dependent variables are assumed:

$$
\begin{aligned}
u &= \varepsilon u_1(r, \theta, \xi) + \varepsilon^2 u_2 + \dots, \\
v &= \varepsilon v_1(r, \theta, \xi) + \varepsilon^2 v_1 + \dots, \\
w &= w_0(r) + \varepsilon w_1(r, \theta, \xi) + \varepsilon^2 w_2 + \dots, \\
p &= p_1(\xi) + \varepsilon p_1(r, \theta, \xi) + \varepsilon^2 p_2 + \dots.
\end{aligned}
\tag{11}
$$

The numerical subscripts should not be confused with partial differentiation. The dominant flow is fully developed and unidirectional, with flow in the cross plane (the secondary flow) being of higher order. We use the notation

$$
\begin{aligned}
h^{-1} &= 1 - \varepsilon f, \\
h &= 1 + \varepsilon f + \varepsilon^2 f^2 + \cdots
\end{aligned}
$$

to shorten the writing somewhat. Substitution of the expansions (11) into the system (5)–(10) and taking the limit as $\varepsilon \to 0$ gives a hierarchy of equations to be solved in turn. The $O(1)$ system corresponds to that of Poiseuille flow

$$p_{0r} = p_{0\theta} = 0, \tag{12a}$$

$$r^{-1}\{rw_{0r}\}_{,r} = dp_0/d\xi, \tag{12b}$$

$$p_0(0) = 1; \qquad p_0(1) = 0, \tag{12c}$$

with the solution

$$w_0 = \tfrac{1}{4}(1 - r)^2, \tag{13a}$$

$$p_0 = 1 - \xi. \tag{13b}$$

The $O(\varepsilon)$ radial and tangential momentum equations again give

$$p_{1\theta} = p_{1r} = 0. \tag{14a}$$

The $O(\varepsilon)$ longitudinal momentum equation can then be written

$$
\begin{aligned}
\nabla_T^2 w_1 &= dp_1/d\xi - f + r^{-1}(fw_0)_{,r} - r^{-2}(fw_0) \\
&= dp_1/d\xi - \tfrac{3}{2} r\kappa \cos\theta
\end{aligned}
\tag{14b}
$$

where ∇_T^2 denotes the transverse Laplacian operator

$$r^{-1}\{r(\ \ )_{,r}\}_{,r} + r^{-2}(\ \ )_{\theta\theta}.$$

We proceed now to show that $p_1 = 0$. The $O(\varepsilon^2)$ continuity equation is

$$\nabla_T \cdot \vec{q}_{T2} = \nabla_T \cdot (f\vec{q}_{T1}) - w_{1\xi} \tag{15}$$

where $\nabla_T \cdot \vec{q}_T$ is transverse divergence $r^{-1}\{(ru)_{,r} + v_\theta\}$. Integrating Eq. (15) over the cross section and applying the boundary conditions (10) then shows the $O(\varepsilon)$ contribution to the flux is independent of ξ.

$$\frac{\partial}{\partial \xi} \iint_A w_1 \, dA = 0. \tag{16}$$

Apply Green's theorem in the form

$$\iint_A \left(w_0 \nabla_T^2 w_1 - w_1 \nabla_T^2 w_0 \right) dA = \int_0^{2\pi} \left(w_0 w_{1r} - w_1 w_{\theta r} \right) d\theta.$$

The right-hand side vanishes by the no slip condition. Using Eqs. (13a) and (14b) reduces the left-hand side to

$$\frac{\pi}{8} \frac{dp_1}{d\xi} + \iint_A w_1 \, dA = 0. \tag{17}$$

Differentiation of Eq. (17) with respect to ξ and using Eq. (16) shows that $dp_1/d\xi$ is a constant. To satisfy the boundary conditions $p_1(0) = p_1(1) = 0$ then requires that

$$p_1 = 0. \tag{18}$$

We can expect that there is no $O(\varepsilon)$ change in the resistance of the curved tube since there is no $O(\varepsilon)$ longitudinal pressure gradient. This is in fact the case since the solution of Eq. (14b) that satisfies the no slip condition is

$$w_1(r, \theta, \xi) = \tfrac{3}{16} \kappa(\xi) r (1 - r^2) \cos\theta \tag{19}$$

and by symmetry no net flux is delivered. We can also show that to within $O(\varepsilon)$ there is no secondary flow.

