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ON THE COMPLETENESS OF THE RAYLEIGH-MARANGONI AND GRAETZ EIGENSPACES AND THE SIMPLICITY OF THEIR EIGENVALUES*

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Abstract. The study of convection in bounded geometries is often facilitated by consideration of model problems which require little numerical computation. Such problems typically result in a sixth-order ordinary differential system which is non-selfadjoint in the inner product associated with the space of square integrable functions in the sense of Lebesgue. Using theorems of Karlin (1971) and Naimark (1967), we prove the simplicity of the eigenvalues and completeness of the eigenspace. We illustrate some of the simplicity results with numerical calculations. The use of completeness in future problems is explained. We also consider the extended Graetz problem with homogeneous reaction and heterogeneous reaction at the wall. Similar results are shown here. Some of the above-mentioned results can be obtained by other means, but we provide this analysis as an interesting alternative.

1. Introduction. This paper deals with the application of some useful theorems due to Karlin (1971), Naimark (1967), and Birkhoff (1908) to two important problems in free and forced convection. In the case of the latter, it is known that the results could also be established by alternate means.

The Rayleigh-Bénard problem in bounded geometries is related to the onset and subsequent maintenance of convection from a basic quiescent conductive state. This onset occurs because of an imbalance between buoyancy and viscous effects and is mathematically connected to a nonlinear quadratic term. The critical parameter that indicates departure from the conductive state is the Rayleigh number which is denoted by Ra. The usual way of obtaining the critical Rayleigh number is by linearizing the governing equations about the known conductive solution and subsequently inspecting the spectrum of the resulting eigenvalue problem. When physically realistic conditions of no-slip are imposed at fluid-solid boundaries, we often need to consider a numerical solution. In this

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regard we quote the work of Davis (1967), Charlson and Sani (1970, 1975), Jennings and Sani (1972), Narayanan (1978), and Vrentas et al. (1981, 1982) amongst several others. It has been shown without computations by Narayanan and Abasaeed (1985) that the critical Rayleigh number is least when stress-free conditions are considered at fluid-solid boundaries. Numerical evidence due to Jones and Moore (1979) also bears out this fact. It is also known (Joseph (1971)) that stress-free boundary conditions on vertical sides of a rectangular parallelepiped allow us to separate variables in the horizontal directions and obtain ordinary differential equations for velocity and temperature in the vertical direction only. It is curious to note that such a variable separation is not immediately apparent in the case of a right circular cylinder. Rosenblat (1982) and Rosenblat et al. (1982) have thus studied the case of convection in circular cylinders with vanishing tangential vorticity at the vertical sides. This gives rise to the same ordinary differential system mentioned earlier. Further simplifications of the algebraic manipulations are possible if one considers stress-free horizontal boundaries of infinite conductivity. In many problems such simplifications are not possible—even in asymptotic limits. An example is the Rayleigh–Marangoni problem, where we are destined to consider only finite conductivity effects. (An infinite conductivity translates to a null Marangoni effect.)

Thus a natural model problem that deserves consideration is the buoyancy-surface tension driven case in a rectangular parallelepiped with stress-free vertical sides. The conditions on the flat horizontal sides may be arbitrarily fixed provided that they are compatible with the physical situation. The resulting sixth-order differential equation has been, in the absence of Marangoni effects, considered by Sparrow et al. (1964). The completeness of the eigenspace follows from the fact that the equations may be recast in matrix differential selfadjoint form. The eigenspace elements are useful in the study of the related nonlinear problem. The purpose of this manuscript is to examine the completeness of the eigenspace and simplicity of eigenvalues for the model Rayleigh–Marangoni problem. The eigenfunctions of this problem have been used by Rosenblat et al. (1982) in order to construct a series representation for the dependent variable in the physically associated nonlinear problem. While they have assumed completeness, we wish to validate their assumption. The simplicity of the critical Rayleigh number assures us that interaction of vertical modes for a fixed horizontal wave number (and thus related to aspect ratio) is impossible. Thus, cascaded bifurcation (cf. Reiss (1983)) can only occur due to interaction of horizontal modes.

