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A REMARK ON POSITIVELY INVARIANT REGIONS FOR PARABOLIC SYSTEMS WITH AN APPLICATION ARISING IN SUPERCONDUCTIVITY*

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0. Introduction. With regard to the maximum principle, elliptic systems have been treated as a particular case of parabolic systems (see, for example, [W]). Therefore, the smallest set (that this approach can offer) which localizes pointwise the steady states is necessarily positively invariant with respect to the flow that the associated parabolic system induces. It turns out that one can do better for the elliptic system and this for a very simple reason: by multiplying through by the inverse of the coefficient matrix corresponding to the highest derivatives, the system is reduced to one with equal diffusion coefficients. Thus, whenever this can be performed, we improve the rectangle to an inscribed ellipsoid. This observation, apparently not noticed before, has an interesting implication in the case that the parabolic system is gradient: the set that localizes the steady states as well as the family of its congruent sets that circumscribe it will not, in general, be positively invariant. Yet any solution corresponding to the initial condition lying in any of these sets has eventually to return to the set and stay in it. In a sense, in this case, the elliptic sets are eventually positively invariant for the parabolic system.

We choose to illustrate these observations in terms of an example. We have not made any effort to identify the most general circumstances where our approach is applicable. The system we have in mind is

$$\begin{gathered}\frac{\partial u}{\partial t} = D\Delta u + [1 - \langle Au, u\rangle] \quad \text{on } \Omega,\\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0,\end{gathered} \tag{1}$$

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$u = \operatorname{col}(u_1, \ldots, u_N)$, D, A symmetric positive definite, $\langle\ ,\ \rangle$ inner product in $\mathbf{R}^N$, $\Omega \subseteq \mathbf{R}^n$, bounded, smooth, connected. Equation (1) is related to some models in superconductivity [AT]. It was introduced from the point of view of invariant regions in [CCS] for the choice $D = A = I$, $N = 2$. In [BDG] the asymptotic behavior was investigated for the [CCS] choice of (1) and under a smallness hypothesis on Ω stabilization of every solution was established.

To understand the meaning of these hypotheses, let $D = \operatorname{diag}(\nu_1, \nu_2)$, ν_i positive constant, and $A = I$. The set $M^c = \{(z_1, z_2)/z_1^2 + z_2^2 \leqslant c^2,\ c \geqslant 1\}$ is positively invariant iff $\nu_1 = \nu_2$ (cf. [CCS]) and so it is only for the choice of equal diffusion coefficients that the results [CCS], [W] imply that the steady states are in M^1, a piece of information crucial in the analysis in [BDG]. If $\nu_1 \neq \nu_2$ the best set that [CCS] provides is the rectangle circumscribing M^1. See [SM] for more information. The effect of the small domain (= large diffusion) is to make all the steady states constants. It should be remarked, however, that the stabilization question even under all these restrictive hypotheses is not straightforward since the set of equilibria is connected. In the present paper we establish that the steady states are in M^c for any choice of diagonal D and that all solutions stabilize without any smallness hypotheses on Ω.

With regard to stabilization we note that compactness of the orbits of solutions corresponding to integrable initial data is easily established (proof of Theorem 4), and in the event that $AD = DA$, a natural Liapunov functional exists which via LaSalle's Invariance Principle renders that the ω-limit sets consist of steady states. However, this is far from establishing stabilization since the set of equilibria ($\equiv$ steady states) has a complicated structure and contains nontrivial connected components. It is at this point that we need to appeal to a recent result in [SI] (the finite-dimensional version of a result that goes back to Lojasiewicz [L]) which asserts the stabilization of bounded solutions to analytic gradient systems.

1. By a *steady state* we mean a time independent solution of system (1) that is in $C^{2+\alpha}(\Omega)$ for some α in $(0,1)$. Our first result localizes pointwise the set of steady states.

THEOREM 1. *Let $DA + AD$ be positive definite. Then all the steady states of (1) take values in*

$$M^1 = \{z \in \mathbf{R}^n \mid \langle Az, z\rangle \leqslant 1\}.$$

Moreover the nonconstant ones lie strictly in M^1.

