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TRANSLATIONAL ADDITION THEOREMS FOR PROLATE SPHEROIDAL VECTOR WAVE FUNCTIONS $\mathbf{M}^r$ AND $\mathbf{N}^r$*

By

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Abstract. The translational addition theorems for prolate spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$ with reference to the spheroidal coordinate system at the origin 0 are obtained in terms of spheroidal vector wave functions with reference to the translated spheroidal coordinate system at the origin $0'$. These addition theorems are absolutely necessary for the study of the multiple scattering of electromagnetic waves when the single electromagnetic scattering solution is also expressed in terms of spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$.

1. Introduction. In the study of multiple scattering of electromagnetic waves, it is necessary to have the solution of the single scattering problem and also addition theorems for the vector wave functions which express the single solution. In the general case, the study of the scattering is limited to the evaluation of the total scattering far field and requires the transformation of the outgoing wave from any body centered at 0 into the incoming wave to the other body centered at $0'$, when the boundary conditions are expressed upon the surface of each scatterer. Consequently, it is only necessary to transform the scalar wave function for a radius vector $\mathbf{r}'$ (with respect to the coordinate system at the origin $0'$), the length of which is smaller than (or equal to) the translational distance $00'$.

Translational addition theorems for spherical vector wave functions were established [1–3] and applied in the studies of the scattering of a plane electromagnetic wave from a set of two spheres [4]. To our knowledge, there are only two recent works [5, 6] about multiple scattering of a plane electromagnetic wave from two prolate spheroids. In these works, translational addition theorems were established by the authors. In the former, the basic solution of the single scattering problem [7, 8] is expressed in terms of $\mathbf{M}^r$ and $\mathbf{N}^r$ generated by the radius vector $\mathbf{r}$ drawn from the origin to an arbitrary point. The two prolate spheriods have the same axis of revolution and their centers are axially displaced.

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Consequently, translational addition theorems are given only in this configuration. In the latter, the two prolate spheroids have their axes of revolution mutually parallel and the displacement of their centers is arbitrary. The single solution [9] is expressed in terms of the spheroidal vector wave function $\mathbf{M}^a$ and $\mathbf{N}^a$ (with $a = x, y, z$) generated by the three Cartesian unit vectors; so translational addition theorems are only given for these vector wave functions [10].

The aim of our paper is to obtain translational addition theorems for the spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$ with an arbitrary translation in order to extend our previous study [5] to a more general location of the two interacting prolate spheroids. This problem is treated in the same way as in [5], namely:

(1) transformation without translation of the spheroidal scalar wave function as an expansion of spherical wave functions;

(2) translation of these spherical scalar wave functions from the coordinate system at the origin 0 to the coordinate system with the origin in $0'$;

(3) transformation of the translated spherical scalar wave function into spheroidal scalar wave functions;

(4) derivation of the vector wave function in terms of translational vector wave functions.

This procedure allows us to have easier computations because a recursive relation can be found for particular configurations; for instance, we have successively considered two prolate spheroids having the same polar axis [5] or having parallel polar axes but with centers located in the perpendicular plane at the common direction of these axes [11]. Then, it is very likely that a recursive formula can be obtained for any translation but the complexity of such a relation can reduce its efficiency. Besides, we do not forget that the recursive relations obtained in [5] and [11] respectively contain five and thirteen terms.

In addition, we specify that the electromagnetic field is described with time-dependence $\exp(-i\omega t)$ and the definitions and notations are the same as those of Stratton [12] and Flammer [13]. These notations were indicated in our previous papers [14, 15, 8, 5, 11] and will not be recalled here.

2. Translation of the spheroidal scalar wave function.

2.1. *Definitions.* Let us consider two translated Cartesian coordinate systems $0xyz$ and $0'x'y'z'$ and define $a = \overline{00'}$. Two spherical and two spheroidal coordinate systems are associated with the two previous Cartesian coordinate systems. In the two prolate spheroidal systems, any point P is respectively characterized by the sets (d, ξ, η, ϕ) and (d', ξ', η', ϕ') where $2d$ and $2d'$ are the interfocal distances. In the same way, in the two spherical coordinate systems, respectively centered at 0 and $0'$, point P is characterized by the sets (r, θ, ϕ) and (r', θ', ϕ').

