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ON THE MECHANICAL INADMISSIBILITY AND THERMODYNAMIC ADMISSIBILITY ACCORDING TO THE CLAUSIUS-DUHEM INEQUALITY OF THE UNBOUNDED SHEAR FLOW OF A SECOND-ORDER FLUID WITH NEGATIVE STRESS POWER*

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Recently, Rajagopal [1] has considered an unsteady shearing flow of the incompressible second-order fluid with the constitutive equation for the extra stress **S** given by

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{1}$$

In (1), μ, α_1, α_2 are constants, with $\mu \geqslant 0$; $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors [2].

Rajagopal [1] showed that the time-dependent plane flow

$$\dot{x} = u(y,t), \qquad \dot{y} = 0, \quad \text{in } (x,y,t) \in (-\infty,\infty)\times(0,\infty)\times(0,\infty), \tag{2}$$

satisfied the equations of motion, including inertia but excluding body forces, for the second-order fluid described by (1) above when

$$u(y,t) = Ue^{-my}\cos(wt - ny). \tag{3}$$

In (2), U (> 0), w (> 0), m and n are constants; moreover, if ρ is the density of the fluid, then

$$2m^2 = \rho w\left\{\left[\mu^2 + \alpha_1^2 w^2\right]^{1/2} + \alpha_1 w\right\}\left[\mu^2 + \alpha_1^2 w^2\right]^{-1}, \tag{4}$$

$$2n^2 = \rho w\left\{\left[\mu^2 + \alpha_1^2 w^2\right]^{1/2} - \alpha_1 w\right\}\left[\mu^2 + \alpha_1^2 w^2\right]^{-1}. \tag{5}$$

The crucial observation made by Rajagopal [1] lies in the fact that when u, m, and n are determined by (3)–(5), it is possible to pick at least one pair $(y,t) \in R^+\times R^+$ so that the stress power $\operatorname{tr}(\mathbf{SA}_1)$ obeys the *strict inequality*: $\operatorname{tr}(\mathbf{SA}_1) < 0$, *irrespective of the sign of* α_1. The constant α_2 does not determine the velocity field, nor does it affect the stress power; indeed, the constant α_2 cannot affect the stress power in any two-dimensional flow—see (7) below. The contribution made by Rajagopal [1] is nevertheless interesting for it is the

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first known example of a dynamically possible motion in a material with dissipation when the stress power is negative. However, as Rivlin has pointed out [3], the power expended over a cycle in the flow (3) is

$$(\pi\mu/w)U^2e^{-2my}(n^2 + m^2), \tag{6}$$

and this is positive. Therefore, while Rajagopal's example is important in that one can now say that the existence of a flow with a negative stress power in a dissipative material is a demonstrable fact, rather than a conjecture [4, 5], the positivity of the cyclic power expended in the flow makes this example less dramatic than it would be otherwise. Also, the flow (3) leads to the stress power being negative only at isolated points $(y, t) \in R^+ \times R^+$.

Hence, the first task of this note is to exhibit a steady flow in the plane wherein $\mathrm{tr}(\mathbf{SA}_1) < 0$ for all $y \in (0, \infty)$, when the fluid is described by the constitutive equation (1), and the constant $\alpha_1 > 0$. The sign of the other constant α_2 is immaterial, for in a plane flow of any incompressible material the Cayley–Hamilton theorem says that [6, eq. (41.19)]

$$\mathbf{A}_1^2 = -(\det \mathbf{A}_1)\mathbf{1}, \tag{7}$$

since $\mathrm{tr}\,\mathbf{A}_1 = 0$, and thus the stress power contribution of the $\mathbf{A}_1^2$ term is zero. Now, because the stress power in the example adduced below is negative everywhere in the flow domain, the example may be used as a convincing proof of the claim concerning the existence of flows with negative stress power in dissipative media.

Leaving these preliminaries aside, let us now consider the unbounded shear flow with suction or injection:

$$\dot{x} = u(y), \qquad \dot{y} = V, \tag{8}$$

in $(x, y) \in (-\infty, \infty) \times (0, \infty)$, with $V(\neq 0)$ being a constant, and

$$u(y) = (1 - e^{Dy})/D, \tag{9}$$

where D is the constant given by

$$2\alpha_1 VD = -\mu - \left(\mu^2 + 4\rho\alpha_1 V^2\right)^{1/2}. \tag{10}$$

Note that if $V > 0$, (10) tells us that $D < 0$ and hence $u(y) \to 1/D$ as $y \to \infty$. If $V < 0$, then (10) implies that $D > 0$ and $u(y)$ becomes unbounded as $y \to \infty$. In both cases, $u(0) = 0$, of course.

It is easily verified that the flow (8)–(10) is dynamically possible in the fluid described by (1), when inertia is included, the pressure field obeys $p = p(y)$ and the body forces are omitted. Now, for the flow (8)–(10),

$$\mathrm{tr}(\mathbf{SA}_1) = 2u'[\mu u' + \alpha_1 V u''] = 2e^{2Dy}[\mu + \alpha_1 VD]. \tag{11}$$

Since (10) tells us that

$$\mu + \alpha_1 VD < 0 \tag{12}$$

because $\alpha_1 > 0$, it follows that $\mathrm{tr}(\mathbf{SA}_1) < 0$ for all $y \in (0, \infty)$.