$$u_1 = v_1 = 0. \tag{20}$$

To see this, we first note that the $O(\varepsilon)$ continuity equation is

$$\nabla_T \cdot \vec{q}_{T1} = 0 \tag{21}$$

which can be satisfied by introducing the streamfunction Ψ_1 such that

$$u_1 = r^{-1} \Psi_{1\theta}; \qquad v_1 = -\Psi_{1r}. \tag{22}$$

The $O(\varepsilon^2)$ transverse momentum equations can be written

$$p_{2r} = -r^{-1} \Omega_{1\theta}, \tag{23a}$$

$$p_{2\theta} = r \Omega_{1r}, \tag{23b}$$

$$\Omega_1 = r^{-1} \{ (r v_1)_r - u_{1\theta} \}. \tag{23c}$$

Elimination f the vorticity Ω_1 by cross differentiation shows that p_2 is tranversely harmonic

$$\nabla_T^2 p_2 = 0. \tag{24}$$

The solution of Eq. (24) that is single valued and regular at the origin is

$$p_2(r, \theta, \xi) = A(\xi) + \sum_{n=1}^{\infty} r^n \{ B_n(\xi) \cos n\theta + C_n(\xi) \sin n\theta \} \tag{25}$$

with $A(\xi)$, $B_n(\xi)$, $C_n(\xi)$ to be determined. Substitution of Eq. (22) into Eq. (23c) leads to

$$\nabla_T^2 \Psi_1 = -\Omega_1. \tag{26}$$

Ω_1 can be determined by integration of Eq.'s (23a), (23b) using Eq. 25. The single valued solution of Eq. (26) that is regular at the origin is

$$\psi_1(r,\theta,\xi) = \frac{1}{4}r^2B(\xi) + \sum_{n=1}^{\infty} \frac{r^{n+2}}{4(n+1)}\{B_n(\xi)\sin n\theta - C_n(\xi)\cos n\theta\} + H(r,\theta,\xi) \tag{27}$$

where H is another transversely harmonic function

$$H(r,\theta,\xi) = \alpha(\xi) + \sum_{n=1}^{\infty} r^n\{\beta_n(\xi)\cos n\theta + \gamma_n(\xi)\sin n\theta\}. \tag{28}$$

Ψ_1 is subject to the boundary conditions

$$\Psi_{1\theta}(1,\theta,\xi) = \Psi_{1r}(1,\theta,\xi) = 0 \tag{29}$$

which show after substitution that

$$B(\xi) = B_n(\xi) = C_n(\xi) = \beta_n(\xi) = \gamma_n(\xi) = 0. \tag{30}$$

Equation (20) then follows from Eq. (22). We also note that

$$p_2(r,\theta,\xi) = A(\xi). \tag{31}$$

$A(\xi)$ and $w_2(r,\theta,\xi)$ can be determined in the following manner. The $O(\varepsilon^2)$ longitudinal momentum equation is

$$\begin{aligned}\nabla_T^2 w_2 &= p_{2\xi} - f^2 + r^{-1}(f^2 w_0)_r + r^{-2}\{f(fw_0)_\theta\}_\theta \\ &\quad + r^{-1}(fw_1)_r + r^{-2}\{w_1 f_\theta\}_\theta \\ &= dA/d\xi + \frac{7}{16}\kappa^2(\xi) - \frac{11}{8}r^2\kappa^2(\xi) - \frac{15}{16}r^2\kappa^2(\xi)\cos 2\theta.\end{aligned} \tag{32}$$

The $O(\varepsilon^3)$ continuity equation is

$$\nabla_T \cdot \vec{q}_{T3} = \nabla_T \cdot (f\vec{q}_{T2}) - w_{2\xi}. \tag{33}$$

