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DISCRETE APPROXIMATION OF THE POISSON-VLASOV SYSTEM*

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Abstract. Solutions of the Poisson–Vlasov system of equations that have spherical symmetry are considered. A numerical scheme which exploits the symmetry is proposed and is shown to converge in a pointwise sense. Prior convergence results due to Neunzert conclude convergence in only a weak topology although they require no symmetry.

The motion of a continuous mass (charge) distribution acting under the influence of its own gravitational (electric) field without collisions is described by the Poisson–Vlasov system of equations:

$$\psi_t + v \cdot \nabla_x \psi + F(x,t) \cdot \nabla_v \psi = 0, \tag{1}$$

$$F(x,t) = -\nabla_x U(x,t), \tag{2}$$

$$U(x,t) = \gamma \int \frac{\sigma(y,t)}{|x-y|}\, dy, \tag{3}$$

$$\sigma(x,t) = \int \psi(x,v,t)\, dv. \tag{4}$$

Here $\psi = \psi(x,v,t)$ ($x \in \mathbf{R}^3$, $v \in \mathbf{R}^3$, $t \geqslant 0$) is a scalar function describing the density in phase space, σ is the mass (charge) density, and F is the gravitational (electric) field. The constant γ is -1 in the gravitational case and $+1$ in the electrical case.

The Cauchy problem is to solve (1), (2), (3), and (4) with the initial condition

$$\psi(x,v,0) = \mathring{\psi}(x,v) \tag{5}$$

for all $t > 0$. Local existence of classical solutions was established by Kurth [4]. Batt [2] showed global existence in the gravitational case for spherically symmetric data (i.e., $\mathring{\psi}$ depends only on $|x|$, $|v|$, and $x \cdot v$). Wollman [6] extended this result to the electrical case, $\gamma = +1$. Horst [3] further extended these results to include data with cylindrical symmetry (i.e., $\mathring{\psi}$ depends only on $\sqrt{x_1^2 + x_2^2}$, $\sqrt{v_1^2 + v_2^2}$, $x_1 v_1 + v_2 v_2$, x_3, and v_3). Recently, Bardos

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and Degond [1] showed global existence for data that is sufficiently small in a certain norm but need not have any symmetry. The general existence problem is still open.

A natural way to generate a discrete approximation to a solution of the above Cauchy problem is to approximate the data function $\mathring{\psi}$ by a large number of point masses and then follow the time evolution of the resulting many-body problem; this is called the particle-in-cell method. There is no hope for obtaining pointwise convergence for such a method since the initial error of approximating the data by point masses immediately introduces a large pointwise error. However, Neunzert [5] has established the convergence of the particle-in-cell method in a weak topology under quite general conditions. Here we consider smooth data with spherical symmetry and propose a discrete model of the continuous system, (1)–(5). In order to preserve the symmetry, the basic unit of approximation will be a spherical shell of mass (charge) rather than a point mass. Preserving the symmetry permits better estimates and this way pointwise convergence is established.

Preliminaries. We will consider only data $\mathring{\psi}$ of the form $\mathring{\psi}(x, v) = \mathring{\phi}(|x|, |v|, \cos^{-1}(\frac{x \cdot v}{|x||v|}))$ where $\mathring{\phi} \in C_0^1((0, \infty) \times (0, \infty) \times (0, \pi))$. Hence it is natural to use the variables $r = |x|$, $u = |v|$, and $\alpha = \cos^{-1}(x \cdot v/|x||v|)$. Batt showed [2] that the resulting solution ψ is continuously differentiable in x, v, and t for all $t > 0$ and that ψ depends only on r, u, α, and t (i.e., the time evolution preserves the symmetry). We will write $\psi(x, v, t) = \phi(r, u, \alpha, t)$ and note that

$$U(x, t) = \gamma \int_0^\infty \frac{\rho(\eta, t)}{\max(r, \eta)} 4\pi\eta^2 \, d\eta \tag{6}$$

and

$$F(x, t) = \gamma M(r, t) r^{-3} x \tag{7}$$

where

$$M(r, t) = \int_0^r \rho(\eta, t) 4\pi\eta^2 \, d\eta \quad \text{and} \quad \rho(r, t) = \sigma(x, t). \tag{8}$$

Given $(r, u, \alpha) \in (0, \infty) \times (0, \infty) \times (0, \pi)$ the characteristic associated with this point is given by $z = z(r, u, \alpha, t)$, $q = q(r, u, \alpha, t)$, and $\theta = \theta(r, u, \alpha, t)$, where

$$z_t = q\cos\theta \quad \text{and} \quad z(r, u, \alpha, 0) = r, \tag{9}$$

$$q_t = \gamma M(z, t) z^{-2} \cos\theta \quad \text{and} \quad q(r, u, \alpha, 0) = u, \tag{10}$$

$$\theta_t = \left(-\gamma M(z, t) z^{-2} q^{-1} - q z^{-1}\right) \sin\theta \quad \text{and} \quad \theta(r, u, \alpha, 0) = \alpha. \tag{11}$$

Note that $zq \sin\theta$ is independent of t, so we also have

$$z_{tt} - (ru\sin\alpha)^2 z^{-3} - \gamma M(z, t) z^{-2} = 0, \tag{12}$$

$z(r, u, \alpha, 0) = r$, and $z_t(r, u, \alpha, 0) = u\cos\alpha$.

Let $\mathscr{S}$ be the support of $\mathring{\phi}$ and recall that $\mathscr{S}$ is assumed to be a compact subset of $(0, \infty) \times (0, \infty) \times (0, \pi)$. Hence there exist positive constants R_0, l_0, and L_0 such that if $(r, u, \alpha) \in \mathscr{S}$ then $r \leqslant R_0$ and $l_0 \leqslant ru \sin\alpha \leqslant L_0$. A consequence of work of Batt [2] and Wollman [6] is that there exists a positive constant U_0 (depending only on $\mathring{\phi}$) such that

