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STABILITY OF TIME-PERIODIC TEMPERATURE FIELDS*

By

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Summary. The energy method developed by Joseph [4], Davis [2], and Homsy [3] is applied to the time-periodic temperature fields considered by Yih and Li [11] to obtain Rayleigh numbers below which the fluid is stable. This is done to see how far the Rayleigh numbers so determined fall below the critical Rayleigh numbers, above which the flow is unstable, as determined by the linear theory [11]. It is found that, unlike the case of classical Bénard cells, the gray area, or area of ignorance, is quite large, indicating the need for some improvement of the energy method to give sharper lower bounds on the Rayleigh number.

1. Introduction. The energy method as applied to hydrodynamic stability is almost as old as the history of hydrodynamic stability itself, having its origin in the work of Orr [6]. The idea of the growth or decay of the kinetic energy of a disturbance was already very evident in Reynolds' work [8]. The method lay dormant for many years, until the paper of Serrin [9] gave it new life. Since then the method has enjoyed extensive development, as witnessed by Joseph's book [5], in which an authoritative account of the method can be found.

The energy method achieves its greatest triumph when it is applied to classical Bénard cells. In this case the Rayleigh number below which the fluid is stable, as determined by the energy method, is exactly the same as the Rayleigh number above which it is unstable, as determined by the linear theory. There is no gray area or area of ignorance. It is then natural to conjecture that for problems of convective instability the gray area between the lower curve for the Rayleigh number provided by the energy theory and the upper curve provided by the linear theory) would be relatively small. But this is a mere conjecture. It is desirable to find, in some specific instances, just how large the gray area is.

In this paper we shall consider the time-periodic temperature distribution treated by Yih and Li [11], who provided the results of linear theory via the Floquet theory. The nonlinear theory developed for time-dependent temperature fields by Joseph [4], Davis [2],

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and Homsy [3] is then applied to find the "best" lower bounds of the Rayleigh number, below which the fluid is stable. The numerical results to be presented will show that the gray area is quite large, indicating the need to improve the energy theory in order to give sharper results for the lower bounds.

The present problem has already been considered by Carmi [1]. Were his results correct, there would be little need for the present paper. As we shall see later, Carmi's results not only differ significantly from ours, but also contradict many well-established results, including the well-known one that the critical Rayleigh number of ordinary Bénard cells according to the linear theory agrees completely with that according to the nonlinear theory. Thus, no reliable results on the Rayleigh number (below which the fluid is stable for arbitrary disturbance) according to the nonlinear theory exist; and, as far as time-periodic temperature fields are concerned, the energy theory promoted by so many previous investigators, straightforward and uncomplicated though it is, is as yet an empty construction without reliable numerical substantiation, and its implied usefulness is not demonstrated. It is this state of affairs that justifies our calculation and the presentation of its results here.

2. The primary temperature distribution. The primary temperature distribution is that considered by Yih and Li [11]. The fluid is bounded above by a plate at $x_3 = d/2$ and below by a plate at $x_3 = -d/2$. Cartesian coordinates (x_1, x_2, x_3) are used, with x_3 measured vertically upward. The temperature at the upper plate is kept at $T_1 + T_2 \cos \omega_* t$ and that at the lower plate at $T_0 - T_2 \cos \omega_* t$, t being the time and ω_* equal to 2π times the frequency of the temperature variation. Only cases of $T_0 \geqslant T_1$ will be considered. The following dimensionless parameters are used:

$$\left.\begin{aligned} \tau &= t\kappa/d^2, \qquad & (x, y, z) &= (1/d)(x_1, x_2, x_3), \\ \omega &= \omega_* d^2/\kappa, \qquad & \bar{\theta} &= (\bar{T} - T_1)/(T_0 - T_1), \end{aligned}\right\} \tag{1}$$

where κ is the thermal diffusivity and $\bar{T}$ the primary temperature. The dimensionless heat-diffusion equation is then

$$\partial\bar{\theta}/\partial\tau = \partial^2\bar{\theta}/\partial z^2. \tag{2}$$