The properties of completeness and simplicity arise from the work of Birkhoff (1908), Naimark (1967), and Karlin (1971). The second part of the paper briefly translates these works into usable form, while the third part is devoted to examples in Rayleigh–Marangoni convection and the forced convection Graetz problem with reaction.

It is necessary to mention that the results of the examples on the Graetz problem could be obtained using the method of Papoutsakis et al. (1980). Their method involves the uncovering of an inner product in which the relevant eigenvalue problem is selfadjoint. We do not claim that the methods espoused in the present paper are superior to theirs. On the other hand they might be seen as a useful alternative. It must be pointed out that the simplicity of eigenvalues is not related to any bifurcation in this case, merely because the latter situation does not even arise. However, simplicity or more generally semisimplicity, is a useful property to justify analytic perturbation methods (Joseph (1976)). Thus, if we are faced with a non-selfadjoint problem and do not wish to search for inner products in which selfadjointness might come about, then a proof for simplicity could be useful for consideration of perturbation results.

2. Preliminary ideas and results. Let

$$Ly = \lambda y \tag{2.1a}$$

represent an eigenvalue problem subject to

$$U_\nu(\mathbf{y}) = 0, \qquad \nu = 1, \ldots, n. \tag{2.1b}$$

Here

$$L \equiv a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^n} + \cdots + a_n(x) \tag{2.1c}$$

and we restrict $a_0 \neq 0$: $x \in (0,1)$. The operators U_ν are the homogeneous boundary conditions.

Ince (1956) shows that y is an integral and analytic function of λ and as such there exist infinite values of λ without any finite limit point.

Let y_i $(i = 1, \ldots, n)$ be the fundamental solutions to (2.1a). Then $y = \sum_{i=1}^{n} c_i y_i$ is the general solution to Eq. (2.1a).

Imposition of the homogeneous boundary conditions leads to the following equation:

$$|\mathscr{U}| = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix} = 0. \tag{2.2}$$

We shall assume that U_ν are such that the rank of $\mathscr{U} = n$. This ensures the linear independence of the boundary conditions and thus Eq. (2.2) implies that all the $c_i \neq 0$. The "rank" condition also means that any value of λ is not an eigenvalue. The polynomial equation that arises from Eq. (2.2) thus generates a countable infinite λ_j. Clearly, if any of the λ_j is of algebraic multiplicity "m" then

$$\left.\frac{d^k\mathscr{U}}{d\lambda^k}\right|_{\lambda=\lambda_j} = 0, \qquad k = 1, \ldots, m-1.$$

We wish to obtain an alternate necessary condition which will prove to be of use in the next sections. Let $c_1 \neq 0$ without loss of generality. Also let each of the minors of order $(n-1)$ (obtained such that the first column of $\mathscr{U}$ and any one of the "n" rows are discarded) not vanish. Both of the above are possible without loss of generality since we could suitably reorder the "n" fundamental solutions.

Suppose the first row is eliminated. Then $c_2, \ldots, c_n$ are uniquely determined in terms of λ. Substitution of c_i into $U_1(y) = 0$ must lead to the polynomial given by Eq. (2.2). Thus, if λ_j is multiple, then

$$\left.\frac{dU_1(y)}{d\lambda}\right|_{\lambda=\lambda_j} = 0. \tag{2.3}$$

The following result is therefore true.

THEOREM 2.1. If λ_j is of multiplicity l, then

$$\left.\frac{d^k U_i(y)}{d\lambda^k}\right|_{\lambda=\lambda_j} = 0, \qquad k = 1,\dots,l-1, \tag{2.4}$$

for any one value of i ($i = 1,\dots,n$).

Now if L^* and U_ν^* are adjoint to L and U_ν within the framework of the usual inner product associated with square integrable functions, Naimark (1967) shows that

$$r = r', \tag{2.5}$$

where r and r' are the ranks of the boundary value problem and its adjoint. Since $\bar{\lambda}_j$ is the eigenvalue of the adjoint, we arrive at an analogue of Eq. (2.3).