Integration over the cross section again shows

$$\frac{\partial}{\partial\xi}\iint_A w_2\, dA = 0. \tag{34}$$

Applying Green's theorem as before (but with w_2 replacing w_1) we obtain

$$\iint_A w_2\, DA = -\iint_A w_0\nabla_T^2 w_2\, dA = \frac{\pi}{8}\left\{\frac{1}{48}\kappa^2(\xi) - dA/d\xi\right\}. \tag{35}$$

From Eqs. (34) and (35) we obtain the differential equation for $A(\xi)$

$$\frac{d}{d\xi}\left\{\frac{dA}{d\xi} - \frac{1}{48}\kappa^2(\xi)\right\} = 0. \tag{36}$$

The solution that satisfies both

$$p_2(r,\theta,0) = p_2(r,\theta,1) = 0 \tag{37}$$

is

$$p_2 = A(\xi) = \frac{1}{48}\left(\int_0^\xi \kappa^2\, d\xi - \xi\int_0^1 \kappa^2\, d\xi\right). \tag{38}$$

The $O(\varepsilon^2)$ contribution to the flux can now be computed from Eq. (35).

$$Q_2 = \iint_A w_2\, dA = \frac{\pi}{384}\int_0^1 \kappa^2\, d\xi. \tag{39}$$

It is quite unexpected that $Q_2 > 0$, implying that a small amount of curvature *decreases* the resistance of the tube. We define the flux ration $\tilde{Q}$ as the ratio of volume flux in the curved tube to that in a straight tube of equal length with the same driving pressure. $\tilde{Q}$ has the expansion

$$\tilde{Q} = 1 + \varepsilon \frac{1}{48}\int_0^1 \kappa^2\, d\xi + \dots. \tag{40}$$

Secondary flow. To determine the dominant secondary flow $\vec{q}_{T2} = (u_2, v_2)$ consider Eq. (15) rewritten as

$$\nabla_T \cdot \vec{q}_{T2} = \frac{-3}{16}\frac{d\kappa}{d\xi} r(1-r)^2 \cos\theta. \tag{41}$$

This indicates that the rate of change of curvature drives the secondary flow. This might have been expected since it is known (Chadwick, [1]) that Stokes flow in a torus (constant curvature) does not have a secondary component. The $O(\varepsilon^2)$ radial and tangential momentum equations are

$$-p_{3r} = r^{-1}\Omega_{2\theta} + (w_{1r} - w_0 f_r)_\xi, \tag{42a}$$

$$-p_{3\theta} = -r\Omega_{2r} + (w_{1\theta} - w_0 f_\theta)_\xi. \tag{42b}$$

p_3 can be eliminated by cross differentiation leaving a forced vorticity equation

$$\nabla_T^2 \Omega_2 = \frac{1}{2}\frac{d\kappa}{d\xi} r \sin\theta. \tag{43}$$

Equations (41) and (43) together with the condition of vanishing flow at the wall constitute a well defined problem for the determination of $\vec{q}_{T2}$. The equations can be uncoupled by using the decomposition

$$\vec{q}_{T2} = \nabla_T \Phi + \nabla_T \Psi \times \vec{e}_\xi \tag{44}$$

where $\vec{e}_\xi$ is a unit vector in the axial direction. We shall also use the gauge condition

$$\nabla_T \cdot (\Psi \vec{e}_\xi) = 0 \tag{45}$$

Then Eqs. (41) and (43) can be put into the uncoupled form

$$\nabla_T^2 \Phi = -\frac{3}{16}\frac{d\kappa}{d\xi} r(1-r^2)\cos\theta, \tag{46a}$$

$$\nabla_T^4 \Psi = \frac{1}{2}\frac{d\kappa}{d\xi} r \sin\theta. \tag{46b}$$

The coupling is now through the boundary conditions

$$u_2 = \Phi_r + r^{-1}\Psi_\theta = 0 \quad \text{on } r = 1, \tag{46c}$$

$$v_2 = r^{-1}\Phi_\theta - \Psi_r = 0 \quad \text{on } r = 1. \tag{46d}$$

The secondary flow system Eq. 46(a)–(d) has the solution

$$u_2 = \frac{1}{24}\frac{d\kappa}{d\xi}(r^2 - 1)^2 \cos\theta, \tag{47a}$$

$$v_2 = \frac{-1}{48}\frac{d\kappa}{d\xi}(r^2 - 1)(r^2 - 2)\sin\theta. \tag{47b}$$