The boundary conditions are

$$\bar{\theta} = 1 - b\cos\omega\tau \quad \text{at } z = -\tfrac{1}{2}, \tag{3}$$

$$\bar{\theta} = b\cos\omega\tau \quad \text{at } z = \tfrac{1}{2}, \tag{4}$$

$$b = T_2/(T_0 - T_1). \tag{5}$$

The solution of the differential system (1)–(5) is

$$\bar{\theta} = \tfrac{1}{2} - z + bF(z, \tau), \tag{6}$$

where

$$F(z, \tau) = (B\cos\omega\tau - C\sin\omega\tau)\sinh\beta z\cos\beta z \\ - (C\cos\omega\tau + B\sin\omega\tau)\cosh\beta z\sin\beta z, \tag{7}$$

with

$$B = -\sinh\beta'\cos\beta'/(\sinh^2\beta' + \sin^2\beta'), \qquad C = -B\tan\beta'\coth\beta', \tag{8}$$

and

$$\beta = \left(\tfrac{1}{2}\omega\right)^{1/2}, \qquad \beta' = \tfrac{1}{2}\beta. \tag{9}$$

If ρ_0 is the density at temperature T_0 and at the prevailing pressure, the density at temperature T not too far from T_0 is

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right], \tag{10}$$

where α is the thermal-expansion coefficient of the fluid. The Boussinesq approximation will be made.

If $T_0 = T_1$, we set

$$\bar{\theta} = \left(\bar{T} - T_0\right)/T_0. \tag{11}$$

Then the primary temperature distribution satisfies

$$\bar{\theta} = \pm\cos\omega\tau \quad \text{at } z = \pm\tfrac{1}{2}, \tag{12}$$

and the solution for the primary temperature field is simply

$$\bar{\theta} = F(z, \tau). \tag{13}$$

The Rayleigh number is defined by

$$R = g\alpha(T_0 - T_1)d^3/(\kappa\nu) \tag{14}$$

if T_0 is not equal to T_1, and by

$$R = g\alpha T_2 d^3/(\kappa\nu) \tag{15}$$

if $T_0 = T_1$, ν being the kinematic viscosity and g the gravitational acceleration.

3. The energy method. The Navier–Stokes equation of motion and the heat equation are, in dimensionless forms, with $\theta = (T - T_1)/(T_0 - T_1)$ or $(T - T_0)/T_0$,

$$\sigma^{-1}(\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p + \nabla^2\mathbf{v} + R\theta\mathbf{k}, \tag{16}$$

$$\partial\theta/\partial t + \mathbf{v}\cdot\nabla\theta = \nabla^2\theta - w(\partial\bar{\theta}/\partial z), \tag{17}$$

where σ is the Prandtl number, $\mathbf{v}$ the velocity vector, p the pressure, and $\mathbf{k}$ the unit vector in the z direction. Then [4, p. 164] we obtain from (16) and (17), by integration in the fluid domain after taking appropriate inner products,

$$\frac{1}{2}\frac{dK(\mathbf{v})}{dt} = \frac{1}{2\sigma}\frac{d}{dt}\left\langle |\mathbf{v}|^2 \right\rangle = -\left\langle \nabla\mathbf{v}:\nabla\mathbf{v} \right\rangle + R\left\langle \omega\theta \right\rangle, \tag{18}$$

$$\frac{1}{2}\frac{d\Theta(\theta)}{dt} = \frac{1}{2}\frac{d}{dt}\left\langle \theta^2 \right\rangle = -\left\langle |\nabla\theta|^2 \right\rangle - \left\langle \theta w\frac{\partial\bar{\theta}}{\partial z} \right\rangle. \tag{19}$$

The sign $\langle\ \rangle$ means integration over the fluid domain. It is implicitly assumed that the integrals converge. If not, one can consider the limits of the integrals divided by the area in the x-y plane over which the integrals are performed (apart from the integration in the z direction).

Following Joseph [4] and Homsy [3], we use the energy functional

$$E' = K + \lambda R\Theta(\theta) \tag{20}$$

and obtain from (18) and (19)

$$\frac{1}{2}\frac{dE'}{dt} = R\left(\left\langle w\theta \right\rangle - \lambda\left\langle w\theta\frac{\partial\bar{\theta}}{\partial z} \right\rangle\right) - \left\langle \nabla\mathbf{v}:\nabla\mathbf{v} + \lambda R|\nabla\theta|^2 \right\rangle. \tag{21}$$

With the substitutions

$$\phi = (\lambda R)^{1/2}\theta, \qquad E = K(\mathbf{v}) + \Theta(\phi), \tag{22}$$

one obtains [3, p. 132]

$$D^{-1}\tfrac{1}{2}dE/dt \leqslant -1 + R^{1/2}/\rho_\lambda, \tag{23}$$

where

$$D \equiv \left\langle \nabla\mathbf{v} : \nabla\mathbf{v} + |\nabla\phi|^2 \right\rangle$$

and ρ_λ is given by

$$\frac{1}{\rho_\lambda} = \max_h \left\{ \frac{\langle w\phi \rangle}{\lambda^{1/2}} - \lambda^{1/2} \left\langle w\phi \frac{\partial \bar{\theta}}{\partial z} \right\rangle \right\} D^{-1}. \tag{24}$$

D is positive definite. Homsy [3] assumed it to be equal to unity for convenience. But that is not necessary. The letter h signifies the Hilbert space in which w and ϕ are allowed to roam, provided they satisfy the boundary conditions.

