QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIII, NUMBER 4
JANUARY 1986, PAGES 509-521

EXISTENCE OF SOLUTIONS IN A POPULATION DYNAMICS PROBLEM*

By

GASTÓN E. HERNÁNDEZ[1]

Pontificia Universidad Católica de Chile, Santiago

Abstract. In this paper we show the existence of a solution for the Gurtin–MacCamy model in population dynamics with age dependence and diffusion. We also discuss the behavior of this solution.

1. Introduction. We consider the population dynamics model with age dependence and diffusion proposed by Gurtin and MacCamy in a series of papers [1–8]. In this theory $\rho(a, t, x)$ represents the population distribution of age "a" at time "t"; x is the space variable. There is a birth law

$$\rho(0, t, x) = \int_0^\infty \beta_0(a, u(x, t))\rho(a, t, x)\, da,$$

where $u(x, t) = \int_0^\infty \rho(a, t, x)\, da$ is the space population density. The initial age-space distribution $\rho(a, 0, x) = \rho_0(a, x)$ is assumed to be nonnegative, and the balance law is:

$$\rho_a(a, t, x) + \rho_t(a, t, x) + \mu(a, u(x, t))\rho(a, t, x) = -q_x(a, t, x)$$

where $q(a, t, x) = -k\rho(a, t, x)u_x(x, t)$ is the diffusion velocity that corresponds to the case in which species disperse to avoid crowding.

It is also assumed that the death modulus $\mu(a, u)$ is independent of a, and that the birth modulus is $\beta_0(a, u) = \beta(u)e^{-\alpha a}$ which models the case in which individuals are more fertile at younger ages. The case in which the birth modulus β and the death modulus μ do not depend on the actual population u, have been treated by MacCamy [8] in the case of limited environment.

* Received November 19, 1984.

[1]Supported in part by Dirección de Investigación de la Universidad Católica de Chile through DIUC Grant and by Fondo Nacional de Desarrollo Científico y Tecnológico.

We integrate the differential equation from 0 to ∞, with respect to a, assuming that $\rho(a, t, x)$ tends to 0 as a tends to ∞. Also, we multiply it by $e^{-\alpha a}$ and integrate from 0 to ∞. We arrive at

$$u_t + \mu(u)u - \beta_0(u)G = (uu_x)_x,$$
$$G_t + [\mu(u) + \alpha + \beta_0(u)]G = (Gu_x)_x,$$
$$u(x,0) = u_0(x),\ G(x,0) = G_0(x),$$

where G is the auxiliary function

$$G(x, t) = \int_0^\infty e^{-\alpha a}\rho(a, t, x)\, da$$

and

$$u_0(x) = \int_0^\infty \rho_0(a, x)\, da, \qquad G_0(x) = \int_0^\infty e^{-\alpha a}\rho_0(a, x)\, da.$$

The natural assumption for β and μ is that they are positive smooth functions, such that $\beta(u)$ is decreasing in u and $\mu(u)$ is increasing in u. However, in this work we will assume only that both functions are uniformly bounded along with their first derivatives.

This problem is of most interest when the initial datum $u_0(x)$ has compact support, so we assume $u_0(x) \geqslant 0$ for $|x| \leqslant x_1$, and $u_0(x) = 0$ for $|x| \geqslant x_1$. Also, from the definition of u_0 and G_0, we can see that if for some x, $\rho_0(a, x) = 0$ almost everywhere in a, then $u_0(x) = G_0(x) = 0$. Similarly if $\rho_0(a, x) > 0$ in a set of finite measure, then $0 < G_0(x) < u_0(x)$.

We introduce the quotient function $q(x, t)$ defined by $q(x, t)u(x, t) = G(x, t)$ and $q_0(x, t)u_0(x, t) = G_0(x, t)$. With this new function the system becomes

$$\begin{cases} (uu_x)_x + [\beta(u)q - \mu(u)]u = u_t, & (1) \\ q_t - u_x q_x = (\beta(u) - \alpha)q - \beta(u)q^2, & (2) \\ u(x,0) = u_0(x),\ q(x,0) = q_0(x), \end{cases}$$

with the assumptions $u_0(x) \geqslant 0$ and $0 < \overline{m} \leqslant q_0(x) \leqslant 1$.

It is well known that even for the porous medium equation $(uu_x)_x = u_t$, with real analytic data, we can not expect to obtain classical solutions for all times, unless the initial data is strictly positive. We introduce a suitable definition of weak solutions for the system (1) and (2), in the following way:

Assume u and q are classical solutions of the system. Multiplying (1) by q, (2) by u, and substracting, we arrive at

$$(uq)_t = \left[\frac{1}{2}(u^2)_x q\right]_x + (\beta(u) - \alpha - \mu(u))\, uq.$$

We let $\Omega_T = (-\infty, \infty) \times (0, T)$ and define a test function $\phi(x, t)$ as a continuously differentiable function in Ω_T, with compact support in Ω_T, and that equals 0 near T.

Then multiplying this equation and Eq. (1) by ϕ and integrating over Ω_T, we obtain

$$\iint_{\Omega_T}\left[\frac{1}{2}(u^2)_x\phi_x - u\phi_t\right] dx\,dt = \iint_{\Omega_T}[\beta(u)q - \mu(u)]u\phi\,dx\,dt + \int_{\mathbf{R}} u_0(x)\phi(x,0)\,dx, \tag{3}$$

$$\iint_{\Omega_T}\left[\frac{1}{2}(u^2)_x q\phi_x - uq\phi_t\right] dx\,dt = \iint_{\Omega_T}[\beta(u) - \alpha - \mu(u)]uq\phi\,dx\,d\xi + \int_{\mathbf{R}} u_0(x)q_0(x)\pi(x,0)\,dx. \tag{4}$$

Thus, we define a weak solution of problem (1) and (2) as a pair of functions u, q such that: u is continuous in Ω_T, u^2 is differentiable with respect to x in the sense of distributions, q belongs to $L^2_{\text{loc}}(\Omega_T)$ and u and q satisfy (4) for any function ϕ continuously differentiable in $\overline{\Omega}_T$, with compact support in $\overline{\Omega}_T$ that equals 0 near T.