The scalar and vector wave functions with reference to the two spherical coordinate systems are marked by a dark point (for example $\dot{\mathbf{M}}^r$). In this way, we distinguish the spherical wave functions from the spheroidal ones when these functions are built with the same radius vector and in their own coordinate systems centered at the same origin.

Let us define $c = kd$ and $c' = kd'$ where k denotes the modulus of the wave propagation vector. The polar coordinates of the origin $0'$ with respect to the origin 0 are (a, β, α) (see figure) and we put $x_0 = a \sin\beta \cos\alpha$, $y_0 = a \sin\beta \sin\alpha$, $z_0 = a \cos\beta$.

2.2. *Translational addition theorem for the scalar wave functions.* Starting with the relation (2) of [5], written with $h = 3$ and after multiplying both sides by $\exp(im\phi)$, we obtain

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\exp(im\phi) = \sum_{s=0,1}^{\infty}{}' i^{s+m-n} d_s^{mn}(c) h_{m+s}^{(1)}(kr) P_{m+s}^{m}(\cos\theta)\exp(im\phi).$$

We translate the scalar spherical function by the use of the equation B-1 given by Cruzan [3] and we obtain

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\exp(im\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu} b_{\mu\nu}^{mn}\exp\left[i(m-\mu)\alpha\right] j(kr')P_{\mu}^{\nu}(\cos\theta')\exp(i\mu\phi'), \qquad r' \leqslant a \tag{1}$$

with

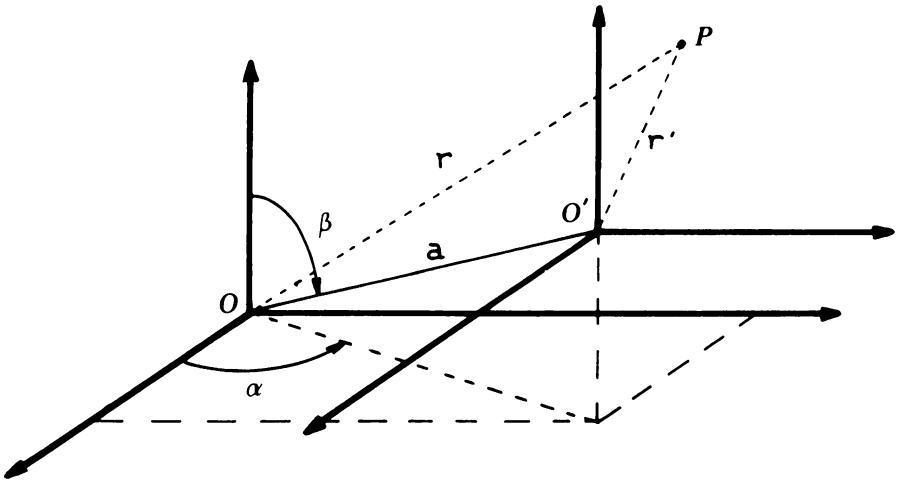
$$b_{\mu\nu}^{mn} = (-1)^{\mu}(2\nu+1)\sum_{s=0,1}^{\infty}{}' i^{s+m-n} d_s^{mn}(c) \cdot \sum_{p} i^{\nu+p-m-s} a(m, m+s|-\mu,\nu|p) h_p^{(1)}(ka) P_p^{m-\mu}(\cos\beta) \tag{2}$$

where the rules governing the choice of the parity of the indices are known [13, 2, 4].

Now, we consider the equivalent form of relation (1), established in Appendix, namely

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\,{}^{\cos}_{\sin}\left[m(\phi-\alpha)\right] = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} {}^{e,0}B_{\mu\nu}^{mn} j_{\nu}(kr')P_{\nu}^{\mu}(\cos\theta')\,{}^{\cos}_{\sin}\left[\mu(\phi'-\alpha)\right]. \tag{3}$$

