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ON AN INEQUALITY RELATED TO CERTAIN TOTALLY POSITIVE GREEN'S FUNCTIONS*

BY

SHMUEL FRIEDLAND[1]

University of Illinois at Chicago

In what follows we use the results and notation of [2, Chapter 10].

Let $\{w_i(x)\}$ $(i = 1, \ldots, n)$ be a set of positive functions of class $C^n[0,1]$, and associate with them the first-order differential operator

$$(D_i f) = \frac{d}{dx}(f(x)/w_i), \quad i = 1, \ldots, n, \tag{1}$$

and the nth-order differential operator

$$L_n f = D_n \cdots D_1 f. \tag{2}$$

Let $G(x, y)$ be the Green's function of the $2n$th-order differential operator

$$M = (-1)^n L_n^* L_n \tag{3}$$

coupled with the boundary conditions

$$\begin{aligned}
&u(0) = 0,\\
&(-1)^{n+1} c_2 (D_1 u)(0) + \left(D_3^* \cdots D_n^* D_n \cdots D_1 u\right)(0) = 0, \ldots,\\
&(-1)^{2n-1} c_n (D_{n-1} \cdots D_1 u)(0) + (D_n \cdots D_1 u)(0) = 0,\\
&u(1) = 0,\\
&(-1)^{n+2} d_2 (D_1 u)(1) + \left(D_3^* \cdots D_n^* D_n \cdots D_1 u\right)(1) = 0, \ldots,\\
&(-1)^{2n} d_n (D_{n-1} \cdots D_1 u)(1) + (D_n \cdots D_1 u)(1) = 0,
\end{aligned}$$

where

$$0 < c_i, d_i \leqslant \infty, \quad i = 2, \ldots, n.$$

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So $G(x, y)$ is a totally positive operator on $[0,1]$. Moreover, the above boundary conditions imply

$$\begin{aligned} &G_x^{(i)}(0, y) = 0, \quad i = 0, \ldots, p-1, \qquad 0 < G_x^{(p)}(0, y), \quad 1 \leqslant p \leqslant n, \\ &G_x^{(j)}(1, y) = 0, \quad j = 0, \ldots, q-1, \qquad 0 < (-1)^q G_x^{(q)}(1, y), \quad 1 \leqslant q \leqslant n, \end{aligned} \tag{4}$$

for all $y \in (0,1)$. Here p and q are the smallest integers for which $c_p = \infty > c_{p+1}$, $d_q = \infty > d_{q+1}$ where $c_{n+1} = d_{n+1} = 0$.

Theorem. Let

$$g(x) = \max\{G(x, y) \colon y \in [0,1]\}, \tag{5}$$

$$h(y) = \inf\left\{\frac{G(x, y)}{g(x)} \colon x \in (0,1)\right\}. \tag{6}$$

Then g and h are strictly positive in the open interval $(0,1)$ and each vanishes at the end points. Moreover, $h \leqslant 1$ and

$$G(x, y) \geqslant g(x)h(y) \quad \text{in } [0,1] \times [0,1]. \tag{7}$$

Proof. Clearly $g(x)$ is positive on $(0,1)$ and vanishes on the boundary points in view of (4). We now prove the main assertion that $h(y)$ is positive on $(0,1)$. Since $G(x, y)$ is totally positive we deduce that

$$G(x_1, y_1)G(x_2, y_2) \geqslant G(x_1, y_2)G(x_2, y_1), \quad 0 < x_1 < x_2 < 1, 0 < y_1 < y_2 < 1.$$

It then follows that for fixed y_1 and y_2 the function

$$f(x) = \frac{G(x, y_1)}{G(x, y_2)}$$

is an increasing function if $0 < y_2 < y_1 < 1$ and is a decreasing function if $1 > y_2 > y_1 > 0$. In the first case let $x \to 0$ and in the second case let $x \to 1$ to deduce

$$\frac{G(x, y_1)}{G(x, y_2)} \geqslant \min\left(\frac{G_x^{(p)}(0, y_1)}{G_x^{(p)}(0, y_2)}, \frac{G_x^{(q)}(1, y_1)}{G_x^{(q)}(1, y_2)}\right). \tag{8}$$

Let

$$g(x) = G(x, \eta(x)), \quad 0 < \eta(x) < 1 \quad \text{for} \quad 0 < x < 1$$

and

$$\mu = \max\left(\max_{1 \leqslant y \leqslant 1} G_x^{(p)}(0, y), \max_{0 \leqslant y \leqslant 1} (-1)^q G_x^{(q)}(1, y)\right).$$

So for any $x \in (0,1)$ we have the inequality

$$\frac{G(x, y)}{g(x)} \geqslant \min\left(G_x^{(p)}(0, y), (-1)^q G_x^{(q)}(1, y)\right)/\mu.$$

Hence $h(y)$ is positive for $y \in (0,1)$. The assertion that $h \leqslant 1$ and the inequality (7) in the domain $(0,1) \times (0,1)$ follows easily from the definitions of $g(x)$ and $h(y)$. Finally, use the continuity of G on $[0,1] \times [0,1]$, the positivity of h on $(0,1)$, and the equalities (4) to deduce that $h(0) = h(1) = 0$. ∎

A special case of this theorem was proved by Day [1].

References

[1] W. A. Day, *Positive deflections of elastic beams*, this journal.
[2] S. Karlin, *Total Positivity*, Stanford University Press, Stanford, California, 1968