Integral curves representing the projection of pathlines onto the transverse plane can be obtained by integration of

$$dr/u_2 = r d\theta/v_2$$

to give

$$r\sin\theta = C(1 - r^2)^{1/4} \tag{48}$$

where C is a constant of integration that determines a particular curve in the family. To complete the solution to $O(\varepsilon^2)$ we give the solution of Eq. (32) that satisfies the no slip condition

$$\begin{aligned} w_2(r,\theta,\xi) = &-\frac{1}{4}\left(\frac{dp_2}{d\xi} + \frac{7}{16}\kappa^2\right)(1 - r^2) \\ &+ \frac{11}{128}\kappa^2(1 - r^4) + \frac{5}{64}\kappa^2 r^2(1 - r^2)\cos 2\theta. \end{aligned} \tag{49}$$

Some higher order results. It would be desirable to calculate higher order terms in the expansion of the flux ratio in Eq. (40), since our main interest is in the resistance of the curved vessel. Unfortunately the algebraic labor increases rapidly. Without too much difficulty it is possible to show that the $O(\varepsilon^3)$ contribution to the flux ratio is zero. The $O(\varepsilon^3)$ longitudinal momentum equation is

$$\begin{aligned} \nabla_T^2 w_3 = &\ r^{-1}(f w_2 + f^2 w_1 + f^3 w_0)_{,r} + r^{-2}(f_\theta w_2 + f_\theta f w_1 + f_\theta f^2 w_0)_{\theta} \\ &+ p_{3\xi} + f p_{2\xi} - f^3 + w_{1\xi\xi}. \end{aligned} \tag{50}$$

Application of Green's theorem again gives the relation

$$Q_3 = \iint_A w_3\, dA = -\iint_A w_0 \nabla_T^2 w_3\, dA. \tag{51}$$

The only term that contributes to the integral is the $p_{3\xi}$ term on the right-hand side of Eq. (50). All the other terms vanish by their angular symmetry. p_3 can be determined by integration of the $O(\varepsilon^3)$ transverse momentum equations (42a, b) with the result

$$p_3(r,\theta,\xi) = \frac{1}{24}\frac{d\kappa}{d\xi}\cos\theta(3r^3 - r) + D_3(\xi). \tag{52}$$

The $O(\varepsilon^4)$ continuity equation again shows that Q_3 is independent of ξ. Substitution of Eq. (52) into (51) thus shows that $dD_3/d\xi$ must also be independent of ξ. The boundary conditions

$$p_3(r,\theta,0) = p_3(r,\theta,1) = 0 \tag{53}$$

can be satisfied with

$$D_3 = 0 \tag{54}$$

in which case

$$Q_3 = 0 \tag{54}$$

provided that

$$\frac{d\kappa}{d\xi}(0) = \frac{d\kappa}{d\xi}(1) = 0. \tag{56}$$

If the geometrical conditions of Eq. (56) are not satisfied, then the boundary conditions cannot be satisfied. This indicates that the problem becomes singular at this stage of approximation. The situation can be remedied by introducing edge layer expansions near $\xi = 0$ and 1 that satisfy Eq. (53) and asymptotically match to Eq. (52). Here we will assume Eq. (56) holds.