In this work we answer in the affirmative the question of existence, proposed by Gurtin in [1] as an open problem.

The main difficulty of the proof lies in the fact that (1) is not a uniformly parabolic equation because $u_0(x) = 0$ for $|x|$ larger than x_1. Thus, first we obtain approximating solutions u_ε and q_ε, with positive initial data $u_0(x) + \varepsilon$ and $q_0(x)$. Still, it is not possible to solve the system directly even with positive smooth initial data because of the presence of the function q in (1) which, being a solution of a first-order differential equation, does not allow good a priori estimates for the function u. Thus, we consider another approximation problem by taking $q_{(n)}$, a "nice" version of q, instead of q. Once we obtain classical solutions to this last problem, we let n tend to ∞ and then let ε tend to 0.

To solve the approximating problems, we integrate the equation for q along the characteristics defined by $-u_x$, obtaining an explicit expression for the solution, together with some estimates in the supremum norm of the difference of two such solutions. Some estimates for the gradient of q are also obtained. Then, we substitute this solution in the first equation and solve for u. To this scheme we apply the Schauder Fixed Point Theorem to obtain a solution for the system.

2. Some lemmas and notations. Through this paper differentiation is indicated by subscripts, $\Omega_T = \{(x,t)/x \in \mathbf{R},\ 0 < t < T\}$, and $\overline{\Omega}_T = \{(x,t)/x \in \mathbf{R},\ 0 \leqslant x \leqslant T\}$. $C^{2,1}(\Omega_T)$ is the Banach Space consisting of functions $u(x,t)$ defined in Ω_T with continuous second derivatives in x, and continuous first derivative in t. $C^\alpha(\Omega_T)$ is the space of functions $u(x,t)$ defined in Ω, for which the α-norm,

$$|u|_\alpha = \sup_\Omega |u| + \sup_\Omega \frac{|u(x,t) - (y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

is bounded in Ω_t; $C^{2+\alpha}(\Omega_T)$ is the space of functions $u(x,t)$ for which the $2+\alpha$ norm $|u|_{2+\alpha} = |u_{xx}|_\alpha + |u_x|_\alpha + |u_t|_\alpha + |u|_\alpha$ is bounded in Ω_T. $K(a, b, c, \ldots)$ are constants that depend on a, b, c, etc. The following results will be used in the proof.

LEMMA 1. (*The Hölder property*). Let $u \in C^{2,1}(\Omega_T)$ be a solution of

$$\begin{aligned} uu_{xx} + (u_x)^2 &= h(x,t,u)(u-\varepsilon) = u_t, \\ u(x,0) &= u_0(x) + \varepsilon, \end{aligned} \tag{5}$$

such that $\varepsilon \leqslant u \leqslant M$, $|h| \leqslant M_1$, u_x, u_t are bounded and $|u_0, u_{0_x}| \leqslant M_0$. Then the α-norm of u is bounded by a constant K_1 depending only on α, M, M_0, M_1 and T, i.e.,

$$\frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}} \leqslant K_1, \quad \text{for any } x, y \in \mathbf{R}, \quad s, t \in [0, T].$$

The proof of this lemma is given in [9] and consists in considering the function

$$g(x,y,t,s) = \frac{|u(x,t) - u(y,s)|^{2/\alpha}}{|x-y|^2 + K|t-s|^{\alpha/2}}$$

at a point of maximum.

LEMMA 2. (*Bounds for the gradient of u*). Let $u \in C^{2,1}(\Omega_T)$ be a solution of

$$\begin{aligned} uu_{xx} + (u_x)^2 + (\beta(u)q - \mu(u))(u-\varepsilon) &= u_t, \\ u(x,0) &= u_0(x) + \varepsilon. \end{aligned} \tag{6}$$

Assume that $|q_x| \leqslant N_1$, $|q| \leqslant 1$, $|h(x,t,u)| = |\beta(u)q - \mu(u)| \leqslant M_1$, $\varepsilon \leqslant u \leqslant M$, and u_x, u_{xx} bounded in Ω_T. Then there exist constants K_2 and K_3 depending only on M, M_0, and M_1 such that for any c positive, $|u_x|^2 \leqslant K_2 + K_3/c + cN_1^2$ in Ω_T.

The proof of this lemma, also given in [9], is a direct application of Bershtein's technique to Eq. (6).

3. The ε-n-approximation problems. To prove existence of solutions of (1) and (2) we solve first the ε-n-approximation problems;

$$uu_{xx} + u_x^2 + \big(\beta(u)q_{(n)} - \mu(u)\big)(u-\varepsilon) = u_t, \tag{7}$$

$$\begin{aligned} q_t - u_x q_x &= (\beta(u) - \alpha)q - \beta(u)q^2, \\ u(x,0) &= u_0(x) + \varepsilon, \qquad q(x,0) = q_0(x), \end{aligned} \tag{8}$$

where $q_{(n)}$ is a "smoothed" version of q. That is to say, if k is a nonnegative C^∞ function in $\mathbf{R}^2$ such that $k(r) = 0$ for $|r| \geqslant 1$, k is symmetric in r and $\iint k(r)\,dr = 1$, we define $k_n(r) = nk(nr)$ and $q_{(n)}(r) = \iint k_n(r-s)q(r)\,ds$. We note that if $q \in L^2$, $q_{(n)}$ is a C^∞ sequence of functions that converge to q in L^2. If q is continuous, $q_{(n)}$ converge to q uniformly on compact sets.

To prove that this system has a solution we first study equation (8) in terms of existence, bounds for the gradient of q and bounds for the difference of two solutions. Then we apply the Leray-Schauder Fixed Point Theorem as presented in Friedman [10] to obtain a classical solution of (7) and (8).

The following two lemmas establish existence of solutions for Eq. (8) given the function u.