COROLLARY 2. *Let $D = \operatorname{diag}(\nu_1, \ldots, \nu_N)$. Then the conclusion of Theorem 1 holds if*

$$\max_{i,j} \sqrt{\frac{\nu_i}{\nu_j}} < \frac{\sqrt{E_A} + 1}{\sqrt{E_A} - 1},$$

$$E_A = \frac{\max_i \lambda_i(A)}{\min_i \lambda_i(A)}, \qquad \lambda_i(A) = i\text{th eigenvalue of } A.$$

Proof of Theorem 1. Without loss of generality we take $D = \operatorname{diag}(\nu_1, \dots, \nu_N)$. Let $G(z) = \langle Az, z\rangle$ and note

$$\Delta G = \operatorname{tr}\{(\nabla u)^T(\partial^2 G)(\nabla u)\} + \sum_{i=1}^{N} G_{u_i}\Delta u_i. \tag{2}$$

Assume that $\max G(u(x)) = G(u(\bar{x})) > 1$.

Case (i). $\bar{x} \in \Omega$. It follows that $\Delta G(u(\bar{x})) \leqslant 0$. The convexity of G implies via (2) that at $\bar{x}$

$$\sum_i G_{u_i}\Delta u_i \leqslant 0. \tag{3}$$

From (1) we obtain

$$\sum_i \left[G_{u_i}\Delta u_i + [1 - G(u)]\frac{u_i}{\nu_i}G_{u_i}\right] = 0 \tag{4}$$

or, equivalently,

$$\sum_i G_{u_i}\Delta u_i - [1 - G(u)]\langle Bu, u\rangle = 0, \qquad B = \tfrac{1}{2}(AD^{-1} + D^{-1}A). \tag{5}$$

The hypothesis implies B is positive definite and so (5) contradicts (3).

Case (ii). $\bar{x} \in \partial\Omega$. By continuity $G(u(x)) > 1$ on some subdomain $\Omega_1 \subset \Omega$. Therefore,

$$\sum_i G_{u_i}\Delta u_i \geqslant 0 \quad \text{on } \Omega_1 \text{ (by (5))}, \tag{6}$$

and so

$$\Delta G \geqslant 0 \quad \text{on } \Omega_1 \text{ (by (2))}. \tag{7}$$

The Hopf lemma gives

$$(\partial/\partial n)(G(u(\bar{x}))) > 0, \tag{8}$$

which contradicts $\partial u/\partial n|_{\partial\Omega} = 0$. So far we have established $G(u(x)) \leqslant 1$. By (2)

$$\Delta(1 - G) + (1 - G)\langle Bu, u\rangle \geqslant 0. \tag{9}$$

The strong maximum principle applies to give the result. □

Proof of Corollary 2. By a result in [S], [N] the hypothesis implies that $DA + AD$ is positive definite. It is clear that ∂M^1 consists of steady states. The next result shows that the set of steady states is considerably richer.

PROPOSITION 3. Let $D = \operatorname{diag}(\nu_1, \dots, \nu_N)$. If $1/\nu_i < \lambda_1$ (λ_j the jth eigenvalue of $-\Delta$ with Neumann condition, $j = 0, 1, \dots$), the ith coordinate of all steady states is a constant. If $1/\nu \in (\lambda_K, \lambda_{K+1}]$, $K \geqslant 1$, there are at least K distinct pairs of nonconstant steady state solutions.

Proof. Assume $1/\nu_1 < \lambda_1$, and consider

$$\nu_1\Delta u_1 + [1 - (u_1^2 + \cdots + u_N^2)]u_1 = 0, \qquad \left.\frac{\partial u_1}{\partial n}\right|_{\partial\Omega} = 0; \tag{10}$$

let $q(x) = 1 - (u_1^2 + \cdots + u_N^2)$. By Theorem 1, $q(x) \geqslant 0$. It is clear enough from (10) that

$$\int q u_1 = 0. \tag{11}$$

Therefore,

$$\nu_1 \int |\nabla w|^2 \leqslant \int w^2 \leqslant \frac{1}{\lambda_1} \int |\nabla w|^2$$

from which we conclude that w is zero and so u_1 is a constant.

For the second part it will be sufficient to establish the statement for the scalar equation

$$\nu \Delta u + [1 - u^2] u = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{12}$$

The associated functional is

$$E(u) = \frac{\nu}{2} \int |\nabla u|^2 - \frac{1}{2} \int u^2 + \frac{1}{4} \int u^4$$

considered on $w^{1,2}(\Omega)$. First take $n < 4$ so that E is C^1. We show later by a standard device how to circumvent this. We apply Ljusternik–Schnirelmann, specifically Theorem 3.1 in [R], and obtain the solutions to (12) as critical points of E. In [R] the applications are for homogeneous Dirichlet problems. The Neumann case is only slightly more complicated for the specific problem at hand.