The $O(\varepsilon^4)$ term for the flux ratio is nonzero. At this order we have as before

$$Q_4 = \iint_A w_4\, dA = -\iint_A w_0 \nabla_T^2 w_4\, dA \tag{57}$$

where

$$\begin{aligned}\nabla_T^2 w_4 = {} & r^{-1}\{ f w_3 + f^2 w_2 + f^3 w_1 + f^4 w_0 \}_r \\ & + r^{-2}\{ f_\theta w_3 + f f_\theta w_2 + f^2 f_\theta w_1 + f^3 f_\theta w_0 \}_\theta \\ & + 2r^{-1}(r f u_2)_{r\xi} + w r^{-1}(f v_2)_{\theta\xi} - w_{2\xi\xi} \\ & - f^4 + f^2 p_{2\xi} + f p_{3\xi} + p_{4\xi}\end{aligned} \tag{58}$$

Eq. (58) is the $O(\varepsilon^4)$ longitudinal momentum equation. The right-hand side of this equation is known except for terms involving p_4 and w_3. The latter can be found by straightforward solution of Eq. (50). Here we shall only note that w_3 containsn terms with the angular dependence of $1, \cos\theta, \cos 3\theta$. Only the $\cos\theta$ terms contribute to the flux integral in Eq. (53). The $O(\varepsilon^4)$ transverse momentum equations are

$$-p_{4r} = f p_{3r} + r^{-1}\Omega_{3\theta} - r^{-1} f \Omega_{2\theta} + w_{2r\xi}, \tag{59}$$

$$-p_{4\theta} = f p_{3\theta} - r\Omega_{3r} + r f \Omega_{2r} + w_{2\theta\xi}. \tag{60}$$

Elimination of Ω_3 by cross differentiation gives the equation for pressure

$$\begin{aligned}\nabla_T^2 p_4 = {} & -r^{-1}(r f p_{3r})_r - r^{-2}(f p_{3\theta})_\theta - \nabla_T^2 w_{2\xi} \\ & + r^{-1}\{(f\Omega_{2\theta})_r - (f\Omega_{2r})_\theta\} \\ = {} & \frac{d}{d\xi}\kappa^2\left(-\frac{37}{96} + \frac{15}{16}r^2 + \frac{21}{32}r^2\cos 2\theta\right).\end{aligned} \tag{61}$$

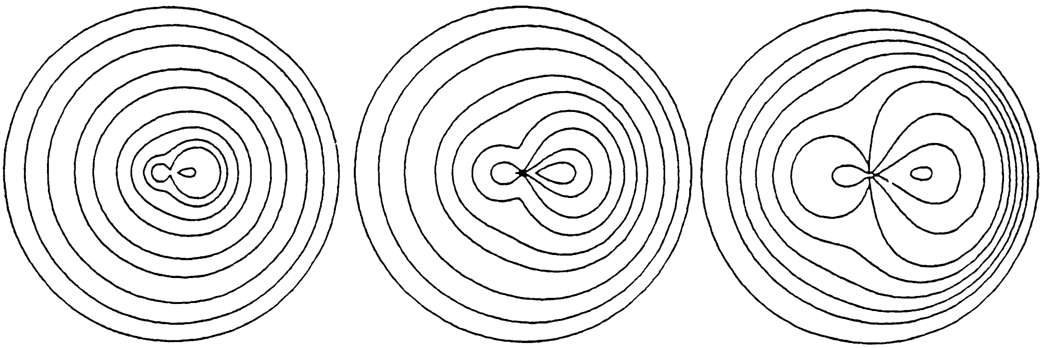
The particular solution corresponding to the first two terms of the right-hand side of Eq. (61) and an unknown function of ξ from the homogeneous solution contribute to the flux integral. This unknown function can be found by requiring that Q_4 be independent of ξ. Finally, after a considerable amount of algebra we obtain the extension of Eq. (40).

$$\tilde{Q} = 1 + \varepsilon^2 \frac{1}{48} \int_0^1 \kappa^2 \, d\xi - \varepsilon^4 \left\{ \frac{121}{9216} \int_0^1 \kappa^4 \, d\xi - \frac{1}{2304} \left(\int_0^1 \kappa^2 d\xi \right)^2 + \frac{1}{768} \int_0^1 \left(\frac{d\kappa}{d\xi} \right)^2 d\xi \right\} + \cdots \qquad (62)$$

IV. Discussion of results. The main effects of curvature on the flow field will now be described. Figure 2 shows contour plots of the axial velocity field. The computation uses the first three terms of the asymptotic expansion for w. In this figure the flow is toward the reader, so that the flow is turning to the left. The location of maximal axial velocity is displaced toward the inside wall by the influence of curvature. The direction of this shift is opposite to that in flows with large Dean number as in Collins and Dennis [8], and consistent with other reported shifts for small perturbations from Poiseuille flow as in Dean [3], Murata, et al. [6] and Wang [5]. The appearance of a secondary peak on the outer side for increasing curvature is quite unexpected and has not previously been reported.