If R is less than ρ_λ^2, the fluid is stable against all disturbances. The task, then, is to determine ρ_λ, with $\mathbf{v}$ subject to the restriction of the continuity equation

$$\nabla \cdot \mathbf{v} = 0. \tag{25}$$

The Euler–Lagrange equations obtained from (24) are

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{1/2}} - \lambda^{1/2}\frac{\partial \bar{\theta}}{\partial z} \right) w + \nabla^2 \phi = 0, \tag{26}$$

$$\frac{\rho_\lambda}{2}\left(\frac{1}{\lambda^{1/2}} - \lambda^{1/2}\frac{\partial \bar{\theta}}{\partial z} \right) \phi\mathbf{k} + \nabla^2 \mathbf{v} - \nabla\tilde{p} = 0, \tag{27}$$

where $\tilde{p}$ is the Lagrangian multiplier of the left-hand side of (25) in the maximizing procedure. We shall now determine the best or greatest ρ_λ from (25)–(27) and the boundary conditions

$$\mathbf{v} = 0 \quad \text{and} \quad \phi = 0 \quad \text{at } z = \pm\tfrac{1}{2}. \tag{28}$$

The equations to be solved permit a spectral analysis. We shall assume

$$(\tilde{p}, \phi, \mathbf{v}) = f(x, y)(\hat{p}(z), \hat{\phi}(z), \hat{\mathbf{v}}(z)), \tag{29}$$

where, with a denoting a wave number,

$$f_{xx} + f_{yy} + a^2 f = 0. \tag{30}$$

In particular,

$$w = f(x, y)\hat{w}(z). \tag{31}$$

Substituting (29) and (30) into (26) and (27) and eliminating $\hat{p}$ as in the linear theory (see, for instance, Pellew and Southwell [7]), one obtains, after a brief calculation,

$$(D^2 - a^2)^2 \hat{w} = \tfrac{1}{4} R_\lambda a^2 H \hat{\theta}, \tag{32}$$

$$(D^2 - a^2)\hat{\theta} = -H\hat{w}, \tag{33}$$

where $D = \partial/\partial z$, R_λ has been written for ρ_λ^2,

$$\hat{\theta} = 2R_\lambda^{-1/2}\hat{\phi}, \tag{34}$$

$$H = \lambda^{-1/2} - \lambda^{1/2}\partial\bar{\theta}/\partial z. \tag{35}$$

The boundary conditions are

$$\hat{w} = D\hat{w} = \hat{\theta} = 0 \quad \text{at } z = \pm\tfrac{1}{2}. \tag{36}$$

The four boundary conditions involving w arise from the no-slip condition at the solid boundaries, the equation of continuity having been used to reach the conditions on $D\hat{w}$. We now endeavor to determine the best R_λ by solving the eigenvalue system defined by (32), (33), and (36).

4. Determination of R_e. The object is to determine, by the best choice of λ, the best or largest R_λ, which, however, must be the minimum for all values of a and the minimum for all values of τ. The parameters are ω, b, τ, λ, and a, and

$$R_\lambda = R_\lambda(\omega, b, \tau, \lambda, a), \tag{37}$$

or, for $T_0 = T_1$,

$$R_\lambda = R_\lambda(\omega, \tau, \lambda, a). \tag{38}$$

The Rayleigh number sought in the energy theory is

$$R_e = \max_\lambda \min_\tau \min_a R_\lambda(\omega, b, \tau, \lambda, a), \tag{39}$$

where b should be dropped if $T_0 = T_1$.