In order to obtain the ${}^{\cos}_{\sin}m\phi$-dependence which appears in the expression of the vector wave function components of $\mathbf{M}^r$ and $\mathbf{N}^r$ [13, table V], we expand on both sides the sine



and cosine functions. So we have

$$\Psi^{(3)}_{e,0mn}(c,\xi,\eta,\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu}\Big\{{}^{e,0}_{c}B^{mn}_{\mu\nu}\cdot\Psi'_{e,0\mu\nu}(r',\theta',\phi') \mp {}^{0,e}_{s}B^{mn}_{\mu\nu}\cdot\Psi'_{0,e\mu\nu}(r',\theta',\phi')\Big\}, \tag{4}$$

for $\mu > 0$ we have

$${}^{e,0}_{c}B^{mn}_{\mu\nu} = b^{mn}_{\mu\nu}\cos[(m-\mu)\alpha] \pm (-1)^{\mu}\frac{(\nu-\mu)!}{(\nu+\mu)!}b^{mn}_{-\mu\nu}\cos[(m+\mu)\alpha],$$

$${}^{e,0}_{s}B^{mn}_{\mu\nu} = b^{mn}_{\mu\nu}\sin[(m-\mu)\alpha] \pm (-1)^{\mu}\frac{(\nu-\mu)!}{(\nu+\mu)!}b^{mn}_{-\mu\nu}\sin[(m+\mu)\alpha],$$

and, for $\mu = 0$,

$${}^{e}_{c}B^{mn}_{0\nu} = b^{mn}_{0\nu}\cos m\alpha, \qquad {}^{e}_{s}B^{mn}_{0\nu} = b^{mn}_{0\nu}\sin m\alpha,$$

where

$$\Psi^{(h)}_{e,0mn}(c,\xi,\eta,\phi) = R^{(h)}_{mn}(c,\xi)S^{(1)}_{mn}(c,\eta)^{\cos}_{\sin}m\phi, \qquad h = 1,3,$$
$$\cdot\Psi'_{e,0\mu\nu}(r',\theta',\phi') = j_{\nu}(kr')P^{\mu}_{\nu}(\cos\theta')^{\cos}_{\sin}\mu\phi'.$$

Then, we transform the spherical scalar wave function in terms of spheroidal scalar wave functions with the help of relation 5.3.9 of [13], written as follows:

$$j_{\nu}(kr')P^{\mu}_{\nu}(\cos\theta')^{\cos}_{\sin}[\mu(\phi'-\alpha)] = \sum_{l=\mu,\mu+1}^{\infty}{}' \Gamma^{\mu\nu}_{\mu l}(c')R^{(1)}_{\mu l}(c',\xi')S^{(1)}_{\mu l}(c',\eta')^{\cos}_{\sin}[\mu(\phi'-\alpha)]$$

with

$$\Gamma^{\mu\nu}_{\mu l}(c') = \frac{2}{2\nu+1}\frac{(\nu+\mu)!}{(\nu-\mu)!}\frac{i^{l-\nu}}{N_{\mu l}}d^{\mu l}_{\nu-\mu}(c').$$

Finally we have

$$\Psi^{(3)}_{e,0mn}(c,\xi,\eta,\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu}\sum_{l=\mu,\mu+1}^{\infty}{}' \Gamma^{\mu\nu}_{\mu l}(c')\Big\{{}^{e,0}_{c}B^{mn}_{\mu\nu}\Psi'^{(1)}_{e,0\mu l}(c',\xi',\eta',\phi') \mp {}^{0,e}_{s}B^{mn}_{\mu\nu}\Psi'^{(1)}_{0,e\mu l}(c',\xi',\eta',\phi')\Big\}. \tag{5}$$

This last formula is the mathematical expression for the addition theorem of the scalar wave function $\Psi^{(3)}_{e,0mn}$ of two prolate spheroidal coordinate systems related to each other by an arbitrary translation.

The addition theorem for $\Psi^{(1)}_{e,0mn}$ is the same as $\Psi^{(3)}_{e,0mn}$ provided that we consider a $b^{mn}_{\mu\nu}$-coefficient written with $j_p(ka)$ instead of $h^{(1)}_p(ka)$.

3. Translational addition theorem for spheroidal vector wave functions.

Let
$$ {}^{(3)}\mathbf{M}^{r}_{e,0mn},\ {}^{(1)}\mathbf{M}^{r'}_{e,0\mu\nu} \text{ and } {}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} $$
be the spheroidal and spherical vector wave functions expressed in their own coordinate systems.