THEOREM 2.2. If λ_j is multiple, then

$$\left.\frac{dU_i^*}{d\bar{\lambda}}(y^*)\right|_{\bar{\lambda}=\bar{\lambda}_j} = 0. \tag{2.6}$$

The preceding two theorems will be of some use in showing simplicity of eigenvalues for the Graetz problem with homogeneous and heterogeneous reaction.

We now briefly quote some theorems due to Karlin (1971) and Krein (1939) on the simplicity of eigenvalues belonging to nonsymmetric kernels. These will be useful in proving simplicity for model problems of Rayleigh–Marangoni. The proofs of the theorems are delicate and well explained in the reference by Karlin (1971).

DEFINITION. A matrix M is said to be sign consistent of order "p" (sC_p) if all $p \times p$ nonzero subdeterminants of M have the same sign.

THEOREM 2.3 (due to Karlin (1971)). Consider an operator

$$L_n(y) = \mu_0(x) D_n D_{n-1} \dots D_1 y, \tag{2.7}$$

where $D_i y = (d/dx)\,\mu_i(x) y$ and $i = 1, 2, \dots, n$ and $\mu_i(x)$ are strictly positive functions in $C^{(n)}(0,1)$ and $y \in C^{(n)}(0,1)$. We say that L_n is an iterated form if it is represented as above.

Let us couple this operator with boundary conditions at the end points of the form

$$B_0 \colon \sum_{\mu=0}^{n-1} A_{\nu\mu}(D^\mu y)(0) = 0, \qquad \nu = 1,\dots,p, \tag{2.8}$$

$$B_1 \colon \sum_{\mu=0}^{n-1} B_{\lambda\mu}(D^\mu y)(1) = 0, \qquad \lambda = 1,\dots,n-p, \tag{2.9}$$

where

$$D^0 = I, \qquad D^\mu = D_\mu D_{\mu-1} \dots D_1.$$

Let

(a) $\tilde{A} = \|A_{\nu\mu}(-1)^\mu\|$ be sign consistent of order p (sC_p) and of rank p,

(b) $B = \|B_{\lambda\mu}\|$ be sC_q and of rank q where $q = n - p$.

We refer to Karlin (1971) and the Appendix for definitions on sign consistency.

Further let $L_n y = 0$ subject to $B_0 \cap B_1$ have only the trivial solution.

Theorem 2.3 [which may be deduced from Karlin (1971)] states that if the Green's function of $(-1)^{n-p}L_n$ exists then the eigenvalues corresponding to this kernel are positive and simple (algebraic and geometric multiplicity are the same and equal to unity).

Theorem 2.3 will be useful in its application to the Rayleigh–Bénard problem. We now briefly refer to some results on completeness due to Birkhoff (1908) and refined by Naimark (1967). The main idea is to classify homogeneous boundary conditions as regular or irregular and subsequently apply theorems which assure completeness of eigenfunctions for problems with regular boundary conditions.

The boundary conditions of interest here are of Stürm type

$$\begin{aligned} U_{j0}(y) &= y_0^{(k_j)} + \sum_{\nu=1}^{k_j-1} \alpha_{j\nu} y_0^{(\nu)} = 0, \\ U_{j1}(y) &= y_1^{(k_j')} + \sum_{\nu=1}^{k_j'-1} \beta_{j\nu} y_1^{(\nu)} = 0, \end{aligned} \qquad j = 1, 2 \ldots \mu \tag{2.10}$$

where y_0 and y_1 are the values of y at the two end points and $n = 2\mu$ (n is the order of the differential equation). $U_{j0}(y)$ and $U_{j1}(y)$ are boundary conditions at the two end points. Note that $U_\nu(y)$ given by Eq. (2.1b) contains boundary conditions at both end points and thus Eq. (2.10) is a special case of Eq. (2.1b).