As pointed out by Homsy [3], λ is a priori a function of τ, but if so, $\lambda(\tau)$ must not increase with τ in order that conclusions about the decrease of $K(\mathbf{v})$ and $\Theta(\theta)$ can be reached. The primary temperature field being time-periodic, this restriction on $\lambda(\tau)$ demands in effect that λ be constant for all τ. With this in mind, we seek to solve (32) and (33) by assuming

$$\hat{\theta} = \sum_{n=1}^{\infty} B_n(\tau)\cos(2n-1)\pi z, \tag{40}$$

$$\hat{w} = \sum_{n=1}^{\infty} A_n(\tau)\phi_n(z), \tag{41}$$

where $\phi_n(z)$ must satisfy the boundary conditions on $\hat{w}$. We choose $\phi_n(z)$ by demanding that it satisfy these boundary conditions and

$$(d^2 - a^2)^2\phi_n = \cos(2n-1)\pi z. \tag{42}$$

A brief calculation then gives

$$\phi_n(z) = P_n\cosh az + Q_n z\sinh az + C_n^2\cos(2n-1)\pi z, \tag{43}$$

where

$$(P_n, Q_n) = \frac{(-1)^n(2n-1)\pi C_n^2}{a + \sinh a}\left(\sinh\frac{a}{2}, 2\cosh\frac{a}{2}\right), \tag{44}$$

$$C_n = \frac{1}{a^2 + (2n-1)^2\pi^2}. \tag{45}$$

We have chosen the cosine functions in (40) and (42) because the corresponding eigenvalue for R_λ is lower than that for sine functions in (40) and (42).

Substituting (40) and (41) into (32), and using (42), we obtain

$$\sum_{n=1}^{\infty}\left[\frac{R_\lambda a^2}{4}HB_n(\tau) - A_n(\tau)\right]\cos(2n-1)\pi z = 0. \tag{46}$$

Because the cosine functions are orthogonal in the range $(-\frac{1}{2}, \frac{1}{2})$, we have

$$A_m(\tau) = R_\lambda \sum_{n=1}^{\infty} b_{mn}B_n(\tau), \tag{47}$$

where

$$b_{mn} = \frac{a^2}{2}\int H\cos(2m-1)\pi z\cos(2n-1)\pi z\,dz, \tag{48}$$

the integration being over the range $(-\frac{1}{2}, \frac{1}{2})$.

Similarly, from (33) we obtain

$$B_m(\tau) = \sum_{n=1}^{\infty} a_{mn}A_n(\tau), \tag{49}$$

where

$$a_{mn} = 2C_m\int H\phi_n\cos(2m-1)\pi z\,dz, \tag{50}$$

the integration being over the same range. Substituting (49) into (47) and truncating at $n = N$, we have

$$B_m(\tau) = R_\lambda \sum_{n=1}^{N} c_{mn}B_n(\tau), \tag{51}$$

where

$$c_{mn} = \sum_{k=1}^{N} a_{mk}b_{kn} \tag{52}$$

and

$$(m, n) = 1, 2, 3, \ldots, N. \tag{53}$$

Let

$$\mathbf{M} = (c_{mn})_{N\times N}, \qquad \mathbf{b} = (B_1, B_2, \ldots, B_N)^T. \tag{54}$$

Then (51) can be written as

$$\mathbf{M}\cdot\mathbf{b} = \frac{1}{R_\lambda}\mathbf{b}, \tag{55}$$

from which the eigenvalue R_λ is determined, since $\mathbf{b}$ is not a zero vector.

Some of the details for calculating a_{mk} and b_{kn} (and therefore c_{mn}) are given in the appendix.