Since the gradient operator of a scalar function is invariant throughout a change of coordinate system [12, 2, 3], we can write from (4):

$$ \begin{aligned} {}^{(3)}\mathbf{M}^{r}_{e,0mn} = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} \Big\{ {}^{e,0}_{c}B^{mn}_{\mu\nu}\Big({}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} + x_0\,{}^{\cdot}\mathbf{M}^{x'}_{e,0\mu\nu} + y_0\,{}^{\cdot}\mathbf{M}^{y'}_{e,0\mu\nu} + z_0\,{}^{\cdot}\mathbf{M}^{z'}_{e,0\mu\nu}\Big) \\ \mp {}^{0,e}_{s}B^{mn}_{\mu\nu}\Big({}^{\cdot}\mathbf{M}^{r'}_{0,e\mu\nu} + x_0\,{}^{\cdot}\mathbf{M}^{x'}_{0,e\mu\nu} + y_0\,{}^{\cdot}\mathbf{M}^{y'}_{0,e\mu\nu} + z_0\,{}^{\cdot}\mathbf{M}^{z'}_{0,e\mu\nu}\Big)\Big\} \end{aligned} \tag{6} $$

Now we must eliminate the spherical wave functions

$$ {}^{\cdot}\mathbf{M}^{x'}_{e,0\mu\nu},\quad {}^{\cdot}\mathbf{M}^{y'}_{e,0\mu\nu},\quad \text{and}\quad {}^{\cdot}\mathbf{M}^{z'}_{e,0\mu\nu}. $$

These functions, which are built with the Cartesian unit vectors of the $0'x'y'z'$ system, are expanded in terms of spherical wave functions

$$ {}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} \qquad \text{and} \qquad {}^{\cdot}\mathbf{N}^{r'}_{e,0\mu\nu} $$

which are written in the same reference spherical system. To this end, we put

$$ \begin{aligned} \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} B^{mn}_{\mu\nu}\,{}^{\cdot}\mathbf{M}^{x'}_{e,0\mu\nu} &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu}\Big(a'_{\mu\nu}\,{}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} + b'_{\mu\nu}\,{}^{\cdot}\mathbf{N}^{r'}_{0,e\mu\nu}\Big), \\ \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} B^{mn}_{\mu\nu}\,M^{y'}_{e,0\mu\nu} &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu}\Big(a''_{\mu\nu}\,{}^{\cdot}\mathbf{M}^{r'}_{0,e\mu\nu} + b''_{\mu\nu}\,{}^{\cdot}\mathbf{N}^{r'}_{e,0\mu\nu}\Big), \\ \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\nu} B^{mn}_{\mu\nu}\,{}^{\cdot}\mathbf{M}^{z'}_{e,0\mu\nu} &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu}\Big(a'''_{\mu\nu}\,{}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} + b'''_{\mu\nu}\,{}^{\cdot}\mathbf{N}^{r'}_{0,e\mu\nu}\Big). \end{aligned} \tag{7} $$

In the limits of the sums, we have taken into account the cancellation [12] of the three components of

$$ {}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} \qquad \text{and} \qquad {}^{\cdot}\mathbf{N}^{r'}_{e,0\mu\nu} $$

for $\nu = \mu = 0$.

In order to determine the coefficients which appear in relations (7) we use the orthogonality properties of spherical vector wave functions [12, p. 417] and the fact that

$$ {}^{\cdot}\mathbf{M}^{r'}_{e,0\mu\nu} $$

has a null-component along the r'-axis for all values of μ and ν. In the calculations, we find some integrals over θ' whose evaluation can be found in [16]. Finally we obtain