Step 1. Check the Palais–Smale condition. P–S: If $\{u_n\} \subset W^{1,2}$ such that $E(u_n) < 0$, $E(u_n)$ bounded from below, $E'(u_n) \to 0$, then $\{u_n\}$ contains a convergent subsequence,

$$K \leqslant \frac{1}{2} \int |\nabla u_n|^2 - \frac{1}{2} \int u_n^2 + \frac{1}{4} \int u_n^4 < 0. \tag{13}$$

Note that the hypothesis of boundedness from below is redundant in (13).

The Schwarz inequality gives boundedness of $\|u_n\|_{L^4}$. Define w_n by

$$\Delta w_n = u_n^3 - u_n - \frac{1}{|\Omega|} \int (u_n^3 - u_n), \tag{14}$$

$$\frac{\partial w_n}{\partial n} = 0, \qquad \int w_n - u_n = 0. \tag{15}$$

Elliptic theory gives

$$\left\| w_n - \int u_n \right\|_{W^{2,p}} \leqslant C \left\| u_n^3 - u_n - \frac{1}{|\Omega|} \int (u_n^3 - u_n) \right\|_{L^p} \tag{16}$$

for $p > 1$. Note that

$$E'(u_n)(v) = \int \nabla(u_n - w_n) \nabla v + \frac{1}{|\Omega|} \int (u_n^3 - u_n) \int v.$$

Define

$$\tilde{W}^{1,2}(\Omega) = \left\{ u \in W^{1,2}(\Omega) \mid \int u\, dx = 0 \right\}.$$

Then on $W^{1,2}(\Omega)$, $E'(u_n) = u_n - w_n$ and so by the hypothesis, $u_n - w_n \to 0$ in $\tilde{W}^{1,2}(\Omega)$ and a posteriori in $W^{1,2}(\Omega)$. Choosing in (16) $p = \frac{4}{3}$ and making use of the bound for $\|u_n\|_{L^4}$ we obtain a bound for $\|w_n\|_{W^{2,p}}$ and so we are set by the compactness of the imbedding. To avoid restrictions on the dimension, extend $f(z) = (1 - z^2)z$ smoothly for $|z| \geqslant 2$ so that $\tilde{f}(z)z < 0$ for $|z| > 1$ and so that $\tilde{f}$ grows at infinity like $|z|^{1+\varepsilon}$. By the maximum principle the solutions to

$$\nu\Delta u + \tilde{f}(u) = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

are in $|z| \leqslant 1$ and so coincide with those of (12). By choosing ε small enough we guarantee the smoothness of $\tilde{E}(u)$. Finally (13) will be altered at essentially one point, $\int u_n^4$ will be replaced by $\int |u_n|^{2+\varepsilon}$. The rest of the argument is identical.

Step 2. There exists a set A in $W^{1,2}(\Omega)$ with genus $k+1$ on which E is negative.

To construct this let $\{\phi_j\}$ be eigenfunctions corresponding to $\lambda_0, \lambda_1, \ldots, \lambda_k$ and take $\int \phi_i \phi_j = \delta_{ij}$. Then set $S_k(R) = \{u = \sum_{j=0}^k \beta_j \phi_j \mid \sum \beta_j^2 = R^2\}$. Note that $\gamma(S_k(R)) = k+1$ (γ = genus). For $u \in S_k(R)$ we have

$$\begin{aligned} E(u) &= \frac{\nu}{2} \sum \beta_j^2 \lambda_j \int \phi_j^2 - \frac{1}{2} \sum \beta_j^2 \int \phi_j^2 + \frac{1}{4} \int \left| \sum \beta_j \phi_j \right|^4 \\ &= \frac{1}{2} \sum \beta_j^2 (\nu\lambda_j - 1) R^2 + \frac{1}{4} \left(\sum \left| \frac{\beta_j}{R} \phi_j \right|^4 \right) R^4 \\ &\leqslant -\frac{1}{2} m R^2 + \frac{1}{4} M R^4 < 0 \quad \text{for } R \text{ small,} \end{aligned}$$

where $\max_j (\nu\lambda_j - 1) = -m < 0$,

$$\max_{|\beta|=1} \sum \int \left| \beta_j \phi_j \right|^4 = M. \qquad \square$$

REMARK. Let $D = \operatorname{diag}(\nu_1, \ldots, \nu_N)$, $A = I$. It is shown in [A] that if $1/\nu_i < \lambda_2$ the ith coordinate of any solution converges to a constant and if this constant is nonzero the solution converges to a vector on S^{N-1}. The second part in Proposition 3 is established in [BDG] (actually it is shown that there are exactly k pairs) in the case Ω is an interval. See also [BB] for related results.