LEMMA 3. Let $v \in C^{2+\alpha}(\Omega_T)$, $\overline{m} \leqslant q_0(x) \leqslant 1$ and $|v|_{2+\alpha} \leqslant K_4$. Then the problem

$$\begin{aligned} q_t - v_x q_x &= (\beta(v) - \alpha)q - \beta(v)q^2, \\ q(x,0) &= q_0(x) \end{aligned} \tag{9}$$

has a unique solution $q(\bar{x}, \bar{t})$ in Ω_T, given by

$$q(\bar{x}, \bar{t}) = e^{\int_0^{\bar{t}}(\beta(v)-\alpha)\,ds}\left[\int_0^{\bar{t}} \beta(v) e^{-\int_0^t(\beta(v)-\alpha)\,ds}\,dt + q_0^{-1}(X(0))\right] \tag{10}$$

where $\beta(v) = \beta(v(x(t), t))$. The integration is along the characteristic curves defined by

$$\frac{\partial X}{\partial t} = -v_X(x,t), \qquad X(\bar{t}; \bar{t}, \bar{x}) = \bar{x}. \tag{11}$$

Proof. We define characteristic functions $X(t; \bar{t}, \bar{x})$ by means of (11). Since v_x is Lipschitz continuous there is always a unique local solution of this equation. Further, since v_x is also bounded in Ω_T, this local solution can be made global by extending it to the boundary of $[0, T] \times \mathbf{R}$. The usual techniques of maximum principles can be applied to show that if q, q_x are bounded in Ω_T and $\overline{m} \leqslant q_0(x) \leqslant 1$, then $\overline{m}e^{-\alpha t} \leqslant q(x,t) \leqslant 1$. In particular we expect q to be a positive function. Let

$$p(x,t) = q^{-1}(x,t)e^{\int_0^t[\beta(v(X(s),s))-\alpha]\,ds}. \tag{12}$$

If we assume that q is a solution of (9), then p satisfies

$$p_t - v_x p_x = \beta(x,t), \qquad p(x,0) = q_0^{-1}(x), \tag{13}$$

where $\beta(x,t) = \beta(v(x,t))e^{\int_0^t[\beta(v(X(t),s))-\alpha]\,ds}$.

Integrating this equation along characteristics from 0 to $\bar{t}$ we obtain

$$p(\bar{x}, \bar{t}) = p(X(0), 0) + \int_0^{\bar{t}} \beta(X(s), s)\,ds. \tag{14}$$

This, together with (12), gives expression (10) for $q(\bar{x}, \bar{t})$. The solution thus obtained is just a formal one, but it can be proved by direct differentiation and by the fact that

$$X_{\bar{t}}(t) - v_x(\bar{x}, \bar{t})X_{\bar{x}}(t) = 0 \quad \text{for every } t, \tag{15}$$

that (10) is a solution of (9). The uniqueness follows from Haar's lemma.

4. Estimates for $|p - q|$ and $|q_x|$.

LEMMA 4. (*Estimates for* $|p - q|$). Let v, $w \in C^{2+\alpha}(\Omega)$, $|v, w|_{2+\alpha} \leqslant K_4$, $|v - w|_{2+\alpha} \leqslant \delta$ and p, q solutions of (9) with v and w, respectively. That is,

$$q_t - v_x q_x = (\beta(v) - \alpha)q - \beta(v)q^2, \tag{16}$$

$$p_t - w_x p_x = (\beta(w) - \alpha)p - \beta(w)p^2, \tag{17}$$

with $q(x,0) = p(x,0) = q_0(x)$.

Then $|p - q| \leqslant K\delta$ where K_5 depends only on K_4, T, and q_0.

Proof. Let $X_1(t; \bar{t}, \bar{x})$ and $X_2(t; \bar{t}, \bar{x})$ be the characteristics defined by

$$\frac{\partial X_1}{\partial t} = -v_X(x_1, t), \qquad X_1(\bar{t}) = \bar{x},$$
$$\frac{\partial X_2}{\partial t} = -w_X(X_2, t), \qquad X_2(\bar{t}) = \bar{x}. \tag{18}$$

Let $X_3(t) = |X_1(t) - X_2(t)|$. Then subtracting both equations in (18) and using Gronwall's inequality we obtain

$$X_3(t) \leqslant T|v_x - w_x|_\infty e^{|v_x|_\infty T} \leqslant Te^{K_4 T}\delta.$$

Now, since p and q are given by the same expression (10) (with $\beta(v)$ for q, $\beta(w)$ for p, etc.), subtracting these formulae we arrive at an expression that contains multiples (depending on K_4, $|\beta|_\infty$, T, etc.) of $\beta(v) - \beta(w)$. This in turn is bounded by β', $v - w$, and $X_1 - X_2$, and is therefore bounded by a constant K times δ.

LEMMA 5. (*Bounds for the gradient of* q). Let u be as in Lemma 1 and q given by (10), with $\partial X/\partial t = u_X(X, t)$ and $X(\bar{t}) = \bar{x}$. Then $|q_{\bar{x}}| \leqslant K_5|u_x|_\infty$ and $|q_{\bar{t}}| \leqslant K_5|u_x|_\infty^2$, where $K_5 = K_5(\varepsilon, T, M_1, M_0)$.

Proof. Differentiating the characteristic equations with respect to the parameter $\bar{x}$, we obtain

$$\partial X_{\bar{x}}/\partial t = -u_{XX}(X, t)X_{\bar{x}}, \qquad X_{\bar{x}}(\bar{t}) = 1,$$

so $X_{\bar{x}} = e^{\int_{\bar{t}}^t u_{XX}(S(s),s)\,ds}$. Using the differential equation for u_{XX} and integrating, we obtain $|X_{\bar{x}}| \leqslant (M/\varepsilon)e^{T(M_1+1)}$. Then by (15), $|X_{\bar{t}}| \leqslant (M/\varepsilon)e^{T(M_1+1)}|u_x|_\infty$. Differentiating the expression for $q(\bar{x}, \bar{t})$, we obtain that $q_{\bar{x}}$ is bounded by multiples of β', u_x, $X_{\bar{x}}$, q_0^{-1}, and q_0', so that $q_{\bar{x}}$ is bounded by a constant $K_5(\varepsilon, T, M_1, M_0)$ times $|u_x|_\infty$. The same argument is valid for $|q_{\bar{t}}|$.