$q(r,u,\alpha,t) \leqslant U_0$ for all $(r,u,\alpha) \in \mathscr{S}$ and $t \geqslant 0$. Hence $z(r,u,\alpha,t) \leqslant R_0 + U_0 t$ if $(r,u,\alpha) \in \mathscr{S}$. Also since $zq \sin\theta$ is independent of t, $l_0 \leqslant ru\sin\alpha = zq \sin\theta \leqslant zq \leqslant zU_0$ so $l_0 U_0^{-1} \leqslant z(r,u,\alpha,t)$ for $(r,u,\alpha) \in \mathscr{S}$ and $t \geqslant 0$. ϕ is constant on characteristics so $\|\phi(\cdot,\cdot,\cdot,t)\|_\infty = \|\mathring{\phi}\|_\infty$; hence $\rho(r,t) = \int_0^{U_0} \int_0^\pi \phi(r,u,\alpha,t) 2\pi \sin\alpha d\alpha u^2\, du \leqslant 2\pi^2 U_0^3 \|\mathring{\phi}\|_\infty$. To summarize, there exist positive constants r_0, R_0, l_0, L_0, U_0, and ρ_0 (depending only on $\mathring{\phi}$) such that

$$r_0 \leqslant z(r,u,\alpha,t) \leqslant R_0 + U_0 t, \tag{13}$$

$$l_0 \leqslant ru \sin\alpha = zq \sin\theta \leqslant L_0, \tag{14}$$

$$q(r,u,\alpha,t) \leqslant U_0 \tag{15}$$

for all $(r,u,\alpha,t) \in \mathscr{S} \times [0,\infty)$ and

$$\rho(r,t) \leqslant \rho_0 \quad \text{for all } r \geqslant 0 \text{ and } t \geqslant 0. \tag{16}$$

The letter C will denote a positive generic constant depending only on $\mathring{\phi}$, and $C(t)$ will denote an increasing positive function of t depending only on $\mathring{\phi}$.

Another fact that will be used is that the time evolution preserves measure in $\mathbf{R}^6$. If the characteristics of the original system are given by $X(x_0,v_0,t)$ and $V(x_0,v_0,t)$, where $X_t = V$, $X(x_0,v_0,0) = x_0$ and $V_t = F(X,t)$, $V(x_0,v_0,0) = v_0$, then this says

$$\begin{aligned}
&\int\int f(r,u,\alpha)\phi(r,u,\alpha,t)\, dv\, dx \\
&= \int\int f\left(|x|, |v|, \cos^{-1}\left(\frac{x\cdot v}{|x||v|}\right)\right)\psi(x,v,t)\, dv\, dx \\
&= \int\int f\left(|X|, |V|, \cos^{-1}\left(\frac{X\cdot V}{|X||V|}\right)\right)\psi(x_0,v_0,0)\, dv_0\, dx_0 \\
&= \int\int f(z,q,\theta)\mathring{\phi}(r,u,\alpha)\, dv_0\, dx_0
\end{aligned} \tag{17}$$

for all $f \in L^\infty((0,\infty)\times(0,\infty)\times(0,\pi))$. If f is taken to be identically equal to one, we get

$$\int\int \phi(r,u,\alpha,t)\, dv\, dx = \int\int \mathring{\phi}(r,u,\alpha)\, dv_0\, dx_0,$$

which is mass conservation. Take $M = \int\int \mathring{\phi}(r,u,\alpha)\, dv_0\, dx_0$ so that

$$M = \int\int \phi(r,u,\alpha,t)\, dv\, dx = \int_0^\infty \rho(r,t) 4\pi r^2\, dr. \tag{18}$$

Another conserved quantity is the energy

$$\begin{aligned}
E &= \int\int \psi(x,v,t) v^2\, dv\, dx + \int \sigma(x,t) U(x,t)\, dx \\
&= \int\int \phi(r,u,\alpha,t) u^2\, dv\, dx + \gamma \int \rho(r,t) \int_0^\infty \frac{\rho(\eta,t) 4\pi\eta^2}{\max(r,\eta)}\, d\eta\, dx.
\end{aligned} \tag{19}$$

Before introducing the discrete model let us consider the following continuous dependence lemma which involves only the continuous solution.

LEMMA I. There exists an increasing function $C_1(t)$ with $C_1(0) \geqslant 1$ (depending only on $\overset{\circ}{\phi}$) such that for all $t \geqslant 0$, $(r, u, \alpha) \in \mathscr{S}$, and $(\eta, w, \beta) \in \mathscr{S}$

$$|z(r,u,\alpha,t) - z(\eta,w,\beta,t)| + |z_t(r,u,\alpha,t) - z_t(\eta,w,\beta,t)| \leqslant C_1(t)|(r,u,\alpha) - (\eta,w,\beta)|. \tag{20}$$

This holds for $\gamma = 1$ and $\gamma = -1$.

Proof. For brevity we will temporarily use the notation $z_1(t) = z(r,u,\alpha,t)$, $z_2(t) = z(\eta,w,\beta,t)$, and $Q(t) = \sup\{|z_1(\tau) - z_2(\tau)| + |z_1'(\tau) - z_2'(\tau)|: 0 \leqslant \tau \leqslant t\}$.

From equation (12) for $t > 0$

$$\begin{aligned}
|z_1''(t) - z_2''(t)| &= \left|(ru\sin\alpha)^2 z_1^{-3}(t) + \gamma M(z_1(t),t)z_1^{-2}(t)\right| \\
&\qquad - \left|(\eta w\sin\beta)^2 z_2^{-3}(t) - \gamma M(z_2(t),t)z_2^{-2}(t)\right| \\
&\leqslant \left|(ru\sin\alpha)^2 - (\eta w\sin\beta)^2\right| z_1^{-3}(t) + (\eta w\sin\beta)^2\left|z_1^{-3}(t) - z_2^{-3}(t)\right| \\
&\quad + |M(z_1(t),t) - M(z_2(t),t)|z_1^{-2}(t) + M(z_2(t),t)\left|z_1^{-2}(t) - z_2^{-2}(t)\right| \\
&\leqslant (ru\sin\alpha + \eta w\sin\beta)|ru\sin\alpha - \eta w\sin\beta| r_0^{-3} \\
&\quad + L_0^2 z_1^{-3}(t)z_2^{-3}(t)\left|z_2^3(t) - z_1^3(t)\right| + |M(z_1(t),t) - M(z_2(t),t)|r_0^{-2} \\
&\quad + M z_1^{-2}(t)z_2^{-2}(t)\left|z_2^2(t) - z_1^2(t)\right| \\
&\leqslant 2L_0 r_0^{-3}|ru\sin\alpha - \eta w\sin\beta| \\
&\quad + L_0 r_0^{-6}\left(z_1^2(t) + z_1(t)z_2(t) + z_2^2(t)\right)|z_1(t) - z_2(t)| \\
&\quad + r_0^{-2}|M(z_1(t),t) - M(z_2(t),t)| + M r_0^{-4}(z_1(t) + z_2(t))|z_1(t) - z_2(t)|.
\end{aligned} \tag{21}$$

Now note that

$$\begin{aligned}
|ru\sin\alpha - \eta w\sin\beta| &\leqslant ru|\sin\alpha - \sin\beta| + r|u - w|\sin\beta + |r - \eta|w\sin\beta \\
&\leqslant L_0|\alpha - \beta| + R_0|u - w| + U_0|r - \eta| \leqslant C|(r,u,\beta) - (\eta,w,\beta)|
\end{aligned}$$

and

$$\begin{aligned}
|M(z_1(t),t) - M(z_2(t),t)| &= \left|\int_{z_1(t)}^{z_2(t)} 4\pi s^2\rho(s,t)\,ds\right| \\
&\leqslant 4\pi(R_0 + U_0 t)^2\rho_0|z_1(t) - z_2(t)|.
\end{aligned}$$