$$
\begin{aligned}
a'_{\mu\nu} &= \frac{k}{2\nu(\nu+1)}\left\{\frac{\nu}{2\nu+3}\left[(\nu+\mu+1)(\nu+\mu+2)B^{mn}_{\mu+1,\nu+1} - B^{mn}_{\mu-1,\nu+1}\right]\right.\\
&\quad\left.+\frac{\nu+1}{2\nu-1}\left[-(\nu-\mu-1)(\nu-\mu)B^{mn}_{\mu+1,\nu-1} + B^{mn}_{\mu-1,\nu-1}\right]\right\},\\
a''_{\mu\nu} &= \frac{\pm k}{2\nu(\nu+1)}\left\{\frac{-\nu}{2\nu+3}\left[(\nu+\mu+1)(\nu+\mu+2)B^{mn}_{\mu+1,\nu+1} + B^{mn}_{\mu-1,\nu+1}\right]\right.\\
&\quad\left.+\frac{\mu+1}{2\nu-1}\left[(\nu-\mu-1)(\nu-\mu)B^{mn}_{\mu+1,\nu-1} + B^{mn}_{\mu-1,\nu-1}\right]\right\},\\
a'''_{\mu\nu} &= \frac{k}{\nu(\nu+1)}\left[\frac{(\nu+1)(\nu-\mu)}{2\nu-1}B^{mn}_{\mu,\nu-1} + \frac{\nu(\nu+\mu+1)}{2\nu+3}B^{mn}_{\mu,\nu+1}\right],\\
b'_{\mu\nu} &= \frac{\pm k}{2\nu(\nu+1)}\left[(\nu-\mu)(\nu+\mu+1)B^{mn}_{\mu+1,\nu} + B^{mn}_{\mu-1,\nu}\right],\\
b''_{\mu\nu} &= \frac{k}{2\nu(\nu+1)}\left[(\nu-\mu)(\nu+\mu+1)B^{mn}_{\mu+1,\nu} - B^{mn}_{\mu-1,\nu}\right],\\
b'''_{\mu\nu} &= \frac{\mp k\mu}{\nu(\nu+1)}B^{mn}_{\mu\nu}.
\end{aligned}
\tag{8}
$$

At this level, we can recall that in the case of two translational spherical systems, Cruzan had defined and used the same coefficients (see in [3] the expressions (16) and (17)).

The structure of these expressions suggests some explanations concerning the variation of the subscripts. The quantities B^{mn}_{pq} have no sense for $p > q$. Moreover, in a'_{01} and a''_{01} it appears B^{mn}_{-10} and B^{mn}_{-20} are related to b^{mn}_{-10} and b^{mn}_{10} and to b^{mn}_{-20} and b^{mn}_{20}, respectively, according to (3). But b^{mn}_{10} and b^{mn}_{20} must be multiplied by some factors containing negative factorials in the denominator, whereas b^{mn}_{-10} and b^{mn}_{-20} (see relation (2)) cancel for the same reasons. In addition, in b'_{01} and b''_{02}, B^{mn}_{-11} occurs, which in fact has no sense because such a coefficient cannot be generated by the development (4).

In these conditions for $\mu = 0$ and $\nu = 1$ we must use the following relations:

$$
\begin{aligned}
a'_{01} &= \frac{3}{10}kB^{mn}_{12}, & b'_{01} &= \pm\frac{1}{2}kB^{mn}_{11},\\
a''_{01} &= \mp\frac{3}{10}kB^{mn}_{12}, & b''_{01} &= \frac{1}{2}kB^{mn}_{11},\\
a'''_{01} &= k\left(B^{mn}_{00} + \frac{1}{5}B^{mn}_{02}\right), & b'''_{01} &= 0.
\end{aligned}
$$

For the other values of μ and ν, formulae (8) are used without difficulty. Then, expressions (7) allow to write the relation (6) in the following form:

$$
{}^{(3)}\mathbf{M}^{r}_{e,0mn} = \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu}\left\{{}^{e,0}X^{mn}_{\mu\nu}\cdot\mathbf{M}^{r'}_{e,0\mu\nu} + {}^{e,0}Y^{mn}_{\mu\nu}\cdot\mathbf{N}^{r'}_{e,0\mu\nu} + {}^{0,e}Z^{mn}_{\mu\nu}\cdot\mathbf{M}^{r'}_{0,e\mu\nu} + {}^{0,e}T^{mn}_{\mu\nu}\cdot\mathbf{N}^{r'}_{0,e\mu\nu}\right\}
\tag{9'}
$$