5. Existence of solutions of the ε-n-approximating problems.

THEOREM 1. Under the given hypotheses for u_0, q_0, β, and μ, there exists a classical solution in Ω_T of

$$uu_{xx} + u_x^2 + \big(\beta(u)q_{(n)} - \mu(u)\big)(u - \varepsilon) = u_t,$$
$$q_t - u_x q_x = (\beta(u) - \alpha)q - \beta(u)q^2,$$
$$u(x,0) = u_0(x) + \varepsilon,\ q(x,0) = q_0(x). \tag{19}$$

Proof. Let V be the convex set in $C^{2+\alpha}(\Omega_T)$ consisting of all functions w with $|w|_{2+\alpha} \leqslant K_6$ and $w \geqslant \varepsilon$. We define T: $V \to C^{2+\alpha}(\Omega_T)$ in the following way. Given $w \in W$, $|w|_{2+\alpha} \leqslant K_4$, by Lemma 3 there exists a unique solution q of

$$q_t - w_x q_x = (\beta(w) - x)q - \beta(w)q^2,$$
$$q(x,0) = q_0(x).$$

By the maximum principle, $me^{-\alpha t} \leqslant q(x, t) \leqslant 1$. Thus $q_{(n)} \in C^\infty(\Omega_T)$ and $|q_{(n)}|_\sigma \leqslant 2n$ for any $\sigma \in (0, 1)$. Now consider the problem

$$E(u)u_{xx} + u_x^2 + \big(\beta(u)q_{(n)} - \mu(u)\big)(u - \varepsilon) = u_t,$$
$$u(x,0) = u_0(x) + \varepsilon, \tag{20}$$

where $E(u)$ is a C^∞ function of u that equals $\frac{\varepsilon}{2}$ for $u \leqslant \frac{\varepsilon}{2}$; it increases up to ε in $(\frac{\varepsilon}{2}, \varepsilon)$ and equals u for $u \geqslant \varepsilon$. This problem has a unique solution $u \in C^{2+\sigma}(\Omega_T)$, $|u|_{2+\sigma} \leqslant K(\varepsilon, n)$. By the maximum principle $\varepsilon \leqslant u \leqslant M(\varepsilon, n)$; then $E(u) = u$. Actually u is the unique solution in $C^{2+\sigma}(\Omega_T)$ of Eq. (20) that satisfies these conditions. We let $u = T(w)$ and choose $K_4 \geqslant K(\varepsilon, n)$ and $\sigma > \alpha$. Then T maps V into V. Since bounded sets in $C^{2+\sigma}(\Omega_T)$ are precompact in $C^{2+\alpha}(\Omega_T)$ with $0 < \alpha < \sigma < 1$, we obtain that $T(V)$ is precompact.

Next we prove that T is continuous. We let $u = T(w)z = T(v)$. If we subtract the corresponding q and p we find that $|q - p|_\infty \leqslant K(K_4)\delta$ when $|v - w|_{2+\alpha} \leqslant \delta$. Thus $|q_{(n)} - p_{(n)}|_\alpha \leqslant K(K_4, n)\delta$. We then set $S = u - z$ and note that S satisfies the linear equation

$$uS_{xx} + (u_x + z_x)S_x + c(x,t)S - S_t = f(x,t),$$
$$S(x,0) = 0,$$

where

$$c(x,t) = \big(\beta(u)q_{(n)} - \mu(u)\big) + (z - \varepsilon)\big(p_{(n)}\beta'(\cdot) - \mu'(\cdot)\big) + z_{xx}$$

and

$$f(x,t) = \beta(u)(z - \varepsilon)(q_{(n)} - p_{(n)}).$$

We also have that $u \geqslant \varepsilon$, $u_x + z_x$ is bounded by K_4, $c(x,t)$ is bounded by a multiple of K_4 and the α-norm of $f(x,t)$ is bounded by a multiple of the α-norm of $q_{(n)} - p_{(n)}$, i.e., it is bounded by $K\delta$. Then by Theorem 5.1 in [11] we have $|S|_{2+\alpha} \leqslant K(K_4, n, \varepsilon)\delta$. This implies continuity.

Thus T has a fixed point in V, i.e., there exist functions u and q solutions of (20). Since $0 < \overline{m}e^{-\alpha T} \leqslant q \leqslant 1$, by the maximum principle we obtain that $\varepsilon \leqslant u \leqslant M_1$, where $M_1 = (M_0 + 1)e^{M_3 T}$ and $M_3 = |\beta|_\infty + |\mu|_\infty$. ($M_1$ is independent of ε and n.)

Also

$$u \in C^{2+\alpha}(\Omega_T), \qquad |u|_{2+\alpha} \leqslant K(n, \varepsilon),$$
$$q \in C^{1+\alpha}(\Omega_T), \qquad |q|_{1+\alpha} \leqslant K(n, \varepsilon),$$
$$|q|_x, |q_t| \leqslant K(|u_x|_\infty, \varepsilon).$$

Existence of solutions of the ε-approximating problems. In this section we prove existence of solutions of the system:

$$(uu_x)_x + (\beta(u)q - \mu(u))(u - \varepsilon) = u_t, \tag{21}$$

$$q_t - u_x q_x = (\beta(u) - \alpha)q - \beta(u)q^2, \tag{22}$$

$$u(x,0) = u_0(x) + \varepsilon, \qquad q(x,0) = q_0(x).$$

THEOREM 2. Under the given hypothesis on β, μ, u_0, and q_0, there exists a (weak) solution of (21) and (22) for each $\varepsilon > 0$. That is to say, there exists a continuous function u, $(u^2)_x \in L^2_{\text{loc}}(\Omega_T)$ and a function q, $q \in L^2_{\text{loc}}(\Omega_T)$, that satisfy

$$\iint_{\Omega_T} \left[\frac{1}{2}(u^2)_x \phi_x - u\phi_t\right] dx\, dt$$
$$= \iint_{\Omega_T} (\beta(u)q - \mu(u))(u - \varepsilon)\, dx\, dt + \int_{\mathbf{R}} (u_0(x) + \varepsilon)\phi(x,0)\, dx \tag{23}$$

and

$$\iint_{\Omega_T}\left[\frac{1}{2}(u^2)_x q\phi_x - uq\phi_t\right] dx\,dt = \iint_{\Omega_T}\left[(\beta(u) - \alpha - \mu(u))uq\phi\right] dx\,dt + \int_{\mathbf{R}}(u_0(x) + \varepsilon)q_0(x)\phi(x,0)\,dx \tag{24}$$

for every test function ϕ over Ω_T.

