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COMMENTS ON THE PAPER *DETERMINATION OF THE STRETCH AND ROTATION IN THE POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT* BY A. HOGER AND D. E. CARLSON*

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Hoger and Carlson [1] have employed the Cayley–Hamilton theorem to derive explicit expressions for $\mathbf{U}$ and $\mathbf{U}^{-1}$ which contain only positive integral powers of $\mathbf{C}$, and the principal invariants of $\mathbf{U}$ and $\mathbf{C}$ [1]. Here, $\mathbf{C}$ is the right Cauchy–Green strain tensor, and $\mathbf{U}$ is the positive-definite symmetric right stretch tensor defined by

$$\mathbf{U}^2 = \mathbf{C}.$$

An objective, stated in [1], is the elimination in the above-mentioned expressions of the dependence on the invariants of $\mathbf{U}$, so that only $\mathbf{C}$ and its invariants appear in them. Attention will be given to the two- and three-dimensional cases.

For the two-dimensional case, with the principal invariants of $\mathbf{U}$ and $\mathbf{C}$ defined by

$$I_U = \operatorname{tr}\mathbf{U}, \qquad II_U = \det\mathbf{U},$$

and

$$I_C = \operatorname{tr}\mathbf{C}, \qquad II_C = \det\mathbf{C},$$

Hoger and Carlson [1] show that the former invariants are unambiguously expressible in terms of the latter by

$$I_U = \sqrt{I_C + 2\sqrt{II_C}}\,, \qquad II_U = \sqrt{II_C}\,.$$

It follows that $\mathbf{U}$ and $\mathbf{U}^{-1}$ can be expressed in terms of I_C, II_C, and $\mathbf{C}$, upon using Eqns. (3.3) and (4.1) of [1].

To consider the three-dimensional case, it is convenient to introduce the principal stretches (extension ratios) $\lambda_i (i = 1, 2, 3)$, which may be regarded either as the roots of the

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[1]See also Marsden & Hughes [2] and Ting [3].

cubic

$$\lambda^3 - I_U\lambda^2 + II_U\lambda - III_U = 0, \tag{1}$$

or as the positive roots of the sextic

$$\lambda^6 - I_C\lambda^4 + II_C\lambda^2 - III_C = 0. \tag{2}$$

The coefficients in (1) and (2) are the principal invariants of **U** and **C**, viz.,

$$I_U = \operatorname{tr}\mathbf{U}, \qquad II_U = \frac{1}{2}\left\{(\operatorname{tr}\mathbf{U})^2 - \operatorname{tr}\mathbf{U}^2\right\}, \qquad III_U = \det\mathbf{U}, \tag{3}$$

and

$$I_C = \operatorname{tr}\mathbf{C}, \qquad II_C = \frac{1}{2}\left\{(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right\}, \qquad III_C = \det\mathbf{C}. \tag{4}$$

From (1)–(4), following [1], we note that

$$I_U = \lambda_1 + \lambda_2 + \lambda_3, \qquad III_U = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad III_U = \lambda_1\lambda_2\lambda_3, \tag{5}$$

and

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad II_C = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad III_C = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{6}$$

Hoger and Carlson [1] employ (5) and (6) directly to obtain

$$III_U = \sqrt{III_C}$$

and

$$I_U^2 = I_C + 2II_U, \qquad II_U^2 = II_C + 2\sqrt{III_C}\,I_U, \tag{7}$$

and they proceed by eliminating II_U^2 from between the equations $(7)_{a,b}$. This yields the quartic

$$I_U^4 - 2I_CI_U^2 - 8\sqrt{III_C}\,I_U + \left(I_C^2 - 4II_C\right) = 0, \tag{8}$$

a solution of which is given in Eqn. (5.5) of [1], under the assumption that a *unique* positive value of I_U exists for arbitrary (allowable) values of I_C, II_C, III_C.

Now, if the quantity $(I_C^2 - 4II_C)$ is nonpositive, the existence of a unique positive root of (8) is apparent upon applying the rule of signs. The other case is covered by a counterexample. For, suppose that $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$. Then $I_C = 21$, $II_C = 84$, $III_C = 64$, $(I_C^2 - 4II_C)$ is positive, and

$$I_U^4 - 2I_CI_U^2 - 8\sqrt{III_C}\,I_U + \left(I_C^2 - 4II_C\right) = (I_U - 7)(I_U - 1)\left(I_U^2 + 8I_U + 15\right),$$

an expression that has two distinct positive zeroes, viz., $I_U = 7, 1$. It follows from $(5)_a$ that the correct value for I_U in this example is seven. Whence, the solution of (8), given in Eqn. (5.5) of [1], requires further qualification and/or clarification, in general.

A different approach, already suggested in [1] for higher-dimensional cases, avoids the ambiguity encountered in the above example. This is to gain a direct solution of (2), based on the knowledge that this equation admits three (possibly repeated) positive values for λ^2. Thus,

$$\lambda_i^2 = \frac{1}{3}\left\{I_C + 2\sqrt{I_C^2 - 3II_C}\cos\left[\frac{1}{3}(\phi - 2\pi i)\right]\right\}, \quad i = 1, 2, 3, \tag{9}$$

where

$$\phi = \cos^{-1}\left\{ \frac{2I_C^3 - 9I_C II_C + 27III_C}{2\left(I_C^2 - 3II_C\right)^{3/2}} \right\}.$$

Then, upon taking $\lambda_i = \sqrt{\lambda_i^2}$, the calculation of I_U, II_U, III_U in terms of I_C, II_C, III_C flows directly from (9) and (5). Finally, the use of Eqns. (3.7) and (4.2) of [1] gives expressions for $\mathbf{U}$ and $\mathbf{U}^{-1}$ containing only $\mathbf{C}$, $\mathbf{C}^2$, and the principal invariants of $\mathbf{C}$.

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REFERENCES

[1] A. Hoger and D. E. Carlson, *Determination of the Stretch and Rotation in the Polar Decomposition of the Deformation Gradient*, Quart. Appl. Math., Vol. XLII, No. 1, 113–117 (April 1984)

[2] J. E. Marsden and T. Hughes, *Mathematical Foundations of Elasticity*, p. 55, Prentice-Hall, Englewood Cliffs, N. J., 1983

[3] T. C. T. Ting, *Determination of* $\mathbf{C}^{1/2}$, $\mathbf{C}^{-1/2}$ *and more general isotropic tensor functions of* $\mathbf{C}$, J. Elasticity (to appear)