Using these in (21) we have

$$\begin{aligned}
&|z_1''(t) - z_2''(t)| \\
&\quad\leqslant 2L_0 r_0^{-3}C|(r,u,\alpha) - (\eta,w,\beta)| + L_0 r_0^{-6}3(R_0 + U_0 t)^2|z_1(t) - z_2(t)| \\
&\qquad + r_0^{-2}4\pi(R_0 + U_0 t)^2\rho_0|z_1(t) - z_2(t)| + M r_0^{-4}2(R_0 + U_0 t)|z_1(t) - z_2(t)| \\
&\quad= C|(r,u,\alpha) - (\eta,w,\beta)| + C(t)|z_1(t) - z_2(t)|.
\end{aligned} \tag{22}$$

By (22) we have for $t \geqslant 0$

$$
\begin{aligned}
&|z_1(t) - z_2(t)| + |z_1'(t) - z_2'(t)| \\
&\qquad \leqslant Q(0) + \int_0^t \left(|z_1'(\tau) - z_2'(\tau)| + |z_1''(\tau) - z_2''(\tau)|\right) d\tau \\
&\qquad \leqslant Q(0) + \int_0^t \left(Q(\tau) + C|(r,u,\alpha) - (\eta,w,\beta)| + C(\tau)Q(\tau)\right) d\tau \\
&\qquad \leqslant |r - u| + |u\cos\alpha - w\cos\beta| + Ct|(r,u,\alpha) - (\eta,w,\beta)| \\
&\qquad\quad + (1 + C(t))\int_0^t Q(\tau)\,d\tau \\
&\qquad \leqslant C(t)|(r,u,\alpha) - (\eta,w,\beta)| + C(t)\int_0^t Q(\tau)\,d\tau.
\end{aligned}
$$

Now for $0 \leqslant \tau \leqslant t$,

$$|z_1(\tau) - z_2(\tau)| + |z_1'(\tau) - z_2'(\tau)| \leqslant C(t)|(r,u,\alpha) - (\eta,w,\beta)| + C(t)\int_0^t Q(s)\,ds$$

so

$$Q(t) \leqslant C(t)|(r,u,\alpha) - (\eta,w,\beta)| + C(t)\int_0^t Q(s)\,ds.$$

Since $C(t)$ is increasing we have

$$Q(t) \leqslant C(T)|(r,u,\alpha) - (\eta,w,\beta)| + C(T)\int_0^t Q(s)\,ds$$

for all $t \in [0,T]$ and $T > 0$. By Gronwall's inequality

$$Q(t) \leqslant C(T)|(r,u,\alpha) - (\eta,w,\beta)|\exp(C(T)t)$$

for $t \in [0,T]$ so $Q(T) \leqslant C(T)|(r,u,\alpha) - (\eta,w,\beta)|$. This completes the proof of Lemma I.

The discrete model. To discretize the continuous mass distribution $\mathring{\phi}$ assume that $\mathscr{S}$ is the union of disjoint connected sets $S_1, \ldots, S_N$. Let

$$\delta = \max_i \sup\left\{\sqrt{(r-\bar{r})^2 + (u-\bar{u})^2 + (\alpha-\bar{\alpha})^2} : (r,u,\alpha), (\bar{r},\bar{u},\bar{\alpha}) \in S_i\right\}$$

and assume that $\delta < \min\{1, r_0\}$. For each $i \in \{1,2,\ldots,N\}$, choose $(r_i,u_i,\alpha_i) \in S_i$ and define $L_i = r_i u_i \sin\alpha_i$, $M_i = \iint_{S_i} \mathring{\phi}(r,u,\alpha)\,dv\,dx$, and $z_i(t) = z(r_i,u_i,\alpha_i,t)$.

From Eq. (12) $z_i''(t) - L_i^2 z_i^{-3}(t) - \gamma M(z_i(t),t)z_i^{-2}(t) = 0$ with $z_i(0) = r_i$ and $z_i'(0) = u_i\cos\alpha_i$, so for the discrete model we define the quantities $\bar{z}_1(t), \bar{z}_2(t), \ldots, \bar{z}_N(t)$ (which approximate $z_1(t), z_2(t), \ldots, z_N(t)$) and $\overline{M}(r,t)$ by the system of differential equations:

$$\bar{z}_i'' - L_i^2(\bar{z}_i)^{-3} - \gamma\overline{M}(\bar{z}_i,t)(\bar{z}_i)^{-2} = 0 \tag{23}$$

with $\bar{z}_i(0) = r_i$ and $\bar{z}_i'(0) = u_i\cos\alpha_i$ for $i = 1,2,\ldots,N$ and

$$\overline{M}(r,t) = \sum_{i=1}^N M_i\xi(r - \bar{z}_i(t)), \tag{24}$$

where we define $\xi: \mathbf{R} \to \mathbf{R}$ by

$$\xi(r) = \begin{cases} 0 & \text{if } r \leqslant 0, \\ r\delta^{-1} & \text{if } 0 < r < \delta, \\ 1 & \text{if } \delta \leqslant r. \end{cases} \tag{25}$$

$\overline{M}(r,t)$ is Lipschitz continuous in r for $r \geqslant 0$ and $r_0 \leqslant \bar{z}_i(0) \leqslant R_0$ for each i so there exists $T > 0$ such that a unique solution to the system (23) and (24) exists for $0 \leqslant t < T$ and satisfies

$$\tfrac{1}{2} r_0 \leqslant \bar{z}_i(t) \leqslant 2(R_0 + U_0 t) \tag{26}$$

for each i and for $0 \leqslant t < T$. We will take T to be the maximal such time and note that T is heavily dependent on the discretization; however it will be shown later that T tends to infinity as δ tends to zero.

THEOREM I. Let $\mathring{\phi} \in C_0^1((0,\infty) \times (0,\infty) \times (0,\pi))$, $z(r,u,\alpha,t)$ as in (9), (10), and (11), $M(r,t)$ as in (8), and $\bar{z}_i(t)$, $\overline{M}(r,t)$ as above. There exists a positive increasing function $C_2(t)$ (which is dependent only on $\mathring{\phi}$ and not on δ) such that for all $i \in \{1,2,\ldots,N\}$, $(r,u,\alpha) \in S_i$, and $t \in [0,T)$:

$$|z(r,u,\alpha,t) - \bar{z}_i(t)| + |z_t(r,u,\alpha,t) - \bar{z}_i'(t)| \leqslant C_2(t)\delta \tag{27}$$

and

$$|M(\eta,t) - \overline{M}(\eta,t)| \leqslant C_2(t)\delta \quad \text{for all } \eta \geqslant 0. \tag{28}$$

This holds for $\gamma = 1$ and $\gamma = -1$.