Proof. By the previous theorem, for each n, we have $u^n \in C^{2+\alpha}(\Omega_T)$, $q^n \in C^1(\Omega_T)$ such that

$$(u^n u_x^n)_x + \left[\beta^n q_{(n)}^n - \mu^n\right](u^n - \varepsilon) = u_t^n, \tag{25}$$

$$q_t^n - u_x^n q_x^n = (\beta^n - \alpha)q^n - \beta(q^n)^2,$$
$$u^n(x,0) = u_0(x) + \varepsilon, \qquad q^n(x,0) = q_0(x). \tag{26}$$

Multiplying (21) by q^n, (22) by u^n, and adding, we obtain

$$(u^n q^n)_t = \tfrac{1}{2}\left[\left((u^n)^2\right)_x q^n\right]_x + (\beta^n - \alpha - \mu^n)u^n q^n + \beta^n u^n q^n\left(q_{(n)}^n - q^n\right) - \varepsilon q^n\left(\beta^n q_{(n)}^n - \mu^n\right). \tag{27}$$

If ϕ is a test function we multiply (25) and (26) by ϕ and integrate over Ω_T to obtain

$$\iint_{\Omega_T}\left[\frac{1}{2}\left((u^n)^2\right)_x \phi_x - u^n\phi_t\right] dx\,dt = \iint_{\Omega_T}\left(\beta^n q_{(n)}^n - \mu^n\right)(u^n - \varepsilon)\,dx\,dt + \int_{\mathbf{R}}(u_0(x) + \varepsilon)\phi(x,0)\,dx, \tag{28}$$

and

$$\iint_{\Omega}\left[\frac{1}{2}\left((u^n)^2\right)_x q^n\phi_x - u^n q^n\phi_t\right] dx\,dt$$
$$= \iint_{\Omega_T}(\beta^n - \alpha - \mu^n)u^n q^n\phi\,dx\,dt + \iint_{\Omega_T}\left[\beta^n u^n q^n\left(q_{(n)}^n - q^n\right)\phi\right] dx\,dt$$
$$-\varepsilon\iint_{\Omega_T}\left[q^n\left(\beta^n q_{(n)}^n - \mu^n\right)\phi\right] dx\,dt + \int_{\mathbf{R}}(u_0(x) + \varepsilon)q_0(x)\phi(x,0)\,dx. \tag{29}$$

Since $|u^n|_\alpha \leqslant K_1$ uniformly in n, there exists a subsequence $\{u^{n_1}\}$ such that $\{u^{n_1}\}$ converges uniformly on compact sets to an α-Hölder continuous function u. Moreover $|u|_\alpha \leqslant K_1$.

By Lemma 5, $|q_x^n| \leqslant K_5(\varepsilon)|u_x^n|_\infty$, thus $N_n = \max|q_x^n| \leqslant K_5(\varepsilon)|u_x^n|_\infty$ is also finite. If we put $c = \frac{1}{2}K_5^{-2}$ in Lemma 2, we obtain

$$|u_x^n|^2 \leqslant K_2 + 2K_3K_5^2 + \left(\frac{1}{2}\frac{1}{K_5^2}\right)\left(K_5^2|u_x^n|_k^2\infty\right).$$

Taking the supremum in the left-hand side we have $|u_x^n|_\infty^2 \leqslant 2(K_2 + 2K_3K_5^2)$ so that $|q_x^n|_\infty^2 \leqslant 2K_5(K_2 + 2K_3K_5^2)$. Thus $|q^n|_\alpha$ and thus $|q_{(n)}^n|_\alpha$ can be bounded above by a constant K_{10} depending only on ε, M_1, T, and M_0. If we put $v = (u^n)^2$ we have $u^n v_{xx} + (\beta^n q_{(n)}^n - \mu^n)v - v_t = \varepsilon(\beta^n q_{(n)}^n - \mu^n)u^n$. Since $|u^n|_\alpha \leqslant K_1$, $u^n \geqslant \varepsilon$ and $|q^n|_\alpha \leqslant K_{11}$, we obtain that $|v|_{2+\alpha}$ is bounded and consequently $|u^n|_{2+\alpha}$ is bounded by a constant $K_{12}(\varepsilon)$. Then for $\alpha' < \alpha$, there exists a subsequence $\{u^{n_2}\}$ that converges to u in $C^{2+\alpha'}(\Omega_T)$. In particular $\{u_x^{n_2}\}$ converges uniformly to u_x on compact sets. Also $\{q_{(n_2)}^{n_2}\}$ converges pointwise to q and $q^{n_2}(\beta^{n_2}q_{(n_2)}^{n_2} - \mu^{n_2})$ converges pointwise to $q(\beta q - \mu)$. Therefore all the integrals converge in (28) and (29) and we have a weak solution $u = u_\varepsilon$, $q = q_\varepsilon$ of the problem (21), (22), that is to say, u_ε and q_ε satisfy

$$\iint_{\Omega_T}\left[\frac{1}{2}(u_\varepsilon)_x\phi_x - u_\varepsilon\phi_t\right]dx\,dt$$

$$= \iint_{\Omega_T}\left[(\beta(u_\varepsilon)q_\varepsilon - \mu(u_\varepsilon))(u_\varepsilon - \varepsilon)\right]\phi\,dx\,dt + \int_{\mathbf{R}}(u_0(x) + \varepsilon)\phi(x,0)\,dx \tag{30}$$

and

$$\iint_{\Omega_T}\left[\frac{1}{2}(u_\varepsilon)_x q_\varepsilon\phi_x - u_\varepsilon q_\varepsilon\phi_t\right]dx\,dt$$

$$= \iint_{\Omega_T}\left[(\beta(u_\varepsilon) - \alpha - \mu(u_\varepsilon))(u_\varepsilon - \varepsilon)q_\varepsilon\phi\right]dx\,dt + \int_{\mathbf{R}}(u_0(x) + \varepsilon)q_0(x)\phi(x,0)\,dx \tag{31}$$

for every test function ϕ over Ω_T.