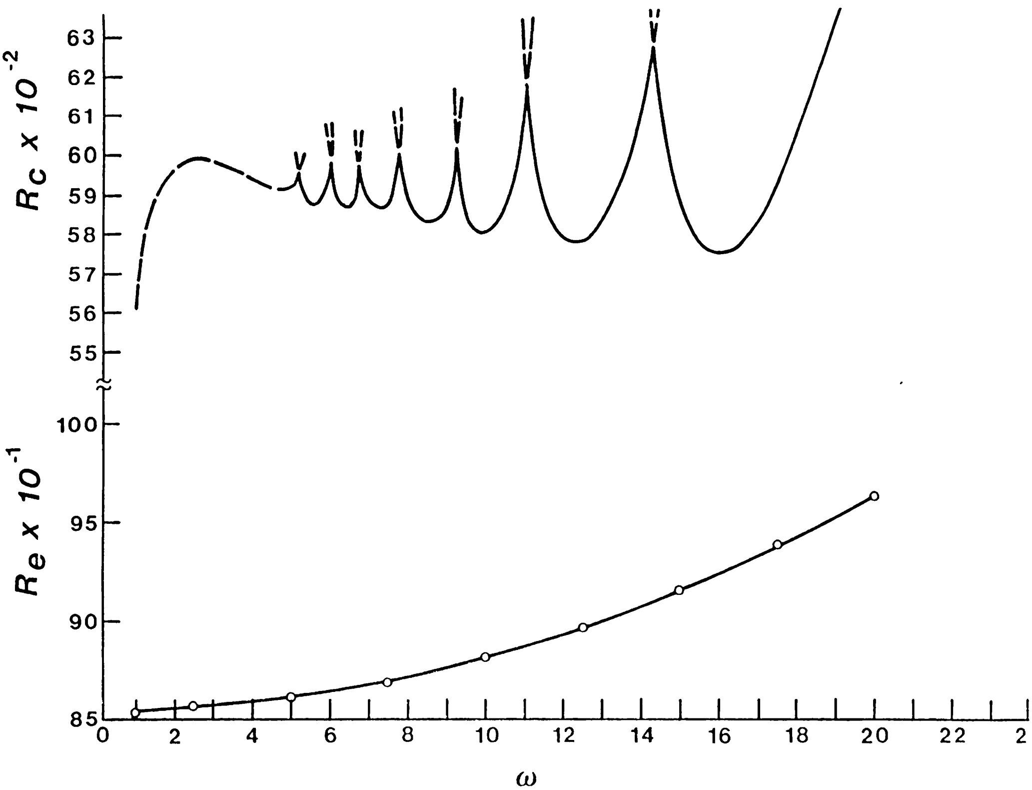

FIG. 1. Comparison of the critical Rayleigh number R_c from the linear theory with the R_e from the energy theory for the special case $T_0 = T_1$ and various values of ω from 1 onward. Note the break of scale of the vertical axis.

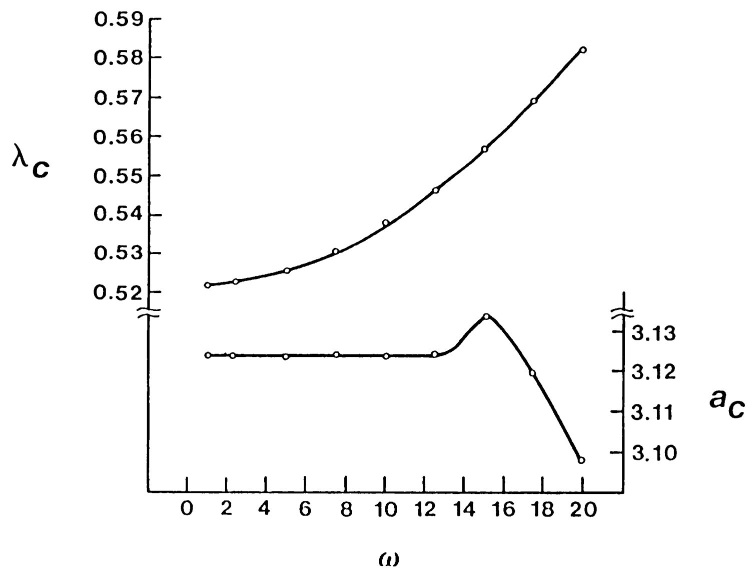

FIG. 2. The optimum λ, denoted by λ_c, and the most "dangerous" or "unstable" a, denoted by a_c, for the case $T_0 = T_1$. Note the break of scale of the vertical axis.

5. Results. By assuming various values of λ for various values of a and τ, but keeping the same λ for all τ, we have obtained the values of R_e defined by (39).

For the special case of $T_0 = T_1$, R_e for various values of ω, starting from $\omega = 1$, is given in Fig. 1. The corresponding values of λ and a are given in Fig. 2. As can be seen from Fig. 1, the R_e values are very much smaller than the R_c values given by the linear theory of Yih and Li [11], reproduced here for comparison.

For $T_0 > T_1$, the R_e values are given in Fig. 3, in which the R_c values determined by the linear theory (the Floquet theory) of Yih and Li [11] are reproduced for comparison. Again, the R_e curve is far below the R_c curve, although at $b = 0$ the two curves meet, as expected, since for $b = 0$ we have the classical Bénard cells. The values of R_e with corresponding λ and a are given in Table 1, for $\omega = 5$.

We also give in Table 2 the values of R_e for various values of b and w. For a definite value of b, R_e does not seem to vary much with ω.

Note that in Figs. 1 and 3, the R_c values given by the linear theory are for $\sigma = 0.73$ (σ being the Prandtl number), but the R_e values given by the energy theory are independent of σ. Also, for $T_0 > T_1$, and $b = 0$, the exact values for λ, a, and R_e are 1, 3.117, and 1707.76, respectively.