Under the additional hypothesis that the matrices A and D commute it can be shown that the system (1) is gradient-like. This point together with other relevant information can be found in [A]. Combining the gradient structure with the fact that the steady states are in M^1, we get the interesting phenomenon that even though M^c ($c \geqslant 1$) is not positively invariant for general D (see [SM]), yet any solution corresponding to an initial condition taking values in M^c returns back to M^c after some time and stays there after.

The stabilization is the content of

THEOREM 4. Let $DA = AD$. Then the solution to (1) corresponding to any initial condition $u_0(\cdot) \in L^p = L^p(\Omega) \times \cdots \times L^p(\Omega)$, $p > 1$, converges uniformly on $\overline{\Omega}$ as $t \to \infty$ to a steady state solution.

Proof. It is shown in [A] that if $AD = DA$ the system is similar via linear transformations to the case $D = \mathrm{diag}(\nu_1, \ldots, \nu_N)$, $A = I$. The elliptic part of the reduced system consists of the Euler–Lagrange equations corresponding to the functional

$$E(u) = \sum \frac{\nu_i}{2} \int |\nabla u_i|^2 - \frac{1}{2} \int u_i^2 + \frac{1}{4} \int \Big(\sum_i u_i^2 \Big)^2$$

which also serves as a Liapunov functional for (1).

Noting that, along solutions

$$\frac{d}{dt} \int |u|^2 \leqslant (\mathrm{meas}\, \Omega)^{1/2} \left(\int_\Omega |u|^2 \right)^{1/2} - \int_\Omega |u|^2,$$

we obtain an L^2 bound for solutions corresponding to L^2 initial conditions. Well-known regularity theory [H] implies that solutions corresponding to L^p initial conditions ($p > 1$) enter L^2 immediately and so the bound above holds eventually for the class of initial conditions we consider. An extra bootstrap argument [A] establishes a uniform L^∞ bound and by further regularity one concludes precompactness of the orbits in the setting of X^α spaces, $X = L^p$, $X^1 = W^{2,p}$ and so in $C(\Omega) \times \cdots \times C(\Omega)$ (see [H]). The Invariance Principle renders that the ω-limit sets are subsets of the set of steady states. Finally, exploiting the analyticity of E, we obtain stabilization of each solution to a steady state (i.e., $\lim_{t \to 0} u(x, t) = s(x)$) by appealing to the relevant result in [SI]. □

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References

[AT] E. Abrahams and T. Tsuneto, *Time variation of the Ginzberg-Landau order parameter*, Phys. Rev. **152**, 416–463 (1966)

[A] N. D. Alikakos, *Quantitative maximum principles and strongly coupled gradient-like reaction-diffusion systems*, Proc. Royal Soc. Edinburgh Sect. A **94**, 265–286 (1983)

[BB] D. L. Barrow and P. W. Bates, *Bifurcation and stability of periodic travelling waves for a reaction-diffusion system*, J. Differential Equations **50**, 218–233 (1983)

[BDG] K. J. Brown, P. C. Dunne, and R. A. Gardner, *A semilinear parabolic system arising in the theory of superconductivity*, J. Differential Equations **40**, 232–252 (1981)

[CCS] K. N. Chueh, C. L. Conley, and J. A. Smoller, *Positively invariant regions for systems of diffusion equations*, Indiana Univ. Math. J. **26**, 373–391 (1977)

[H] D. Henry, *Geometric theory of parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin, 1981.

[L] S. Łojasiewicz, *Une propriété topologique des sous-ensembles analytiques réels*, Colloques International x du C.N.R.S. #117, Les équations aux dérivées partielles, 1963

[N] D. W. Nicholson, *Eigenvalue bounds for $AB + BA$, with A, B positive definite matrices*, Linear Algebra Appl. **24**, 173–183 (1979)

[R] P. H. Rabinowitz, *Variational methods form linear eigenvalue problems*, Course lectures, CIME, Varenna, Italy, 1974

[SI] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math., (2) **118**, 525–517 (1983)

[SM] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1983

[S] W. G. Strang, *Eigenvalues of Jordan products*, Amer. Math. Monthly, 37–40 (1982)

[W] H. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat. (6) **8**, 295–310 (1975)