Since $\alpha \in (0,1)$ is arbitrary instead of α' we can write again α. Moreover u_ε and q_ε satisfy: $u_\varepsilon \in C^{2+\alpha}(\Omega_T)$, $|u_\varepsilon|_{2+\alpha} \leqslant K_{12}(\varepsilon)$, $q_\varepsilon \in C^\alpha(\Omega_T)$, $|q_\varepsilon|_\alpha \leqslant K_{11}(\varepsilon)$. In particular u_ε is a classical solution of (6) or (21).

6. Convergence of the u_ε, q_ε solutions. The main theorem.

THEOREM 3. Under the given hypotheses on β, μ, u_0, and q_0, there exists a weak solution of problem (1) and (2), that is, there is a continuous function u, $(u^2)_x \in L^2_{\text{loc}}(\Omega_T)$, and a bounded function q, $q \in L^2_{\text{loc}}(\Omega_T)$, that satisfy (3) and (4) for every test function ϕ over Ω_T.

Proof. Again by the Hölder property, $|u_\varepsilon|_\alpha \leqslant K_1$, so we can extract a subsequence $\{u_{\varepsilon_k}\}$ that converges uniformly on compact sets to an α-Hölder continuous function u. Moreover, $|u|_\alpha \leqslant K_1$. Since $\{q_{\varepsilon_k}\}$ is uniformly bounded by 1, we can extract a subsequence converging weakly to a function q in $L^2_{\text{loc}}(\Omega_T)$. We call these sequences again $\{u_k\}$ and $\{q_k\}$ and will prove that a subsequence $\{(u_{k_n}^2)_x\}$ of $\{(u_k^2)_x\}$ converges pointwise to $(u^2)_x$, so that all the integrals in (30) and (31) will converge as $\varepsilon_{k_n} \to 0$ proving that u and q are solutions of (1) and (2).

Let Q_3 be a rectangle of length $\frac{1}{2}$ and width $\frac{1}{16}$ centered at $(x, t) = (0, 1)$, $Q_2 = 2Q_3$ and $Q_1 = 2Q_2$. Let $f_1(x, t)$, $f_2(x, t)$ be bounded in Q_1 with $|f_1, f_2| \leqslant K_{21}$. If z is a continuous weak solution in Q_1 of

$$\frac{d}{dx}(wz_x + f_1(x, t)) + f_2(x, t)z = z_t$$

with $\frac{1}{2} \leqslant w \leqslant \frac{3}{2}$ and $|z|_{L^2(Q_2)} \leqslant K_{22}$, then by Aronson–Serrin [13] (Theorems 2 and 3) there exist constants C_1, C_2 depending only on K_{21}, K_{22} such that $|z| \leqslant C_1$ and $|z|_\sigma \leqslant C_3$ in Q_3, for some $\sigma > 0$. We use these facts to prove the next lemma.

LEMMA 6 (*Regularity*). If $0 < u_k(x_0, t_0) = \delta < M$, then

$$|(u_k^2)_x(x_0, t_0)| \leqslant C_4(u_k(x_0, t_0))^{1/4}$$

and

$$|(u_k^2)_x|_\sigma = C_5(K_2, M, M_1, t_0, D_0)$$

in a neighborhood D_0 of (x_0, t_0).

Proof. We define a linear change of variables

$$x(\xi) = x_0 + b\delta^{(2+\alpha)/2}\alpha\xi,$$
$$t(s) = t_0 + b^2\delta^{2/\alpha}(s - 1),$$

where $b = \min\{(8K_1)^{-1/\alpha}, t_0, M_1^{-2}\}$, that maps Q_1 onto a neighborhood D_0 of (x_0, t_0).

Let $w(\xi, s) = \delta^{-1}u_k(x(\xi), t(s))$ for $(\xi, s) \in Q_1$. Then w satisfies: $ww_{\xi\xi} + w_\xi^2 + b^2\delta^{2/\alpha}h(x, t, u_k)(w - \varepsilon\delta^{-1}) = w_s$. We put $\bar{h} = b^2\delta^{2/\alpha}h(x, t, u_k)$ and $z = (w^2)_\xi$. Differentiating with respect to ξ we get

$$z_s = \left[wz_\xi + 2\bar{h}(w^2 - \varepsilon\delta^{-1}w)\right]_\xi.$$

By the Hölder property of u_k and by the choice of b, $|u_k(x(\xi), t(s)) - u_k(x_0, t_0)| \leqslant \delta/4$. Thus dividing by δ, we obtain $|w(\xi, s) - 1| \leqslant \frac{1}{4}$, and consequently $\frac{1}{2} \leqslant w \leqslant \frac{3}{2}$.

Since w is a solution of (7) in Q_1 we also have

$$\iint_{Q_1} \frac{1}{2}(w^2)_\xi G_\xi - wG_s \, d\xi \, ds = \iint_{Q_1} \bar{h}(w - \varepsilon\delta^{-1})G \, d\xi \, ds \tag{32}$$

for every test function G that vanishes on the (total) boundary of Q_1. Let $g(\xi, s)$ be a C^∞ function that takes values between 0 and 1, that is identically 1 in Q_2 and with derivatives g_ξ, g_s bounded by 4. Then the function $G = w^2g^2$ is an admissible function in (32). This gives

$$\iint_{Q_1} g^2((w^2)_\xi)^2 \, d\xi \, ds \leqslant C_3(K_1, M_1, t_0).$$

In particular,

$$|z|_{L^2(Q_2)} = \iint_{Q_2} ((w^2)_\xi)^2 \, d\xi \, ds \leqslant C_3.$$

Thus by the Aronson–Serrin theorems, $|z| = |(w^2)_\xi| \leqslant C_1$ and $|z|_\sigma = |(w^2)_\xi|_\sigma \leqslant C_2$ in Q_3. Hence,

$$|(u_k^2)_x(x_0, t_0)| = 2b^{-1}\delta^{(3\alpha-2)/2}\alpha|(w^2)_\xi(0,1)| \leqslant \frac{2C_1^{1/4}}{b} = C_4(u_k(x_0, t_0))^{1/4}$$

for $\alpha \geqslant 4/5$. Also we can assume that σ is smaller than $(3\alpha - 2)/2\alpha$. Then

$$\begin{aligned}\frac{|(u_k^2)_x(x_1, t_1) - (u_k^2)_x(x_2, t_2)|}{|x_1 - x_2|^\sigma + |t_1 - t_2|^{\sigma/2}} &\leqslant \frac{2b^{-1}\delta^{(3\alpha-2)/2}\alpha|(w^2)_\xi(\xi_1, s_1) - (w^2)_\xi(\xi_1, s_2)|}{b^\sigma\delta^{\sigma(2+\alpha)/2\alpha}(|\xi_1 - \xi_2|^\sigma + |s_1 - s_2|^{2/2})} \\ &\leqslant 2b^{-1-\sigma}\delta^{[(3\alpha-2)-\sigma(2+\alpha)]/2\alpha}C_2 \leqslant 2b^{-1-\sigma}C_2M^2 \\ &= C_5(K_2, M_1, M, t_0, D_0).\end{aligned}$$

In particular, C_4 and C_5 do not depend on δ.