where

$$
\begin{aligned}
{}^{e,0}X_{\mu\nu}^{mn} &= {}_{c}^{e,0}B_{\mu\nu}^{mn} + x_0{}_{c}^{e,0}a'_{\mu\nu} + z_0{}_{c}^{e,0}a'''_{\mu\nu} \mp y_0{}_{s}^{0,e}a''_{\mu\nu},\\
{}^{e,0}Y_{\mu\nu}^{mn} &= y_0{}_{c}^{e,0}b''_{\mu\nu} \mp \left(x_0{}_{s}^{0,e}b'_{\mu\nu} + z_0{}_{s}^{0,e}b'''_{\mu\nu}\right),\\
{}^{0,e}Z_{\mu\nu}^{mn} &= y_0{}_{c}^{e,0}a''_{\mu\nu} \mp \left({}_{s}^{0,e}B_{\mu\nu}^{mn} + x_0{}_{s}^{0,e}a'_{\mu\nu} + z_0{}_{s}^{0,e}a'''_{\mu\nu}\right),\\
{}^{0,e}T_{\mu\nu}^{mn} &= x_0{}_{c}^{e,0}b'_{\mu\nu} + z_0{}_{c}^{e,0}b'''_{\mu\nu} \mp y_0{}_{s}^{0,e}b''_{\mu\nu},
\end{aligned}
$$

In the last relations, the various coefficients (typically a', a'', a''', b', b'', and b''') are defined by (8), in which the coefficients ${}_{c}^{e,0}B_{\mu\nu}^{mn}$ and ${}_{s}^{e,0}\mathrm{B}_{\mu\nu}^{mn}$ replace the $B_{\mu\nu}^{mn}$,

$$
\begin{aligned}
U_3 &= x_0 b'_{\mu\nu} + z_0 b'''_{\mu\nu},\\
U_4 &= y_0 b''_{\mu\nu}.
\end{aligned}
$$

The relation of transformation of the spheroidal wave function

$$
{}^{(3)}\mathbf{N}_{e,0mn}^{r},
$$

noted (9″), can be easily deduced from (9′). It is only sufficient to permute **M** and **N** in the expression (9′) owing to the form of the related relations between **M** and **N** in a homogeneous, isotropic and free source medium [12, 2, 3].

Relations (9′) and (9″) are the mathematical expressions of the addition theorems for the transformation by translation of the spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$ into spherical vector wave functions $'\mathbf{M}^{r'}$ and $'\mathbf{N}^{r'}$.

The last step consists of transforming, without translation, the spherical vector wave functions into spheroidal vector wave functions by using the expression 5.3.9 of Flammer [13] and the invariance property of the gradient operator. We obtain

$$
{}'\mathbf{M}_{e,0\mu\nu}^{r'}(r',\theta',\phi') = \sum_{l=\mu,\mu+1}^{\infty}{}' \Gamma_{\mu l}^{\mu\nu}(c')^{(1)}\mathbf{M}_{e,0\mu l}^{r'}(c',\xi',\eta',\phi') \tag{10′}
$$

and consequently

$$
{}'\mathbf{N}_{e,0\mu\nu}^{r'}(r',\theta',\phi') = \sum_{l=\mu,\mu+1}^{\infty}{}' \Gamma_{\mu l}^{\mu\nu}(c')^{(1)}\mathbf{N}_{e,0\mu l}^{r'}(c',\xi',\eta',\phi'). \tag{10″}
$$

Putting the last results in (9′) and (9″) we finally have

$$
\begin{aligned}
{}^{(3)}\mathbf{M}_{e,0mn}^{r} &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu}\sum_{l=\mu,\mu+1}^{\infty}{}' \Big(A_{\mu\nu l}^{mn(1)}\mathbf{M}_{e,0\mu l}^{r'} + B_{\mu\nu l}^{mn(1)}\mathbf{N}_{0,e\mu l}^{r'}\\
&\quad + C_{\mu\nu l}^{mn(1)}\mathbf{M}_{0,e\mu l}^{r'} + D_{\mu\nu l}^{mn(1)}\mathbf{N}_{e,0\mu l}^{r'}\Big)\\
{}^{(3)}\mathbf{N}_{e,0mn}^{r} &= \sum_{\mu=1}^{\infty}\sum_{\mu=0}^{\nu}\sum_{l=\mu,\mu+1}^{\infty}{}' \Big(A_{\mu\nu l}^{mn(1)}\mathbf{N}_{e,0\mu l}^{r'} + B_{\mu\nu l}^{mn(1)}\mathbf{M}_{0,e\mu l}^{r'}\\
&\quad + C_{\mu\nu l}^{mn(1)}\mathbf{N}_{0,e\mu l}^{r'} + D_{\mu\nu l}^{mn(1)}\mathbf{M}_{e,0\mu l}^{r'}\Big),
\end{aligned} \tag{11}
$$