Our conclusion is that for time-dependent temperature fields the gray area between the R_c curve and the R_e curve is so large that other nonlinear theories, such as that first used by Stuart [10] for stability of parallel flows, which could reduce the gray area by providing sharper lower bounds for R (for sufficient conditions for stability), are still desirable for convective-instability problems involving time-dependent temperature fields.

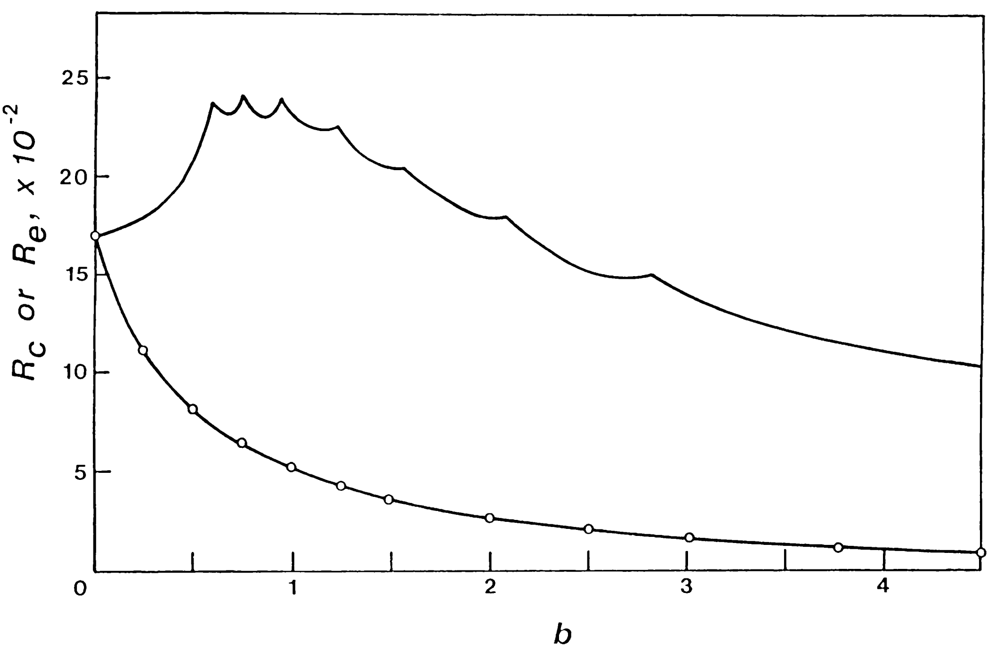

FIG. 3. Comparison of the critical Rayleigh number R_c from the linear theory with the R_e from the energy theory for $\omega = 5$ and various values of b, for $T_0 > T_1$.

TABLE 1. R_e with corresponding λ and a, for $\omega = 5$ and $T_0 > T_1$.

b	λ	a	R_e
0.0	1.0121	3.1240	1707.73
0.25	0.6862	3.1240	1141.47
0.50	0.5239	3.1240	857.04
0.75	0.5072	3.1240	677.31
1.00	0.3698	3.1240	571.03
1.25	0.2030	3.1240	476.27
1.50	0.1729	3.1240	414.87
2.00	0.1410	3.1240	333.16
2.50	0.1169	3.1240	277.44
3.00	0.0758	3.1240	222.13
3.75	0.0712	3.1240	189.78
4.50	0.0488	3.1240	151.34

TABLE 2. R_e for various values of b and ω, for $T_0 > T_1$.

b \ ω	1.0	2.5	5.0	10.0	20.0
0.0	1707.73	1707.73	1707.73	1707.73	1707.73
0.50	853.64	854.39	857.04	867.33	904.41
0.75	674.02	674.44	677.31	687.34	730.94
1.00	567.99	568.66	571.03	580.30	614.40
1.25	472.55	473.34	476.27	486.39	512.92
1.50	412.35	412.94	414.87	422.20	464.36
2.50	275.76	276.21	277.44	283.18	304.46
4.50	157.59	157.79	151.34	155.91	175.26

6. Discussion. Finally, because Carmi [1] has already carried out a calculation for the problem investigated by us, we feel obliged to examine his numerical results and compare them with ours. Carmi's definition of the Rayleigh number, denoted by R_a, is the same as that used by everybody else, and his R is the square root of his R_a. His σ is our ω, and his ε is our b. Examination of his results reveals the following:

1. For the linear theory, the upper curve in his Fig. 3 gives a value for R (of R_L, the subscript indicating "linear") greater than 183, as far as one can read from the vertical axis, on which R_L is 183 and $\varepsilon = 0$. That makes his

$$R_a = R^2 = 33{,}490 \quad \text{for } \varepsilon = 0,$$

which greatly contradicts the well-known fact that R_a for $\varepsilon = 0$ is 1,708. We do not know where Carmi got his curve for R_L. It is not the curve given by Yih and Li [11] in their linear theory, which shows alternating loops for synchronicity and half-frequency and shows a critical Rayleigh number of 1,708 at $b = 0$, b being Carmi's ε.