Proof. Define

$$\overline{\overline{M}}(r,t) = \sum_{i=1}^{N} M_i \xi(r - z_i(t)) \tag{29}$$

for $r \geqslant 0$ and $t \geqslant 0$. The proof is divided into parts A, B, and C. Part A will estimate $M - \overline{\overline{M}}$ and part B will estimate $\overline{\overline{M}} - \overline{M}$.

Part A. Fix $t > 0$ and $(\eta, w, \beta) \in \mathscr{S}$. Choose i so that $(\eta,w,\beta) \in S_i$ and define $I = \{ j: z_j(t) \leqslant z(\eta,w,\beta,t)\}$. Note that if $(r,u,\alpha) \in S_j$, then by Lemma I $|z(r,u,\alpha,t) - z_j(t)| \leqslant C_1(t)\delta$. Hence, if $(r,u,\alpha) \in S_j$ and $j \in I$,

$$z(r,u,\alpha,t) \leqslant z_j(t) + C_1(t)\delta \leqslant z(\eta,w,\beta,t) + C_1(t)\delta,$$

that is, if $j \in I$, then

$$S_j \subset \{(r,u,\alpha): z(r,u,\alpha,t) \leqslant z(\eta,w,\beta,t) + C_1(t)\delta\}$$

and hence

$$\bigcup_{j \in I} S_j \subset \{(r,u,\alpha): z(r,u,\alpha,t) \leqslant z(\eta,w,\beta,t) + C_1(t)\delta\}. \tag{30}$$

Now denoting $z = z(\eta,w,\beta,t)$,

$$\overline{\overline{M}}(z,t) = \sum_{j=1}^{N} M_j \xi(z - z_j(t)) \leqslant \sum_{j \in I} M_j$$

since $j \notin I$ implies $z - z_j(t) \leqslant 0$ which implies $\xi(z - z_j(t)) = 0$ by (25). Also by (30)

$$\begin{aligned}\sum_{j\in I} M_j &= \iint_{\cup_{j\in I} S_j} \mathring{\phi}(r,u,\alpha)\,dv\,dx\\ &\leqslant \iint_{\{(r,u,\alpha):\ z(r,u,\alpha,t)\leqslant z+C_1(t)\delta\}} \mathring{\phi}(r,u,\alpha)\,dv\,dx\\ &= \iint_{\{r\leqslant z+C_1(t)\delta\}} \phi(r,u,\alpha,t)\,dv\,dx = M(z+C_1(t)\delta,t)\\ &= M(z,t) + \int_z^{z+C_1(t)\delta} 4\pi s^2\rho(s,t)\,ds \leqslant M(z,t) + C(t)\delta.\end{aligned}$$

Therefore

$$\overline{\overline{M}}(z,t) \leqslant M(z,t) + C(t)\delta. \tag{31}$$

To bound $\overline{\overline{M}}$ from below define $J = \{j\colon z_j(t) \leqslant z - C_1(t)\delta\}$, where $z = z(\eta, w, \beta, t)$ as before. If $(r,u,\alpha) \in S_j$ and $z(r,u,\alpha,t) \leqslant z - 2C_1(t)\delta$, then by Lemma I, $z_j(t) \leqslant z(r,u,\alpha,t) + C_1(t)\delta \leqslant z - C_1(t)\delta$, that is $j \in J$. Therefore,

$$\mathscr{S} \cap \{(r,u,\alpha)\colon z(r,u,\alpha,t) \leqslant z - 2C_1(t)\delta\} \subset \bigcup_{j\in J} S_j.$$

If $j \in J$ then $z_j(t) \leqslant z - C_1(t)\delta \leqslant z - \delta$, so $z - z_j(t) \geqslant \delta$, and hence $\xi(z - z_j(t)) = 1$. Therefore

$$\begin{aligned}\overline{\overline{M}}(z,t) &= \sum_{j=1}^{N} M_j \xi(z - z_j(t)) \geqslant \sum_{j\in J} M_j = \iint_{\cup_{j\in J} S_j} \mathring{\phi}(r,u,\alpha)\,dv\,dx\\ &\geqslant \iint_{\mathscr{S}\cap\{(r,u,\alpha):\ z(r,u,\alpha,t)\leqslant z-2C_1(t)\delta\}} \mathring{\phi}(r,u,\alpha)\,dv\,dx\\ &= \iint_{\{(r,u,\alpha):\ z(r,u,\alpha,t)\leqslant z-2C_1(t)\delta\}} \mathring{\phi}(r,u,\alpha)\,dv\,dx\\ &= \iint_{\{r\leqslant z-2C_1(t)\delta\}} \phi(r,u,\alpha,t)\,dv\,dx\\ &= M(\max\{0, z-2C_1(t)\delta\},t)\\ &= M(z,t) - \int_{\max\{0,z-2C_1(t)\delta\}}^{z} 4\pi s^2\rho(s,t)\,ds\\ &\geqslant M(z,t) - C(t)\delta.\end{aligned}$$

Combining this and (31) we have

$$\left|M(z,t) - \overline{\overline{M}}(z,t)\right| \leqslant C(t)\delta \tag{32}$$

for $z = z(\eta, w, \beta, t)$ and $(\eta, w, \beta) \in \mathscr{S}$.

We will need (32) for all z, not just for $z \in z(\mathscr{S}, t)$. Note that if $r_1 \geqslant R_0 + U_0 t + \delta$ then $M(r_1,t) = \overline{\overline{M}}(r_1,t) = M$, and if $0 \leqslant r_1 \leqslant r_0$ then $M(r_1,t) = \overline{\overline{M}}(r_1,t) = 0$. Hence we consider $t \geqslant 0$ and $r_1 \in [r_0, R_0 + U_0 t + 1]$, such that $r_1 \notin z(\mathscr{S}, t)$. Define $r_2 = \inf\{\eta\colon \eta \geqslant r_0$ and $(\eta, r_1) \cap z(\mathscr{S}, t) = \varnothing\}$ and note that $r_0 \leqslant r_2 \leqslant r_1$ and $(r_2, r_1) \cap z(\mathscr{S}, t) = \varnothing$.

We have

$$M(r_1,t) - M(r_2,t) = \int_{r_2}^{r_1} 4\pi s^2 \rho(s,t)\,ds$$
$$= \iint_{\{(x,v):\ |x|\in(r_2,r_1)\}} \phi(r,u,\alpha,t)\,dv\,dx$$
$$= \iint_{\{(\eta,w,\beta):\ z(\eta,w,\beta,t)\in(r_2,r_1)\}} \mathring{\phi}(\eta,w,\beta)\,dv\,dx$$
$$= 0 \text{ since } \{(\eta,w,\beta):\ z(\eta,w,\beta,t)\in(r_2,r_1)\}\cap\mathscr{S} \tag{33}$$

is empty.