where, for sake of simplicity, we have put

$$
\begin{aligned}
A_{\mu\nu l}^{mn} &= {}^{e,0}X_{\mu\nu}^{mn}\Gamma_{\mu l}^{\mu\nu}(c'), \quad & C_{\mu\nu l}^{mn} &= {}^{0,e}Z_{\mu\nu}^{mn}\Gamma_{\mu l}^{\mu\nu}(c'),\\
D_{\mu\nu l}^{mn} &= {}^{e,0}Y_{\mu\nu}^{mn}\Gamma_{\mu l}^{\mu\nu}(c'), \quad & B_{\mu\nu l}^{mn} &= {}^{0,e}T_{\mu\nu}^{mn}\Gamma_{\mu l}^{\mu\nu}(c').
\end{aligned} \tag{12}
$$

In fact, these coefficients are functions of d and d' (the semi-interfocal distances) through c and c' and also of the geometrical parameters of the translation $\overline{00'}$ (length a and angular coordinates β and α). In addition, we recall that

$$^{(3)}\mathbf{M}^{r}_{e,0mn} \quad \text{and} \quad ^{(3)}\mathbf{N}^{r}_{e,0mn},$$

which appear on the left side of (11), are functions of (d, ξ, η, ϕ) while

$$^{(1)}\mathbf{M}^{r'}_{e,0\mu l} \quad \text{and} \quad ^{(1)}\mathbf{N}^{r'}_{e,0\mu l},$$

on the right side, are functions of (d', ξ', η', ϕ').

Relations (11) express the addition theorems of the spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$ concerning two prolate spheroidal coordinate systems which correspond to each other by translation. We must point out that these relations are not independent, contrary to the similar relations recently obtained by Sinha and MacPhie [10] for $\mathbf{M}^a$ and $\mathbf{N}^a$ with $a = x, y, z$. Also, it is to note that we start from (4) instead of (5) in the derivation of the relations (11). Starting from (5), the calculus will introduce the vector function grad $[\Psi'^{(1)}_{e,0}(c', \xi', \eta', \phi')] \wedge \mathbf{u}_b$ where $\mathbf{u}_b$ (with $b = \xi', \eta', \phi'$) denotes any unit vector of the spheroidal coordinate system at the origin $0'$. Then, in this assumption, we shall express these vector functions in terms of vector wave functions $\mathbf{M}^a$ and $\mathbf{N}^a$ (with $a = x', y', z', r'$) again built in the same coordinate system. We think this last step is more difficult to realize because the spheroidal vector wave functions do not satisfy the orthogonality properties contrary to the spherical ones [12].

Moreover, we have given in [5] a particular form for (11), in the case of a translation along the z-axis in the study of multiple scattering by two axially displaced prolate spheroids. With our present notations, we have $m = \mu$, (consequently the summation over μ is suppressed) and in addition $x_0 = y_0 = 0$. As a result, the coefficients given by (12) become

$$A^{mn}_{m\nu l} = \left({}^{e,0}_{c}B^{mn}_{m\nu} + z_0\,{}^{e,0}_{c}a'''_{m\nu}\right)\Gamma^{m\nu}_{ml}, \qquad D^{mn}_{n\nu l} = 0,$$
$$B^{mn}_{m\nu l} = z_0\,{}^{e,0}_{c}b'''_{m\nu}\Gamma^{m\nu}_{ml}, \qquad C^{mn}_{m\nu l} = 0.$$

With these new coefficients, relations (11) are identical to relations (6) in [5], where kz_0, $a'''_{m\nu}$, and $b'''_{m\nu}$ respectively correspond to $c\alpha$, α^{mn}_t, and $\mp\beta^{mn}_t$ and where subscript ν corresponds to t.

4. Conclusion. The addition theorems which are derived in this paper are the main part of a theoretical study of the multiple scattering of electromagnetic waves from prolate spheroids with parallel axes in the case where the associated solution corresponding to a single scatterer is expressed with the spheroidal vector wave functions $\mathbf{M}^r$ and $\mathbf{N}^r$. In the particular case where $x_0 = y_0 = 0$, a special solution is obtained (Eq. (6) in [5]). Another form, corresponding to $z_0 = 0$ will find a straight-forward application in the study of multiple scattering from prolate spheroids having their centers lying in a same plane. In these two particular geometrical configurations, we recall that a recursive formula relating the $b^{mn}_{\mu\nu}$-coefficients was established [5, 11] in order to simplify and make easier the multiple scattering computations, which otherwise would be too complicated.