Let $\lambda = -\rho^n$ and ω_i $(i = 1, \ldots, n)$ represent the n roots of -1 and ordered in such a way that

$$\text{Real}(\rho\omega_1) \leqslant \text{Real}(\rho\omega_2) \leqslant \cdots \leqslant \text{Real}(\rho\omega_n),$$

and further define γ_1 and γ_{-1} such that

$$\gamma_1 = \pm \begin{vmatrix} \omega_1^{k_1} & \cdots & \omega_{\mu-1}^{k_1} & \omega_{\mu+1}^{k_1} \\ \omega_1^{k_\mu} & \cdots & & \omega_{\mu+1}^{k_\mu} \end{vmatrix} \begin{vmatrix} \omega_\mu^{k_1'} & \omega_{\mu+2}^{k_1'} & \cdots & \omega_n^{k_1'} \\ \omega_\mu^{k_\mu'} & \omega_{\mu+2}^{k_\mu'} & \cdots & \omega_n^{k_\mu'} \end{vmatrix} \tag{2.11a}$$

and

$$\gamma_{-1} = \pm \begin{vmatrix} \omega_1^{k_1} & \cdots & \omega_\mu^{k_1} \\ \omega_1^{k_\mu} & \cdots & \omega_\mu^{k_\mu} \end{vmatrix} \begin{vmatrix} \omega_{\mu+1}^{k_1'} & \cdots & \omega_n^{k_1'} \\ \omega_{\mu+1}^{k_\mu'} & \cdots & \omega_n^{k_\mu'} \end{vmatrix}. \tag{2.11b}$$

Birkhoff (1908) and Naimark (1967) describe boundary conditions (2.10) as regular if γ_1 and γ_{-1} are each nonzero. Here $n - 1 \geqslant k_1 > k_2 > \cdots > k_\mu \geqslant 0$ and $n - 1 \geqslant k_1' > k_2' > \cdots > k_\mu' \geqslant 0$.

THEOREM 2.4 (cf. Naimark (1967)). If $L(y)$ is an ordinary differential operator of the type given by relation (2.1b) and coupled with regular homogeneous boundary conditions, then the following expression is possible:

$$f(x) = \sum_{n=1}^{\infty} \int_0^1 f(\xi)\, \overline{y_n^*}(\xi)\, d\xi \cdot y_n(x), \tag{2.12}$$

where $\{y_n\}$ and $\{\overline{y_n^*}\}$ are sets of the eigenfunctions and conjugate adjoint eigenfunctions and are biorthogonal. This "perfect" representation means that $\{y_i\}$ forms a complete space (a closed linear manifold which coincides with the Hilbert space $L_{(2)}(0,1)$. Completeness is in the metric generated by the usual $L_{(2)}$ inner product). The theorem requires inspection only of the boundary conditions. In the next section we consider two classical examples for the use of such earlier established theorems. We will also have occasion to refer to the work of other investigators who might have obtained similar results through alternate means.

3. Examples.

Example 1. *The Rayleigh–Marangoni* (*R–M*) *problem*. The critical Rayleigh number in the R–M problem, in a finite rectangular bounded geometry, is itself bounded below by the analogous problem with stress-free vertical sides. Separation of horizontal space variables leads to a sixth-order ordinary differential equation in the perturbed temperature field "θ_0" (see Joseph (1976)). The principle of exchange of stability is invoked (cf. Petty and Stevens (1980)).

$$(D^2 - \alpha^2)^3\theta_0 = -\mathrm{Ra}_0\alpha^2\theta_0, \tag{3.1}$$

$$D\theta_0 = \mathrm{Bi}\,\theta_0, \qquad \text{at } z = 0, \tag{3.2}$$

$$D^2\theta_0 = \alpha^2\theta_0, \qquad \text{at } z = 0, \tag{3.3}$$

$$D^4\theta_0 = \alpha^4\theta_0 + \mathrm{Ma}\,\alpha^2\theta_0, \quad \text{at } z = 0, \tag{3.4}$$

where $D \equiv d/dz$ and "$z = 0$" represents the top horizontal surface.