$z_j(t)\notin(r_2,r_1)$ for all j so by (25)

$$\overline{\overline{M}}(r_1,t) - \overline{\overline{M}}(r_2,t) = \sum_{j=1}^{N} M_j\big(\xi\big(r_1 - z_j(t)\big) - \xi\big(r_2 - z_j(t)\big)\big)$$
$$\leqslant \sum_{\{j:\ z_j(t)\in(r_2-\delta,r_2]\}} M_j.$$

If $z_j(t)\in(r_2-\delta, r_2]$ then by Lemma I

$$z(S_j,t)\subset\big[z_j(t) - C_1(t)\delta,\ z_j(t) + C_1(t)\delta\big]$$
$$\subset\big[r_2 - \delta - C_1(t)\delta,\ r_2 + C_1(t)\delta\big]$$

so

$$0 \leqslant \overline{\overline{M}}(r_1,t) - \overline{\overline{M}}(r_2,t)$$
$$\leqslant \iint_{\substack{\cup S_j \\ j:\ z_j(t)\in(r_2-\delta,r_2]}} \mathring{\phi}(r,u,\alpha)\,dv\,dx$$
$$\leqslant \iint_{\{(r,u,\alpha):\ |z(r,u,\alpha,t)-r_2|\leqslant\delta+C_1(t)\delta\}} \mathring{\phi}(r,u,\alpha)\,dv\,dx$$
$$= \iint_{\{(x,v):\ ||x|-r_2|\leqslant\delta+C_1(t)\delta\}} \phi(r,u,\alpha,t)\,dv\,dx$$
$$= \int_{\max\{0,\,r_2-\delta-C_1(t)\delta\}}^{r_2+\delta+C_1(t)\delta} 4\pi s^2\rho(s,t)\,ds$$
$$\leqslant 4\pi\big(r_2 + 1 + C_1(t)\big)^2\rho_0 2\big(1 + C_1(t)\big)\delta$$
$$\leqslant C(t)\delta, \text{ since } r_2 \leqslant R_0 + U_0 t + 1. \tag{34}$$

Recall that by definition $r_2 = \inf\{\eta:\ \eta\geqslant r_0 \text{ and } (\eta,r_1)\cap z(\mathscr{S},t) = \varnothing\}$. The map $(r,u,\alpha)\to z(r,u,\alpha,t)$ is continuous by Lemma I and $\mathscr{S}$ is compact; hence $z(\mathscr{S},t)$ is compact. If $r_2 > r_0$ then $r_2\in z(\mathscr{S},t)$ and so by (32)

$$\big|M(r_2,t) - \overline{\overline{M}}(r_2,t)\big| \leqslant C(t)\delta. \tag{35}$$

If $r_2 = r_0$ then $M(r_2, t) = \overline{\overline{M}}(r_2, t) = 0$ so (35) still holds. Now combining (33), (34), and (35) we have

$$\begin{aligned}
&\left|M(r_1, t) - \overline{\overline{M}}(r_1, t)\right| \\
&\quad \leqslant \left|M(r_1, t) - M(r_2, t)\right| + \left|M(r_2, t) - \overline{\overline{M}}(r_2, t)\right| + \left|\overline{\overline{M}}(r_2, t) - \overline{\overline{M}}(r_1, t)\right| \\
&\quad \leqslant 0 + C(t)\delta + C(t)\delta.
\end{aligned}$$

Therefore

$$\left|M(r, t) - \overline{\overline{M}}(r, t)\right| \leqslant C(t)\delta \quad \text{for all } r \geqslant 0 \text{ and } t \geqslant 0. \tag{36}$$

Part B. In order to estimate $\overline{M} - \overline{\overline{M}}$ let us define for $0 \leqslant t < T$

$$\|z - \bar{z}\|(t) = \sup_{0 \leqslant \tau \leqslant t} \max_{1 \leqslant i \leqslant N} |z_i(\tau) - \bar{z}_i(\tau)|. \tag{37}$$

Let $r \geqslant 0$ and $t \in [0, T)$. If $z_j(t) \geqslant r + \|z - \bar{z}\|(t)$,

$$r - z_j(t) \leqslant -\|z - \bar{z}\|(t) \leqslant 0,$$

and

$$r - \bar{z}_j(t) \leqslant r - \left(z_j(t) - \|z - \bar{z}\|(t)\right) \leqslant 0,$$

so

$$\xi\left(r - z_j(t)\right) = \xi\left(r - \bar{z}_j(t)\right) = 0.$$

Similarly, if $z_j(t) \leqslant r - \|z - \bar{z}\|(t) - \delta$, then

$$r - z_j(t) \geqslant \|z - \bar{z}\|(t) + \delta \geqslant \delta$$

and

$$r - \bar{z}_j(t) \geqslant r - \left(z_j(t) + \|z - \bar{z}\|(t)\right) \geqslant \delta,$$

so

$$\xi\left(r - z_j(t)\right) = \xi\left(r - \bar{z}_j(t)\right) = 1.$$

Hence

$$\begin{aligned}
\left|\overline{M}(r, t) - \overline{\overline{M}}(r, t)\right| &= \left|\sum_{j=1}^{N} M_j\left(\xi\left(r - \bar{z}_j(t)\right) - \xi\left(r - z_j(t)\right)\right)\right| \\
&\leqslant \sum_{j \in K} M_j
\end{aligned} \tag{38}$$

where

$$K = \left\{ j \colon \left|r - z_j(t)\right| < \|z - \bar{z}\|(t) + \delta \right\}.$$

Now if $j \in K$ and $(\eta, w, \beta) \in S_j$ then

$$\left| z_j(t) - r \right| < \| z - \bar{z} \|(t) + \delta$$

and by Lemma I

$$\left| z(\eta, w, \beta, t) - z_j(t) \right| \leqslant C_1(t)\delta$$

so

$$\left| z(\eta, w, \beta, t) - r \right| < \| z - \bar{z} \|(t) + \delta + C_1(t)\delta.$$