From Rajagopal's example [1] and the one exhibited here, it is clear that the second-order fluid described by (1) above can support "mechanically inadmissible flows," i.e., those motions where the stress power is strictly negative, at least at one point in the domain of the flow; moreover, the second-order fluid, described by (1) above, sustains such motions, whatever the sign of the constant α_1. We assume $\alpha_1 > 0$ in this paper.

Hence from the point of view of a mechanical theory, we may have to conclude that the flow (9) is not possible in the real world. In order to resolve this, we appeal to "rational thermodynamics" and investigate whether the *isothermal* flow (9) satisfies the conservation of mass, the balance of linear momentum and the balance of energy equations as well as the second law of thermodynamics. If the isothermal flow satisfies all the four laws, then it may be accepted that the flow can occur in the "real world," despite the contrary outcome from a purely mechanical theory.

Now, it is not difficult to show that the isothermal flow (9), with an absolute temperature $\theta = $ const., meets the conservation of mass equation and the balance of linear momentum principle when the coefficients μ, α_1, and α_2 are all functions of θ. Again, the pressure is a function of y along with the fixed θ; the stress power as given by (11) is also negative.

Turning to the balance of energy equation which has the form

$$\rho\dot{\varepsilon} = \frac{1}{2}\operatorname{tr}(\mathbf{S}\mathbf{A}_1) - \operatorname{div}\mathbf{q} + \rho r, \tag{13}$$

where ε is the internal energy, q is the heat flux and r is the radiant heat supply, it is clear that in an isothermal process, $\mathbf{q} \equiv \mathbf{0}$. Hence in any isothermal process, irrespective of the sign of the stress power, it is generally the rule that $r \neq 0$; indeed, $r \neq 0$ even when the material is an ideal gas, as any textbook on classical thermodynamics shows. Since the fluid described by (1) is dissipative, we should expect that the radiant heating $r \neq 0$, and, in fact, we find that in order to satisfy the equation (13), with the stress power given by (11), one needs

$$\rho r(y,\theta) = -\mu(\theta)e^{2Dy}, \tag{14}$$

when the constitutive equation for the internal energy function is assumed to be [5, Eq. (6.1)]

$$\varepsilon = \varepsilon(\theta,\mathbf{A}_1) = \bar{\varepsilon}(\theta) + \frac{\bar{\alpha}_1(\theta)}{4\rho}\operatorname{tr}\mathbf{A}_1^2. \tag{15}$$

We see, therefore, that the isothermal flow (9) meets the three balance laws, when the constitutive equations for the stress and the internal energy are given by (1) and (15), respectively, and the radiant heat supply obeys (14). We note that (14) means that energy is being radiated away from the fluid, as it undergoes the steady flow (9). In order for the isothermal flow (9) to be fully compatible with thermodynamics, i.e., to satisfy as well the second law in the form of the Clausius–Duhem inequality, it is necessary and sufficient that [5, Thm. 6]

$$\mu = \bar{\mu}(\theta) \geqslant 0, \tag{16}$$

$$\alpha_i = \bar{\alpha}_i(\theta),\ i = 1,2;\ \alpha_1 + \alpha_2 = 0, \tag{17}$$

that the free energy function ψ has the form

$$\psi = \bar{\psi}(\theta) + \frac{\bar{\alpha}_1(\theta)}{4\rho} \operatorname{tr} \mathbf{A}_1^2, \tag{18}$$

and the heat flux vector function $\tilde{\mathbf{q}}$ and the viscosity function $\bar{\mu}(\theta)$ obey

$$\frac{\tilde{\mathbf{q}}(\theta, \mathbf{g}, \mathbf{A}_1, \mathbf{A}_2) \cdot \mathbf{g}}{\theta} \leqslant \bar{\mu}(\theta) \operatorname{tr} \mathbf{A}_1^2, \tag{19}$$

where $\mathbf{g}$ is the temperature gradient. We refer the reader to [5] for conditions on $\tilde{\mathbf{q}}$; here these restrictions are not crucial because $\mathbf{g} \equiv \mathbf{0}$.

We shall not only assume that (16)–(19) hold, but demand that

$$\bar{\alpha}_1(\theta) > 0, \tag{20}$$

whence the free energy has a strict minimum [5, Thm. 6, Cor. 1] in equilibrium.

Therefore, the isothermal flow (9) is fully compatible with all three balance laws and the Clausius–Duhem inequality, provided (1), (16)–(20) hold along with (14)–(15).

The velocity and temperature fields described here seem to be the first example of a thermodynamic process in which the stress power is negative everywhere, while the Clausius–Duhem inequality holds. That such a situation might arise in continuum thermodynamics was conjectured by Coleman [4] in 1962.

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