In the case of rectangular containers,

$$\alpha^2 = \pi^2\left(i^2/L_1^2 + j^2/L_2^2\right),$$

where i, j are integers and L_1, L_2 are the horizontal dimensions.

In the case of axisymmetric motion in cylindrical containers, we have α_n as the roots of $J_1(\alpha_n R) = 0$, where the cylinder radius is R and the height is unity. If we have nonaxisymmetric motion in a right circular cylinder, we may impose the condition of vanishing tangential vorticity in order to separate the horizontal spatial variables. This case is a lower bound on the convection problem with rigid vertical sides and can be shown in a manner similar to Narayanan and Abasaeed (1985). Calculations of Jones and Moore (1979) also bear out this fact.

In the above equations, Bi is the Biot number, Ma is the Marangoni number and represents the ratio of surface tension gradient driving effects to viscous, thermometric dissipative effects, and Ra_0 is the Rayleigh number. The bottom horizontal surface may be rigid or stress free and highly conducting or poorly conducting to temperature. In our example we have assumed a stress-free conducting bottom for the sake of convenience. Any other boundary conditions could be chosen.

Thus,

$$\theta_0 = 0, \quad \text{at } z = 1, \tag{3.5}$$

$$D^2\theta_0 = 0, \quad \text{at } z = 1, \tag{3.6}$$

$$D^4\theta_0 = 0, \quad \text{at } z = 1. \tag{3.7}$$

Ra_0 represents the eigenvalues for fixed α, Bi, and Ma. For a further discussion of these boundary conditions, we refer to Joseph (1976). We note that the eigenfunctions are integral analytic functions of Ra_0 and that a countable infinite set of Ra_0 will therefore arise. On the other hand, if Ra_0 is fixed then it might appear that only certain Ma will give rise to nontrivial θ_0. We may substitute the boundary operators from Eqs. (3.2)–(3.7) into Eq. (2.2) and separate the matrix into two parts—one which is independent of Ma and one which is linear and homogeneous in the first power of Ma. Since $\det \mathcal{U} = 0$ we would obtain Ma as a ratio of two determinants and is thus unique. This situation would naturally occur for all other types of boundary conditions at the bottom surface. Rosenblat et al. (1981) have demonstrated this fact numerically from the equations of Nield (1964).

Now Joseph (1976) shows that the operator $(D^2 - \alpha^2)^3$ may be written in an iterated form as required by Theorem 2.3. It is clear that we can therefore obtain the operator $(D^2 - \alpha^2)^3$ also in an iterated form with some positive weight function $\mu_i(z)$. Here $\mu_i(z)$ are defined as

$$\mu_1 \equiv \frac{1}{\cosh(\alpha z)}, \qquad \mu_2 \equiv \cosh^2(\alpha z), \qquad \mu_3 = \frac{1}{\cosh^2(\alpha z)},$$

$$\mu_4 \equiv \cosh^2(\alpha z), \qquad \mu_5 \equiv \frac{1}{\cosh^2(\alpha z)}, \qquad \mu_6 \equiv \cosh^2(\alpha z),$$

and

$$\mu_0 \equiv \frac{1}{\cosh \alpha z}.$$

Thus, $(D^2 - \alpha^2)^3$ is of the form L_n (ref. Theorem 2.3) with $n = 6$. A simple calculation will show that the boundary conditions (3.2)–(3.7) satisfy the requirements on $A_{\nu\mu}$ and $B_{\nu\mu}$ in Theorem 2.3. Thus we obtain as a result from Theorem 2.3 the following property (we refer to the Appendix for explicit forms of these matrices).

THEOREM 3.1. The Rayleigh–Marangoni eigenvalue problem given by equations (3.1)–(3.7) with arbitrary Marangoni effects will give rise to algebraically simple eigenvalues Ra_0 and if Ra is fixed, Ma is necessarily unique.

This result is of importance in the understanding of cascaded bifurcation. The possibility of multiple solutions with respect to horizontal eigenmodes still exists (see Reiss (1983)). However, our result indicates that multiple vertical eigenmodes cannot coexist for the same eigenvalue for all values of Bi and Ma. Naturally this is also of computational interest. Table 1 gives numerical evidence, due to earlier researchers.