Differentiability of u^2. Now we will prove that u^2 is differentiable with respect to x, for every $t > 0$. We divide the proof in two cases:

(i) If $u(x_0, t_0) = 0$, since u is α-Hölder continuous in x, $|u(x, t)| \leqslant K_1|x - x_0|^\alpha$ and

$$\left|\frac{u^2(x_0, t_0) - u^2(x_0, t_0)}{h}\right| \leqslant K_1h^{2\alpha-1} \to 0$$

as $h \to 0$ for $\alpha > 1/2$.

(ii) If $u(x_0, t_0) = \delta > 0$, by the uniform continuity of u and the uniform convergence of $\{u_k\}$, there is an integer k_0 and a neighborhood D_0 of (x_0, t_0) in which

$$\tfrac{3}{4}\delta \leqslant u_k(x, t) \leqslant \tfrac{5}{4}\delta \text{ for } k \geqslant k_0.$$

By the previous lemma, $|(u_k)_x|_\sigma \leqslant C_5$ in a subdomain D_{01}. Thus we can extract a subsequence $\{u_{k_n}\}$ such that $\{(u_{k_n}^2)_x\}$ converges uniformly in a compact subdomain of D_{01} containing (x_0, t_0).

Since $\{u_{k_n}^2\}$ converges uniformly to u^2 we conclude that $(u^2)_x = \lim_{k_n\to\infty}(u_{k_n}^2)_x$, for every x in that subdomain. Therefore $(u^2)_x$ exists everywhere.

Completion of the proof. Now we fix R and t_1. We prove that there exists a subsequence $\{u_{k,n}\}$ of $\{u_k\}$ such that $\{(u_{k,n}^2)_x\}$ converges pointwise to $(u^2)_x$ in D_{R,t_1}.

Let $n \geqslant 2$ and $B_n = \{(x, t) \in D_{R,t_1}/u(x, t) \geqslant 1/n\}$. For $(x_0, t_0) \in B_n$, with $u(x_0, t_0) = \delta \geqslant 1/n$ as before, there is an integer k_0 and a neighborhood D_0, in which $\frac{3}{4}\delta \leqslant u_k(x, t) \leqslant \frac{5}{4}\delta$ for $k \geqslant k_0$. Then by the regularity lemma there is a $D_{01} \leqslant D_0$ such that $|(u_k^2)_x|_\sigma \leqslant C_5$ in D_{01}. By compactity we can cover B_n with finitely many of these neighborhoods and take C_6 to be the maximum of the C_5's. Then $|(u_k^2)_x|_\sigma \leqslant C_6$ in B_n, so there exists a subsequence $\{u_{k,n}\}$ such that $\{(u_{k,n}^2)_x\}$ converges uniformly in B_n. Then, out of this sequence and by the same arguments we extract a subsequence $\{u_{k,n+1}\}$ such that $\{(u_{k,n+1}^2)_x\}$ converges uniformly in B_{n+1}. We repeat for every n and pick a diagonal subsequence $\{u_{k,k}\}$. We claim that $\{(u_{k,k}^2)_x\}$ converges pointwise to $(u^2)_x$ in D_{R,t_1}. Let $\eta > 0$ be given. If $u(x_0, t_0) = 0$, by the differentiability of u^2, $(u^2)_x(x_0, t_0) = 0$. Since $\{u_{k,k}^2\}$ converges uniformly to u^2 and $u_{k,k}(x_0, t_0) = \delta_k \geqslant \varepsilon_k > 0$, by the regularity lemma 6 and since $\delta_k \to 0$, $|(u_{k,k}^2)_x(x_0, t_0)| \leqslant C_4\delta_k^{1/4} \leqslant \eta$ for k large. If $u(x_0, t_0) > 0$ we pick a positive integer r such that $u(x_0, t_0) \geqslant 1/r$. Then $(x_0, t_0) \in B_r$ and since $\{u_{k,k}\}$ is a

subsequence of $\{u_{k,r}\}$ for $k > r$, and $\{(u_{k,r}^2)_x\}$ converges uniformly in B_r to $(u^2)_x$, we have $|(u_{k,r}^2)_x(x_0, t_0) - (u^2)_x(x_0, t_0)| \leqslant \eta$ for k large, and thus $|(u_{k,k}^2)_x(x_0, t_0) - (u^2)_x(x_0, t_0)| \leqslant \eta$ for k large.

We now finish the proof of the main theorem. We let t_1 go to 0 and R go to ∞ and pick a diagonal subsequence, that we call again $\{u_k\}$. Then the corresponding $\{q_k\}$ converges weakly to q, $\{(u_k^2)_x\}$ converges pointwise to $(u^2)_x$, $\{u_k\}$ converges uniformly on compact sets to u, and $\varepsilon_k(B(u_k)q_k - \mu(u_k))$ converges to 0. In this form all the integrals converge in (30), (31) and we obtain a weak solution of problem (1) and (2).

The solution we have obtained has the following property: If $u_0(x) > 0$ in (x_1, x_2) and $u_0(x) = 0$ otherwise, then the set $P[u] = \{(x, t)/u(x, t) > 0\}$ is bounded by a decreasing curve $x = \xi_1(t)$ through $(x_1, 0)$ and an increasing curve $x = \xi_2(t)$ through $(x_2, 0)$. We prove this assertion by means of the next two lemmas.