Therefore

$$\bigcup_{j \in K} S_j \subset \left\{ (\eta, w, \beta) : \left| z(\eta, w, \beta, t) - r \right| < \varepsilon \right\}$$

where $\varepsilon = \|z - \bar{z}\|(t) + \delta + C_1(t)\delta$. Now by (38) we have

$$\begin{aligned}
\left| \overline{M}(r,t) - \overline{\overline{M}}(r,t) \right| &\leqslant \sum_{j \in K} M_j \\
&= \iint_{\bigcup_{j \in K} S_j} \mathring{\phi}(\eta, w, \beta)\, dv\, dx \\
&\leqslant \iint_{\{(\eta, w, \beta) :\, |z(\eta, w, \beta, t) - r| < \varepsilon\}} \mathring{\phi}(\eta, w, \beta)\, dv\, dx \\
&= \iint_{\{(x, v) :\, ||x| - r| < \varepsilon\}} \psi(x, v, t)\, dv\, dx \\
&= \int_{\max\{0, r - \varepsilon\}}^{r + \varepsilon} 4\pi s^2 \rho(s,t)\, ds \leqslant 4\pi (r + \varepsilon)^2 \rho_0 2\varepsilon.
\end{aligned} \tag{39}$$

Recall that $\varepsilon = \|z - \bar{z}\|(t) + \delta + C_1(t)\delta$. Now $|z_j(\tau) - \bar{z}_j(\tau)| \leqslant |z_j(\tau)| + |\bar{z}_j(\tau)| \leqslant 3(R_0 + U_0 t)$ for all $\tau \in [0, t]$ so $\|z - \bar{z}\|(t) \leqslant 3(R_0 + U_0 t)$ and $\varepsilon \leqslant 3(R_0 + U_0 t) + 1 + C_1(t)$. If $r \leqslant 2R_0 + 2U_0 t + 1$ we have by (39),

$$\begin{aligned}
\left| \overline{M}(r,t) - \overline{\overline{M}}(r,t) \right| &\leqslant 4\pi \left(5(R_0 + U_0 t) + 2 + C_1(t)\right)^2 \rho_0 2\varepsilon \\
&= C(t)\varepsilon = C(t)\left(\|z - \bar{z}\|(t) + \delta + C_1(t)\delta \right) \\
&= C(t)\|z - \bar{z}\|(t) + C(t)\delta.
\end{aligned}$$

If $r > 2R_0 + 2U_0 t + 1$, then $\overline{M}(r,t) = \overline{\overline{M}}(r,t) = M$ so for all $r \geqslant 0$ and $t \in [0, T)$

$$\left| \overline{M}(r,t) - \overline{\overline{M}}(r,t) \right| \leqslant C(t)\|z - \bar{z}\|(t) + C(t)\delta. \tag{40}$$

Combining (40) and (36) we get for $r \geqslant 0$ and $t \in [0, T)$

$$\begin{aligned}
\left| M(r,t) - \overline{M}(r,t) \right| &\leqslant \left| M(r,t) - \overline{\overline{M}}(r,t) \right| + \left| \overline{\overline{M}}(r,t) - \overline{M}(r,t) \right| \\
&\leqslant C(t)\delta + C(t)\|z - \bar{z}\|(t) + C(t)\delta \\
&= C(t)\delta + C(t)\|z - \bar{z}\|(t).
\end{aligned} \tag{41}$$

Part C. In this section we use (41) and the differential equations for z_i and $\bar{z}_i$ to estimate $\|z - \bar{z}\|(t)$. From Eqs. (12) and (23)

$$\begin{aligned}
|z_i''(t) - \bar{z}_i''(t)| &= \left|L_i^2 z_i^{-3} + \gamma M(z_i,t) z_i^{-2} - L_i^2 (\bar{z}_i)^{-3} - \gamma \overline{M}(\bar{z}_i,t)(\bar{z}_i)^{-2}\right| \\
&\leqslant L_i^2 \left|z_i^{-3} - (\bar{z}_i)^{-3}\right| + M(z_i,t)\left|z_i^{-2} - (\bar{z}_i)^{-2}\right| \\
&\quad + (\bar{z}_i)^{-2}\left|M(z_i,t) - \overline{M}(\bar{z}_i,t)\right| \\
&\leqslant L_0^2 z_i^{-3}(\bar{z}_i)^{-3}\left|z_i^3 - (\bar{z}_i)^3\right| + M z_i^{-2}(\bar{z}_i)^{-2}\left|z_i^2 - (\bar{z}_i)^2\right| \\
&\quad + (\bar{z}_i)^{-2}\left(|M(z_i,t) - M(\bar{z}_i,t)| + |M(\bar{z}_i,t) - \overline{M}(\bar{z}_i,t)|\right) \qquad (42)
\end{aligned}$$

for $0 \leqslant t < T$. Note that

$$\begin{aligned}
|M(z_i,t) - M(\bar{z}_i,t)| &= \left|\int_{\bar{z}_i}^{z_i} 4\pi s^2 \rho(s,t)\, ds\right| \\
&\leqslant 4\pi(\max\{z_i, \bar{z}_i\})^2 \rho_0 |z_i - \bar{z}_i| \leqslant C(t)\|z - \bar{z}\|(t)
\end{aligned}$$

and by (41)

$$|M(\bar{z}_i,t) - \overline{M}(\bar{z}_i,t)| \leqslant C(t)\delta + C(t)\|z - \bar{z}\|(t)$$

so

$$|M(z_i,t) - M(\bar{z}_i,t)| + |M(\bar{z}_i,t) - \overline{M}(\bar{z}_i,t)| \leqslant C(t)\delta + C(t)\|z - \bar{z}\|(t).$$

Using this in (42) yields for $0 \leqslant t < T$

$$\begin{aligned}
&|z_i''(t) - \bar{z}_i''(t)| \\
&\quad \leqslant L_0^2 z_i^{-3}(\bar{z}_i)^{-3}\left(z_i^2 + z_i(\bar{z}_i) + (\bar{z}_i)^2\right)|z_i - \bar{z}_i| \\
&\qquad + M z_i^{-2}(\bar{z}_i)^{-2}(z_i + \bar{z}_i)|z_i - \bar{z}_i| + (\bar{z}_i)^{-2}\left(C(t)\delta + C(t)\|z - \bar{z}\|(t)\right) \\
&\quad \leqslant L_0^2 r_0^{-3}\left(\tfrac{1}{2}r_0\right)^{-3} 7(R_0 + U_0 t)^2 \|z - \bar{z}\|(t) \\
&\qquad + M r_0^{-2}\left(\tfrac{1}{2}r_0\right)^{-2} 3(R_0 + U_0 t)\|z - \bar{z}\|(t) + \left(\tfrac{1}{2}r_0\right)^{-2}\left(C(t)\delta + C(t)\|z - \bar{z}\|(t)\right) \\
&\quad = C(t)\delta + C(t)\|z - \bar{z}\|(t). \qquad (43)
\end{aligned}$$

Define for $0 \leqslant t < T$

$$Q(t) = \sup_{0 \leqslant \tau \leqslant t} \max_{1 \leqslant i \leqslant N}\left(|z_i(\tau) - \bar{z}_i(\tau)| + |z_i'(\tau) - \bar{z}_i'(\tau)|\right).$$