TABLE 1. The first three critical Rayleigh numbers at various Bi for Ma = 0 and $\alpha = 2.5$ (These are simple eigenvalues.)

Bi	Ra_{01}	Ra_{02}	Ra_{03}
0	694.75	15,625	138,625
0.1	704.88	15,746	139,215
1	777.93	16,674	143,874
10	991.15	20,164	164,890
100	1092.55	22,380	181,714
1000	1106.60	22,723	184,659

An important observation is that equations (3.1)–(3.7) are non-selfadjoint in the inner product $\int_0^1 \bar{\mathbf{a}}' \cdot \mathbf{b}\, dz$ value where **a**, **b** are two vectors. The non-selfadjointness is apparent for all values of Ma. Vrentas et al. (1982) have investigated the special case of Ma = 0 and rigid conditions at $z = 1$. They have shown that the geometric multiplicity is unity and identify this property as a prerequisite for completeness of the eigenfunction set. It is not, however, a necessary prerequisite that the eigenvalues are simple in order that the eigenfunction set be complete (cf. Birkhoff (1908)). The adjoint boundary conditions are

$$\theta_0^* = 0 \quad \text{at } z = 0, \tag{3.8}$$

$$D^2\theta_0^* = 0 \quad \text{at } z = 0, \tag{3.9}$$

$$\mathrm{Bi}\, D^4\theta_0^* - \alpha^2 \mathrm{Ma}\, D\theta_0^* = \left(D^5\theta_0^* + \alpha^4 D\theta_0^* - 2\alpha^2 D^3\theta_0^*\right) \quad \text{at } z = 0, \tag{3.10}$$

$$\theta_0^* = 0 \qquad \text{at } z = 1, \tag{3.11}$$

$$D^2\theta_0^* = 0 \quad \text{at } z = 1, \tag{3.12}$$

$$D^4\theta_0^* = 0 \quad \text{at } z = 1. \tag{3.13}$$

Here superscript " $*$ " represents adjoint eigenfunctions.

Application of boundary conditions (3.2)–(3.7) to Eqs. (2.11a)–(2.11b) yields γ_{-1} and γ_1 to be nonzero. In accordance with Theorem (2.4) we see that the set of $\theta_0(z)$ is complete. This result is important for construction of solutions to the nonlinear Rayleigh–Marangoni problem. The values of γ_{-1}, γ_1 for the case of a rigid conducting surface at $z = 1$ are also nonzero. It is precisely these eigenfunctions which have been used by Rosenblat et al. (1981) for the construction of solutions to a nonlinear R–M problem (with the special case of Ra = 0). Since it turns out that γ_i (in Eqs. 2.11a and 2.11b) are all nonzero for varied combinations of boundary conditions on the fluid (rigid or stress-free) and temperature (conducting and nonconducting), the following property results.

THEOREM 3.2. The eigenfunction set for the sixth-order operator (3.1), subject to all combinations of boundary conditions for the fluid (stress-free, rigid, or Marangoni type) and temperature (Dirichlet or Robin) is complete in $L_{(2)}(0,1)$.

We remark that the method of DiPrima and Habetler (1969), which has been applied to Orr–Sommerfeld equations, uses a theorem due to Mikhlin (1965) which states that a positive linear operator which is bounded below and with domain dense in a Hilbert space can be extended to a selfadjoint operator that possesses an inverse on the entire space. It, however, involves the search for the selfadjoint extension.