Appendix. Starting with Eq. (1), respectively written for the sets (m, μ) and $(-m, -\mu)$, we deduce

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\exp[\mathrm{i}m(\phi-\alpha)] = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu} b^{mn} j_\nu(kr')P_\nu^\mu(\cos\theta')\exp[i\mu(\phi'-\alpha)] \tag{A-1}$$

and

$$R_{-mn}^{(3)}(c,\xi)S_{-mn}^{(1)}(c,\eta)\exp[-\mathrm{i}m(\phi-\alpha)] = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu} b_{-\mu\nu}^{-mn} j_\nu(kr')P_\nu^{-\mu}(\cos\theta')\exp[-i\mu(\phi'-\alpha)]. \tag{A-2}$$

According to [13, p. 22] and [6], we transform A-2 into

$$\begin{aligned} R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\exp[-im(\phi-\alpha)] &= \sum_{\mu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu}(-1)^m\frac{(n+m)!}{(n-m)!}(-1)^\mu\frac{(\nu-\mu)!}{(\nu+\mu)!} \\ &\quad\cdot b_{-\mu\nu}^{-mn} j_\nu(kr')P_\nu^\mu(\cos\theta')\exp[-i\mu(\phi'-\alpha)] \end{aligned} \tag{A-3}$$

with

$$b_{-\mu\nu}^{-mn} = (-1)^{-\mu}(2\nu+1)\sum_{s=0,1}^{\infty}{}' i^{s+m-n}d_s^{-mn}\sum_p i^{\nu+p-m-s}a(-m, m+s|\mu,\nu|p)$$
$$\cdot h_p^{(1)}(ka)P_p^{\mu-m}(\cos\beta).$$

In this last relation, the limits of subscript p are the same as in (2) since they are independent of m and μ.

Using Eq. 3.1.36 in [13] and A-2 in [12] and according to the rules of symmetry of the $3-j$ symbols [17] we established without difficulty that $b_{-\mu\nu}^{-mn}$ is related to $b_{\mu\nu}^{mn}$ by

$$b_{-\mu\nu}^{-mn} = (-1)^{m-\mu}\frac{(n-m)!}{(n+m)!}\frac{(\nu+\mu)!}{(\nu-\mu)!}b_{\mu\nu}^{mn}.$$

Then, relation A-3 can be written as:

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)\exp[-\mathrm{i}m(\phi-\alpha)] = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu} b_{\mu\nu}^{mn} j_\nu(kr')P_\nu^\mu(\cos\theta')\exp[-i\mu(\phi'-\alpha)]. \tag{A-4}$$

Finally, from A-1 and A-4 we obtain

$$R_{mn}^{(3)}(c,\xi)S_{mn}^{(1)}(c,\eta)_{\sin}^{\cos}[m(\phi-\alpha)] = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{+\nu} b_{\mu\nu}^{mn} j_\nu(kr')P_\nu^\mu(\cos\theta')_{\sin}^{\cos}[\mu(\phi'-\alpha)]. \tag{A-5}$$

In order to have only a summation over positive values of μ, we put

$$ {}^{e,0}B_{\mu\nu}^{mn} = b_{\mu\nu}^{mn} \pm (-1)^{\mu} \frac{(\nu-\mu)!}{(\nu+\mu)!} b_{-\mu\nu}^{mn} \qquad \text{for } \mu > 0 $$

and

$$ {}^{e}B_{0\nu}^{mn} = b_{0\nu}^{mn} \qquad \text{for } \mu = 0 $$

and so we deduce the following addition theorem:

$$ R_{mn}^{(3)}(c,\xi) S_{mn}^{(1)}(c,\eta)_{\sin}^{\cos}[m(\phi-\alpha)] = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{+\nu} {}^{e,0}B_{\mu\nu}^{mn} j_{\nu}(kr') P_{\nu}^{\mu}(\cos\theta')_{\sin}^{\cos}[\mu(\phi'-\alpha)]. $$

In addition, we indicate that the same procedure leads to the same transformations when it is applied to the B-1 and B-2 forms of the addition theorems for the scalar wave functions given in [3] by Cruzan.

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