Note that $z_i(0) = \bar{z}_i(0) = r_i$ and $z_i'(0) = \bar{z}_i'(0) = u_i \cos\alpha_i$, so by (43)

$$\begin{aligned}|z_i(t) - \bar{z}_i(t)| + |z_i'(t) - \bar{z}_i'(t)| &\leqslant \int_0^t \left(|z_i'(\tau) - \bar{z}_i'(\tau)| + |z_i''(\tau) - \bar{z}_i''(\tau)|\right) d\tau \\ &\leqslant \int_0^t (Q(\tau) + C(\tau)\delta + C(\tau)Q(\tau))\, d\tau \\ &\leqslant C(t)\delta + C(t)\int_0^t Q(\tau)\, d\tau.\end{aligned}$$

Hence for all $\tau \in [0, t]$ and $i \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}|z_i(\tau) - \bar{z}_i(\tau)| + |z_i'(\tau) - \bar{z}_i'(\tau)| &\leqslant C(\tau)\delta + C(\tau)\int_0^\tau Q(s)\, ds \\ &\leqslant C(t)\delta + C(t)\int_0^t Q(s)\, ds\end{aligned}$$

and therefore $Q(t) \leqslant C(t)\delta + C(t)\int_0^t Q(s)\, ds$ for $0 \leqslant t < T$. For $\tau \in [0, t]$ and $t < T$, $Q(\tau) \leqslant C(t)\delta + C(t)\int_0^\tau Q(s)\, ds$ since $C(t)$ is increasing. By Gronwall's inequality we get for $0 \leqslant \tau \leqslant t$, $Q(\tau) \leqslant C(t)\delta \exp(C(t)\tau)$ so

$$Q(t) \leqslant C(t)\delta \quad \text{for } 0 \leqslant t < T. \tag{44}$$

Finally by (44) and Lemma I if $(r, u, \alpha) \in S_i$ and $0 \leqslant t < T$ then

$$\begin{aligned}&|z(r, u, \alpha, t) - \bar{z}_i(t)| + |z_t(r, u, \alpha, t) - \bar{z}_i'(t)| \\ &\qquad \leqslant |z(r, u, \alpha, t) - z_i(t)| + |z_i(t) - \bar{z}_i(t)| \\ &\qquad\quad + |z_t(r, u, \alpha, t) - z_i'(t)| + |z_i'(t) - \bar{z}_i'(t)| \\ &\qquad \leqslant C_1(t)\delta + C(t)\delta = C(t)\delta.\end{aligned}$$

Also by (41) and (44) for $0 \leqslant r$ and $0 \leqslant t < T$

$$\begin{aligned}|M(r, t) - \overline{M}(r, t)| &\leqslant C(t)\delta + C(t)\|z - \bar{z}\|(t) \\ &\leqslant C(t)\delta + C(t)C(t)\delta = C(t)\delta.\end{aligned}$$

This completes the proof of Theorem I.

COMMENT. Recall that the existence time T depends on δ. We can now use Theorem I to show that T tends to infinity as δ tends to zero: Let $T_0 > 0$ and $\delta \in (0, r_0/(3C_2(T_0)))$. Then for all j and for $0 \leqslant t < \min(T, T_0)$ we have by (27)

$$\begin{aligned}\tfrac{2}{3}r_0 &\leqslant r_0 - C_2(T_0)\delta \leqslant r_0 - C_2(t)\delta \leqslant z_j(t) - |\bar{z}_j(t) - z_j(t)| \\ &\leqslant \bar{z}_j(t) \leqslant z_j(t) + |\bar{z}_j(t) - z_j(t)| \\ &\leqslant R_0 + U_0 t + C_2(t)\delta \leqslant R_0 + U_0 t + \tfrac{1}{3}r_0 < \tfrac{3}{2}(R_0 - U_0 t).\end{aligned}$$

Now if $T \leqslant T_0$ the solution $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N$ can be continued beyond $t = T$, which is a contradiction. Therefore $T > T_0$.

Recall from (19) that the energy is conserved; we may write

$$E = \int\int \phi(r,u,\alpha,t)u^2\,dv\,dx + \gamma\int \rho(r,t)\int_0^\infty \frac{\rho(\eta,t)}{\max(r,\eta)}4\pi\eta^2\,d\eta\,dx$$

$$= \int\int \phi(r,u,\alpha,t)\left(u^2 + \gamma\int_0^\infty \frac{\rho(\eta,t)}{\max(r,\eta)}4\pi\eta^2\,d\eta\right)dv\,dx$$

$$= \int\int \phi(r,u,\alpha,t)\left(u^2 + \gamma\int\int \frac{\phi(r_0,u_0,\alpha_0,t)}{\max(r,r_0)}\,dv_0\,dx_0\right)dv\,dx$$

$$= \int\int \mathring{\phi}(r,u,\alpha)(q^2(r,u,\alpha,t))$$

$$+\gamma\int\int \frac{\mathring{\phi}(r_0,u_0,\alpha_0)\,dv_0\,dx_0}{\max(z(r,u,\alpha,t),\ z(r_0,u_0,\alpha_0,t))}\,dv\,dx. \tag{45}$$

Also since

$$z_t(r,u,\alpha,t) = q(r,u,\alpha,t)\cos(\theta(r,u,\alpha,t))$$

and

$$z(r,u,\alpha,t)q(r,u,\alpha,t)\sin(\theta(r,u,\alpha,t)) = ru\sin\alpha,$$

$$q^2(r,u,\alpha,t) = q^2(r,u,\alpha,t)\cos^2(\theta(r,u,\alpha,t)) + q^2(r,u,\alpha,t)\sin^2(\theta(r,u,\alpha,t))$$

$$= z_t^2(r,u,\alpha,t) + (ru\sin\alpha)^2 z^{-2}(r,u,\alpha,t). \tag{46}$$

Substituting (46) into (45) yields

$$E = \int\int \mathring{\phi}(r,u,\alpha)\Bigg(z_t^2(r,u,\alpha,t)$$

$$+\frac{(ru\sin\alpha)^2}{z^2(r,u,\alpha,t)} + \gamma\int\int \frac{\mathring{\phi}(r_0,u_0,\alpha_0)\,dv_0\,dx_0}{\max(z(r,u,\alpha,t),\ z(r_0,u_0,\alpha_0,t))}\Bigg)dv\,dx. \tag{47}$$

(47) suggests a quantity which will be nearly conserved by the discrete system.