Example 2. *The extended Graetz problem with homogeneous and heterogeneous reaction.* The problem of forced convective laminar flow of a reactant down a cylindrical tube and undergoing homogeneous first-order reaction, as well as a heterogeneous first-order reaction, has been considered by Dang (1983). The following eigenvalue problem results:

$$\frac{d^2C_n}{d\eta^2} + \frac{1}{\eta}\frac{dC_n}{d\eta} + \left[(1-\eta^2)\left(K + \frac{1}{Pe}\right)\beta_n^2 - K + \frac{\beta_n^4}{Pe^2}\left(K + \frac{1}{Pe}\right)^2\right]C_n = 0, \tag{3.14}$$

subject to

$$dC_n/d\eta = 0 \quad \text{at } \eta = 0, \tag{3.15}$$

$$dC_n/d\eta + \alpha C_n = 0 \quad \text{at } \eta = 1, \tag{3.16}$$

where $\alpha \geqslant 0$ and $K \geqslant 0$. Equation (3.15) follows from (3.14) in the limit of $\eta \to 0$.

We may define $\lambda_n \equiv (K + 1/Pe)\beta_n^2$ and prove that λ_n is real. Equation (3.14) may be rewritten as

$$\frac{d}{d\eta}\left(\eta \frac{dC_n}{d\eta}\right) + \eta\left[(1 - \eta^2)\lambda_n - K + \lambda_n^2\right]C_n = 0 \tag{3.17}$$

subject to Eqs. (3.15) and (3.16).

On multiplication of Eq. (3.17) with C_n^* (the conjugate of C_n) and integration, we get

$$\int_0^1 \eta\left[(1 - \eta^2)\lambda_n + \lambda_n^2\right]|C_n|^2\, d\eta > 0. \tag{3.18}$$

Let $\lambda_n \equiv a + ib$. The inequality (3.18) will yield the following:

$$b\int_0^1 \eta\left[(1 - \eta^2)|C_n|^2 + 2a|C_n|^2\right] d\eta = 0, \tag{3.19}$$

and

$$\int_0^1 \eta\left[(1 - \eta^2)a|C_n|^2 + (a^2 - b^2)|C_n|^2\right] d\eta > 0. \tag{3.20}$$

From Eq. (3.19), we see that the integral must vanish (and $a < 0$) if $b \neq 0$. Using this fact in relationship (3.20) we get

$$-\int_0^1 \eta|C_n|^2(a^2 + b^2)\, d\eta > 0. \tag{3.21}$$

This is a contradiction and therefore $b = 0$, i.e., λ_n *is real.*

In order to show that λ_n is simple, we assume the contrary (cf. Davis (1974)):

$$\left|\frac{d\tilde{C}_n}{d\eta}\right|_{\eta=1} + |\alpha\tilde{C}_n|_{\eta=1} = 0, \tag{3.22}$$

where the "~" (tilde) overbar represents derivatives with respect to λ_n.

The differential equation satisfied by $\tilde{C}_n$ is given by

$$\frac{d}{d\eta}\left(\eta \frac{d\tilde{C}_n}{d\eta}\right) + \eta\left[(1 - \eta^2)\lambda_n - K + \lambda_n^2\right]\tilde{C}_n = -\eta\left[(1 - \eta^2)C_n + 2\lambda_n C_n\right]. \tag{3.23}$$

The right-hand side of (3.23) in conjunction with (3.17) may be represented by

$$\frac{1}{\lambda_n}\left[\frac{d}{d\eta}\left(\eta \frac{dC_n}{d\eta}\right) - KC_n - \lambda_n^2 C_n\right].$$

On executing the solvability condition between equations (3.23) and (3.17) and on using the proven fact that λ_n is real, we obtain

$$\frac{1}{\lambda_n}\int_0^1 \left[C_n^* \frac{d}{d\eta}\left(\eta \frac{dC_n}{d\eta}\right) - K|C_n|^2 - \lambda_n^2|C_n|^2\right] = 0.$$

This is not possible, since the integral is negative; hence λ_n are simple. Thus we have the following theorem.

THEOREM 3.3. The eigenspace of the extended Graetz problem in a tube with homogeneous and heterogeneous first-order reactions contains real and simple eigenvalues.

Boundary conditions of the type (3.15) and (3.16) have been clearly discussed by Naimark (1967) for second-order operators. These are Stürm conditions and Eq. (3.14) is of the generalized eigenvalue problem form with coefficients dependent on λ_n. The corresponding values of γ_1 and γ_{-1} in Eqs. (2.11a) and (2.11b) are nonzero. Naimark indicates that the generalized eigenfunctions are complete. Thus we have the following theorem.