2. From Carmi's Fig. 3, one gets

$$R_E = 63 \quad \text{at } \varepsilon = 0,$$

which makes his R_a by the energy theory, which we denote here $(R_a)_E$ for clarity, equal to

$$63^2 = 3969,$$

which is much greater than 1,708, the value it should agree with.

3. In any case for $\varepsilon = 0$ his R_E and R_L are not equal, as they should be, but differ by a factor of 183/63, or nearly 3. The corresponding Rayleigh numbers differ by nearly a factor of 9. Equality of these Rayleigh numbers is strictly required, since he used $\lambda = 1$, which is the *correct* λ to use for $\varepsilon = 0$, as is well known. In our Fig. 3, the Rayleigh number for the linear theory is equal to that for the nonlinear theory, for the case b (Carmi's ε) equal to zero, and our curve for R_c is taken from the paper of Yih and Li [11]. Our Fig. 3 (for $\omega = 5$) and Carmi's Fig. 3 (for $\sigma = 1$, σ being our ω) are very different.

We emphasize that the main difficulty in determining our R_e (which corresponds to Carmi's R_E^2) lies in determining the optimum λ, which should give the maximum of the minimum (for various values of τ and a) of R_e. This is laborious work, which Carmi, taking $\lambda = 1$, did not do. But in view of the items stated above, this is not the source of the errors in Carmi's results. After taking $\lambda = 1$, his calculation should have produced a Rayleigh number equal to 1,708 for $\varepsilon = 0$.

7. Appendix. The details for calculating a_{mk} and b_{kn}, given by (50) and (48), are given below.

From (6), we have

$$\frac{\partial \bar{\theta}}{\partial z} = a_0 + b_0 \frac{\partial F(z, \tau)}{\partial z},$$

where

$$\begin{aligned} a_0 &= -1, \quad b_0 = b \quad \text{if } T_0 > T_1, \\ a_0 &= 0, \quad\; b_0 = 1 \quad \text{if } T_0 = T_1. \end{aligned}$$

Hence

$$H = \lambda^{-1/2} - \lambda^{1/2} \frac{\partial \bar{\theta}}{\partial z} = a_0 + c_0 \cosh \beta z \cos \beta z + s_0 \sinh \beta z \sin \beta z,$$

where λ, for $T_0 > T_1$, satisfies

$$\begin{aligned} a_0 &= \lambda^{-1/2} + \lambda^{1/2}, \\ c_0 &= -b_0 \beta [(B - C) \cos \omega\tau - (B + C) \sin \omega\tau] \lambda^{1/2}, \\ s_0 &= -b_0 \beta [(B + C) \cos \omega\tau + (B - C) \sin \omega\tau] \lambda^{1/2}, \end{aligned}$$

with b, B, C, and β given by (5), (8), and (9).

Then

$$a_{mk} = 2C_m\left\{a_0\left[P_k I_1(a,m) + Q_k I_4(a,m) + \frac{C_k^2}{2}\delta_{mk}\right]\right.$$
$$+c_0\left[P_k I_5(a,\omega,m) + Q_k I_7(a,\omega,m) + C_k^2 I_2(m,k)\right]$$
$$\left.+s_0\left[P_k I_6(a,\omega,m) + Q_k I_8(a,\omega,m) + C_k^2 I_3(m,k)\right]\right\},$$

$$b_{kn} = \frac{a^2}{2}\left\{\frac{a_0}{2}\delta_{kn} + c_0 I_2(n,k) + s_0 I_3(m,k)\right\},$$

where P_k, Q_k, and C_k are defined by (44) and (45), and the integrals I are defined as follows (in complex form for brevity):

$$h(m,z) \equiv \cos(2m-1)\pi z,$$

$$I_1(a,m) + iI_4(a,m) = \int (\cosh az + iz\sin az)h(m,z)\,dz,$$

$$I_2(n,k) + iI_3(n,k) = \int \cosh(1+i)\beta z h(n,z)h(k,z)\,dx,$$

$$I_5(a,\omega,m) + iI_6(a,\omega,m) = \int \cosh(1+i)\beta z\cdot\cosh az\cdot h(m,z)\,dz,$$

$$I_7(a,\omega,m) + iI_8(a,\omega,m) = \int \cosh(1+i)\beta z\cdot z\sinh az\cdot h(m,z)\,dz.$$

The limits of integration are from $-\frac{1}{2}$ to $\frac{1}{2}$.