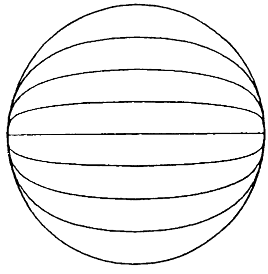
In the present theory, secondary flow is generated by the rate of change of curvature and not by the centripetal acceleration terms that are neglected in Stokes flow. Here the secondary flow is weaker in the sense that it appears at $O(\varepsilon^2)$ and not to the first order in Dean number for small but finite Dean number with constant curvature, Dean [3]. Figure 3a shows the projection of pathlines onto the transverse plane as computed from Eq. (48). The strength of the transverse drift is zero at the crests and troughs of a sinusoid, and maximal at the nodes. The direction of the drift is sketched in Fig. 3b for a centerline having the shape of a sinusoid. Though the magnitude of the transverse drift is quite small, typically a few percent of the axial velocity for the types of buckled vessels seen in the microcirculation, the transit times of cellular components could be significantly altered.

The ratio of volume flux carried by the tortuous vessels to that of a straight one having the same centerline length is the relative conductance, and is given by Eq. (62). For constant curvature Eq. (62) reduces to that given by Chadwick [1] based on an expansion

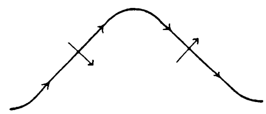


(a) $\varepsilon\kappa = 1/4$. (b) $\varepsilon\kappa = 1/2$. (c) $\varepsilon\kappa = 3/4$.

FIG. 2. Contours of axial velocity.



(a) projection of pathlines onto cross sectional plane.



(b) direction of drift.

FIG. 3. Transverse drift.

of the exact Stokes flow solution obtained in toroidal coordinates. Eq. (62) extends that relation to include the effect of curvature change. For small enough curvature the sign of the $O(\varepsilon^2)$ term of Eq. (58) indicates that curvature tends to *reduce* the resistence, although the magnitude of this change is hardly significant physiologically. The tendencey for reduced resistance was also reported by Murata et al., but their coefficient is smaller by a factor of two, evidently due to a typographical error. They reasoned this effect can be explained in terms of the shift in axial velocity towards the inner wall (cf. Fig. 2a), resulting in a decreased flow path for the mean axial flow. They then argued that this could have been anticipated by invoking minimum energy dissipation for flow at low Reynolds number. We would like to point out that this theorem does not imply a decreased mean flow path length due to curvature even for flows with negligible inertia. The corresponding problem in two dimensions, Van Dyke [2], provides a counter example at $O(\varepsilon^2)$, as does the present problem when carried to $O(\varepsilon^4)$, implying a tendency for increased resistance or increased mean flow path length at larger curvature. The direction of this tendency is shown by the increasing prominence of the secondary peak shown in the axial velocity contours in Fig. 2. An important effect of variable curvature is that $\varepsilon\kappa$ can exceed unity for physiologically relevant geometries. The domain of validity of Eq. 62 should be improved by recasting the series into its reciprocal form, which as the advantage of keeping the relative conductance positive for large values of $\varepsilon\kappa$. For the case $\kappa = \sqrt{2}\,\kappa_0 \cos 2\pi\xi$, having mean square curvature κ_0^2, the reciprocal form of Eq. (62) predicts a decrease in relative conductance of 9.0%, using the physiologically relevant values $\varepsilon\kappa_0 = 5/3$, and $\kappa_0 = 7$. While this case requires the centerline to have a slight torsion to accomodate the transverse dimension, this result suggests that a model experiment would be worthwhile to verify what appears to be a physiologically significant increase in resistance for the sinusoidal mode of buckled microvessel.

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