COROLLARY I. There exists a positive increasing function $C_3(t)$ (depending only on $\mathring{\phi}$) such that for all $t \in [0,T)$

$$\left|E - \sum_{i=1}^N M_i\left(\left(\bar{z}_i'(t)\right)^2 + L_i^2(\bar{z}_i(t))^{-2} + \sum_{j=1}^N \frac{\gamma M_j}{\max(\bar{z}_i(t),\bar{z}_j(t))}\right)\right| \leqslant C_3(t)\delta. \tag{48}$$

Proof. For brevity denote $z = z(r,u,\alpha,t)$ and $z_0 = z(r_0,u_0,\alpha_0,t)$. Note that

$$\left|\frac{1}{\max(z,z_0)} - \frac{1}{\max(\bar{z}_i(t),\bar{z}_j(t))}\right| \leqslant \left|\frac{1}{\max(z,z_0)} - \frac{1}{\max(\bar{z}_i(t),z_0)}\right|$$

$$+\left|\frac{1}{\max(\bar{z}_i(t),z_0)} - \frac{1}{\max(\bar{z}_i(t),\bar{z}_j(t))}\right|$$

$$\leqslant \frac{|z-\bar{z}_i(t)|}{\left(\frac{1}{2}r_0\right)^2} + \frac{|z_0-\bar{z}_j(t)|}{\left(\frac{1}{2}r_0\right)^2}. \tag{49}$$

If $(r, u, \alpha) \in S_i$ and $(r_0, u_0, \alpha_0) \in S_j$, then by Theorem I and (49)

$$\left|\frac{1}{\max(z, z_0)} - \frac{1}{\max(\bar{z}_i(t), \bar{z}_j(t))}\right| \leqslant \frac{C_2(t)\delta}{(r_0/2)^2} + \frac{C_2(t)\delta}{(r_0/2)^2} = \frac{8C_2(t)\delta}{r_0^2}.$$

Therefore if $(r, u, \alpha) \in S_i$, then

$$\left|\int\int \frac{\mathring{\phi}(r_0, u_0, \alpha_0)}{\max(z, z_0)}\, dv_0\, dx_0 - \sum_{j=1}^{N} \frac{M_j}{\max(\bar{z}_i(t), \bar{z}_j(t))}\right|$$

$$= \left|\sum_{j=1}^{N}\left(\iint_{S_j} \frac{\mathring{\phi}(r_0, u_0, \alpha_0)}{\max(z, z_0)}\, dv_0\, dx_0 - \frac{M_j}{\max(\bar{z}_i(t), \bar{z}_j(t))}\right)\right|$$

$$= \left|\sum_{j=1}^{N} \iint_{S_j} \mathring{\phi}(r_0, u_0, \alpha_0)\left(\frac{1}{\max(z, z_0)} - \frac{1}{\max(\bar{z}_i(t), \bar{z}_j(t))}\right) dv_0\, dx_0\right|$$

$$\leqslant \sum_{j=1}^{N} \iint_{S_j} \mathring{\phi}(r_0, u_0, \alpha_0)\frac{8C_2(t)\delta}{r_0^2}\, dv_0\, dx_0 = 8Mr_0^{-2}C_2(t)\delta = C(t)\delta. \tag{50}$$

Again for $(r, u, \alpha) \in S_i$ we have

$$\left|\left(z_t^2 + \frac{(ru\sin\alpha)^2}{z^2}\right) - \left((\bar{z}_i'(t))^2 + \frac{L_i^2}{(\bar{z}_i(t))^2}\right)\right|$$

$$\leqslant \left|z_t^2 - (\bar{z}_i'(t))^2\right| + z^{-2}\left|(ru\sin\alpha)^2 - L_i^2\right| + L_i^2\left|z^{-2} - (\bar{z}_i(t))^{-2}\right|$$

$$\leqslant \left|z_t + \bar{z}_i'(t)\right|\left|z_t - \bar{z}_i'(t)\right| + r_0^{-2}(ru\sin\alpha + L_i)|ru\sin\alpha - L_i|$$

$$+ L_0^2 z^{-2}(\bar{z}_i(t))^{-2}(\bar{z}_i(t) + z)|\bar{z}_i(t) - z|.$$

By Theorem I, $|z_t - \bar{z}_i'(t)| \leqslant C_2(t)\delta$ so

$$\left|z_t + \bar{z}_i'(t)\right| \leqslant |2z_t| + \left|\bar{z}_i'(t) - z_t\right| \leqslant 2U_0 + C_2(t)\delta$$

and

$$\left|z_t^2 + \frac{(ru\sin\alpha)^2}{z^2} - \left((\bar{z}_i'(t))^2 + \frac{L_i^2}{(\bar{z}_i(t))^2}\right)\right|$$

$$\leqslant (2U_0 + C_2(t)\delta)C_2(t)\delta + r_0^{-2}2L_0C\delta + L_0^2 r_0^{-2}\left(\tfrac{1}{2}r_0\right)^{-2}3(R_0 + U_0 t)C_2(t)\delta$$

$$\leqslant C(t)\delta. \tag{51}$$

Now to finish the proof we rewrite E as in (47) and use the bounds given by (50) and (51):

$$\left| E - \sum_{i=1}^{N} M_i \left(\left(\bar{z}_i'(t) \right)^2 + L_i^2 \left(\bar{z}_i(t) \right)^{-2} + \sum_{j=1}^{N} \frac{\gamma M_j}{\max\left(\bar{z}_i(t), \bar{z}_j(t) \right)} \right) \right|$$

$$= \left| \sum_{i=1}^{N} \iint_{S_i} \mathring{\phi}(r,u,\alpha) \left(z_t^2 + \frac{(ru \sin\alpha)^2}{z^2} + \gamma \iint \frac{\mathring{\phi}(r_0,u_0,\alpha_0)}{\max(z,z_0)} \, dv_0 \, dx_0 \right. \right.$$

$$\left. \left. - \left(\bar{z}_i'(t) \right)^2 - L_i^2 \left(\bar{z}_i(t) \right)^{-2} - \gamma \sum_{j=1}^{N} \frac{M_j}{\max\left(\bar{z}_i(t), \bar{z}_j(t) \right)} \, dv \, dx \right| \right.$$

$$\leqslant \sum_{i=1}^{N} \iint_{S_i} \mathring{\phi}(r,u,\alpha) \left(\left| z_t^2 + \frac{(ru \sin\alpha)^2}{z^2} - \left(\bar{z}_i'(t) \right)^2 - \frac{L_i^2}{\left(\bar{z}_i(t) \right)^2} \right| \right.$$

$$\left. + \left| \iint \frac{\mathring{\phi}(r_0,u_0,\alpha_0)}{\max(z,z_0)} \, dv_0 \, dx_0 - \sum_{j=1}^{N} \frac{M_j}{\max\left(\bar{z}_i(t), \bar{z}_j(t) \right)} \right| \right) dv \, dx$$

$$\leqslant \sum_{i=1}^{N} \iint_{S_i} \mathring{\phi}(r,u,\alpha) \left(C(t)\delta + C(t)\delta \right) dv \, dx$$

$$= 2MC(t)\delta.$$

This completes the proof of Corollary I.

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