To evaluate the I's, let

$$R_0(\alpha,\gamma) + iR_1(\alpha,\gamma) = \int \cosh(\alpha+i\gamma)z\,dz,$$

$$R_3(\alpha,\gamma) + iR_2(\alpha,\gamma) = \int z\sinh(\alpha+i\gamma)z\,dz,$$

$$\hat{A}(\alpha,\gamma) = \sinh\frac{\alpha}{2}\sin\frac{\gamma}{2}, \qquad \hat{B}(\alpha,\gamma) = \sinh\frac{\alpha}{2}\cos\frac{\gamma}{2},$$

$$\hat{C}(\alpha,\gamma) = \cosh\frac{\alpha}{2}\sin\frac{\gamma}{2}, \qquad \hat{D}(\alpha,\gamma) = \cosh\frac{\alpha}{2}\cosh\frac{\gamma}{2},$$

$$E(\alpha,\gamma) = \alpha^2+\gamma^2.$$

Then

$$R_0(\alpha,\gamma) = \frac{2\alpha}{E}\left(\hat{B} + \frac{\gamma}{\alpha}\hat{C}\right),$$

$$R_1(\alpha,\gamma) = \frac{2}{\alpha}\hat{C} - \frac{\gamma}{\alpha}R_0(\alpha,\gamma),$$

$$R_2(\alpha,\gamma) = \frac{\alpha}{E}\left[\hat{A} - \frac{2}{\alpha}\hat{C} - \frac{\gamma}{\alpha}\hat{D} + \frac{2\gamma}{\alpha}R_0(\alpha,\gamma)\right],$$

$$R_3(\alpha,\gamma) = \frac{\gamma}{E}\left[A - \frac{2}{\alpha}\hat{C} + \frac{\alpha}{\gamma}\hat{D} + \frac{\gamma^2-\alpha^2}{\alpha\gamma}R_0(\alpha,\gamma)\right].$$

Finally,

$$
\begin{aligned}
I_1(a, m) &= R_0(a, (2m-1)\pi], \\
I_2(n, k) &= \tfrac{1}{4}\{ R_0[\beta, \beta + 2(n+k-1)\pi] + R_0[\beta, \beta - 2(n+k-1)\pi] \\
&\qquad + R_0[\beta, \beta + 2(n-k)\pi] + R_0[\beta, \beta - 2(n-k)\pi]\}, \\
I_3(n, k) &= \tfrac{1}{4}\{ R_1[\beta, \beta + 2(n+k-1)\pi] + R_1[\beta, \beta - 2(n+k-1)\pi] \\
&\qquad + R_1[\beta, \beta + 2(n-k)\pi] + R_1[\beta, \beta - 2(n-k)\pi]\}, \\
I_4(a, m) &= R_3[a, (2m-1)\pi], \\
I_5(a, \omega, m) &= \tfrac{1}{4}\{ R_0[\beta + a, \beta + (2m-1)\pi] + R_0[\beta + a, \beta - (2m-1)\pi] \\
&\qquad + R_0[\beta - a, \beta + (2m-1)\pi] + R_0[\beta - a, \beta - (2m-1)\pi]\}, \\
I_6(a, \omega, m) &= \tfrac{1}{4}\{ R_1[\beta + a, \beta + (2m-1)\pi] + R_1[\beta + a, \beta - (2m-1)\pi] \\
&\qquad + R_1[\beta - a, \beta + (2m-1)\pi] + R_1[\beta - a, \beta - (2m-1)\pi], \\
I_7(a, \omega, m) &= \tfrac{1}{4}\{ R_3[\beta + a, \beta + (2m-1)\pi] + R_3[\beta + a, \beta - (2m-1)\pi] \\
&\qquad - R_3[\beta - a, \beta + (2m-1)\pi] - R_3[\beta - a, \beta - (2m-1)\pi]\}, \\
I_8(a, \omega, m) &= \tfrac{1}{4}\{ R_2[\beta + a, \beta + (2m-1)\pi] + R_2[\beta + a, \beta - (2m-1)\pi] \\
&\qquad - R_2[\beta - a, \beta + (2m-1)\pi] - R_2[\beta - a, \beta - (2m-1)\pi]\}.
\end{aligned}
$$

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