LEMMA 7. Let $t_0 \in [0, T)$. If $u(x_0, t_0) = \eta > 0$, then there is a positive constant K, independent of ε, such that $u_\varepsilon(x_0, t) \geqslant K\eta^2$ for every $t \geqslant t_0$. In particular, if $u_0(x_0) > 0$ then $u(x_0, t) > 0$ for every $t \geqslant 0$.

Proof. Let $L_1[z] = (zz_x)_x - Nz - z_t$, $N \geqslant 2M_1$, and assume we have a bounded solution v in Ω_T of

$$(vv_x)_x - Nv = v_t, \; v(x, 0) = u_0(x) + \varepsilon. \tag{32}$$

Then $L_1[u_\varepsilon] \leqslant 0 = L_1[v]$ and $v(x, 0) \leqslant u_\varepsilon(x, 0)$, so by the maximum principle, $v(x, t) \leqslant u_\varepsilon(x, t)$ in Ω_T. We can obtain solutions of (32) by considering the transformation

$$w(x, \tau) = \frac{1}{1 - N\tau} v(x, t(\tau)), \; t(\tau) = -\frac{1}{N}\log(1 - N\tau).$$

Then (32) transforms into

$$w_\tau = (ww_x)_x, \qquad w(x, 0) = u_0(x) + \varepsilon. \tag{33}$$

Let w be the unique solution of (33) (that exists by Theorem 2 in [12]). This solution is obtained as the limit of solutions to the boundary value problems:

$$(w^n w_x^n)_x = w_\tau^n, \quad w^n(x, 0) = u_0(x) + \varepsilon, \quad w^n(\pm n, t) = u_0(\pm n) + \varepsilon.$$

Let $a = x_0 - \eta/2M_0$, $b = x_0 + \eta/2M_0$. Since u_0 is Lipshitz with constant M_0, $u_0(x) \geqslant \eta/2$ in $[a, b]$. We consider

$$\overline{w} = \frac{-2\eta}{(b - a)^2 + 4\eta\tau}(x - a)(x - b) \quad \text{and} \quad L_2[z] = (zz_x) - z\tau.$$

Then $L_2[w^\eta] = 0 \leqslant L_2[\overline{w}]$, $\overline{w}(x, 0) \leqslant \eta/2 \leqslant w^\eta(x, 0)$ and $\overline{w}(a, \tau) = 0 \leqslant w^\eta(a, \tau)$. Further, $\overline{w}(b, \tau) = 0 \leqslant w^\eta(b, \tau)$, so $w \leqslant w^\eta$ in $[a, b] \times [0, \tau)$ for any $\tau < 1/N$.

At $x = x_0$ we obtain

$$\overline{w}(x_0, \tau) = \frac{\eta(b - a)^2}{2(b - a)^2 + 8\eta\tau} \leqslant w^\eta(x_0, \tau).$$

Since this holds for every n, the same is true for w. Thus

$$u_\varepsilon(x_0, t) \geqslant \frac{\eta(b - a)^2(1 - N\tau)}{2(b - a)^2 + 8\eta\tau} \geqslant \frac{\eta^2}{2} \cdot \frac{Ne^{-NT}}{N + 2M_0^2},$$

and the result follows with $K = Ne^{-NT}/2(N + 2M_0^2)$.

LEMMA 8. For $t_0 \in [0, T)$, the support of $u(x, t_0)$ is an interval $[\xi_1(t_0), \xi_2(t_0)]$.

Proof. Suppose not. Then there exists a point (x_0, t_0) for which $u(x_0, t_0) = 0$ and $u(x, t)$ is positive for some values of x before x_0 and after x_0. Pick a point $a > x_0$ with $u(a, t_0) = \eta > 0$. By Lemma 1, $x_0 > x_2$ and $u(x_0, t) = 0$ for $t \leqslant t_0$. By the continuity of u and the uniform convergence of $\{u_{\varepsilon_k}\}$ for k larger than a certain k_0, $u_{\varepsilon_k}(x_0, t) \leqslant \eta/4$ and there is a neighborhood D_0 containing (a, t_0) in which $u_{\varepsilon_k}(x, t) > \eta/2$. We also pick k_0 large such that $\varepsilon_k \leqslant \eta/4$. Then in the domain $[x_0, \infty) \times [0, t_0)$, $u_{\varepsilon_k} \leqslant \eta/4$ on the lower and left boundaries, and u_{ε_k} tends to $\varepsilon_k \leqslant \eta/4$ as $|x| \to \infty$, so by the maximum principle, u cannot be larger than $\eta/2$. This contradiction proves the lemma.

REFERENCES

[1] M. E. Gurtin, *Some questions and open problems in continuum mechanics and population dynamics*, J. Diff. Eq. **48**, 293–312 (1983)

[2] M. E. Gurtin, *A system of equations for age dependent population diffusion*, J. Theor. Biol. **40**, 389–392 (1973)

[3] M. E. Gurtin and R. C. MacCamy, *Nonlinear age dependent population dynamics*, Arch. Rat. Mech. Anal. **54**, 281–300 (1974)

[4] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological populations*, Math. Bioscience **33**, 35–49 (1977)

[5] M. E. Gurtin and R. C. MacCamy, *Population dynamics with age dependence*, Nonlinear Analysis and Mech. Heriot-Watt Symposium, Vol. III, Pitman, 1979

[6] M. E. Gurtin and R. C. MacCamy, *Some simple models for nonlinear age-dependent population dynamics*, Math. Biosciences **43**, 199–211 (1979)

[7] M. E. Gurtin and R. C. MacCamy, *Diffusion models for age-structured population*, Math. Biosciences **54**, 49–59 (1981)

[8] R. C. MacCamy, *A population model with nonlinear diffusion*, J. Diff. Equations **39**, 52–72 (1981)

[9] G. E. Hernandez, *Existence of solutions of population dynamics problems with diffusion*, Thesis, University of Minnesota, 1983

[10] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, 1964

[11] O. A. Ladyženskaja, V. A. Solonikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, vol. 23, Amer. Math. Soc., 1968

[12] D. G. Aronson, *Regularity properties of flows through porous media: a counterexample*, SIAM J. Appl. Math. **19** (1970)

[13] D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*. Arch. Rat. Mech. Anal. **25**, 81–122 (1967)