THEOREM 3.4. The eigenspace of the extended Graetz problem is complete.

Discussion and conclusions. We have used some theorems of Naimark to prove the completeness of eigenspaces for problems that arise in natural and forced convection. In the case of natural convection with buoyancy and surface tension effects, the model problem that arises has been discussed by earlier workers such as Rosenblat et al. (1981). These authors have assumed the completeness property and we have justified their assumption. We have also shown, in the general case of the model problem, that the critical Marangoni number is unique without recourse to any numerical calculations.

In the discussion of the forced convection Graetz problem, the completeness of eigenfunctions is also easily shown by appealing to the same general theorems of Naimark. Deavours (1971) has discussed a problem which bears resemblance to the Graetz problem and proved completeness, and Pearlstein (1975) has shown how one might calculate coefficients in the eigenfunction expansion. However Papoutsakis et al. (1980) have also shown a similar result by uncovering an unusual inner product in which a vector problem derived from Eq. (3.14) is selfadjoint. Thus an advantage of such a technique resides in being able to deal satisfactorily with the set of eigenfunctions which are naturally complete and wherein consideration of the adjoint to Eq. (3.14) is quite unnecessary. The disadvantage lies in the fact that the inner product is problem specific and the methodology of its derivation is not apparently extended for other different types of problems (such as Example 1). The method of Naimark that is espoused in this paper, however, can be applied to a number of general problems, and the Graetz problem is a simple example of its use. The disadvantage lies in the fact that the solution of the inhomogeneous extension of Eq. (3.14) requires the knowledge of adjoint eigenfunctions in order to suitably form the biorthogonal set.

We have also considered simplicity of eigenvalues for both the Rayleigh–Marangoni problem as well as the extended Graetz problem. In the case of the former, this result would imply that multiple vertically stacked cells cannot coexist, and thus cascaded bifurcation will not occur for fixed values of the horizontal wave number α_n. We may point out that simplicity of the Graetz eigenspace could also be observed if it could be shown that algebraic and geometric multiplicity are the same. Thus selfadjointness in some inner product would assure us of its simplicity for the types of boundary conditions that are discussed in this paper (see Appendix for a proof). We have, however, presented an alternate method in Sec. 3 for proving simplicity, and this does not require the need for uncovering any unusual inner product.

Appendix.

A. *Alternate proof that* λ *are simple for the extended Graetz problem with reaction.* We assume, here, that Eqs. (3.14)–(3.16) may be recast in a selfadjoint form using an unusual inner product in a manner similar to Papoutsakis et al. (1981). In this case the geometric and algebraic multiplicity of the eigenspace are identical.

Proof. Suppose the eigenvalue λ_n is multiple and $X_1(\eta)$ and $X_2(\eta)$ are two eigenfunctions corresponding to it. Let

$$X = X_1(0)X_2(\eta) - X_2(0)X_1(\eta). \tag{A.1}$$

Then

$$\frac{d}{d\eta}\left[\eta\frac{dX}{d\eta}\right] + \eta\left[(1-\eta^2)\lambda_n - K + \lambda_n^2\right]X = 0, \tag{A.2}$$

$$X(\eta = 0) = 0, \tag{A.3}$$

$$\frac{dX}{d\eta}(\eta = 0) = 0. \tag{A.4}$$

Now (A.2)–(A.4) constitute an initial value problem which is homogeneous. Therefore, $X = 0$ is the only possible solution and from (A.1) it is clear that $X_1(\eta)$ is linearly dependent on $X_2(\eta)$.

Therefore, the geometric multiplicity (and hence the algebraic multiplicity) is one.

B. The matrices

$$\tilde{A} \equiv \begin{bmatrix} \mathrm{Bi} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -a^2\mathrm{Ma} & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

will be used in Theorem 2.3 and reflect stress-free conducting bottom (matrix B) and Marangoni conditions at the top ($z = 0$ and matrix A).

A matrix is sign consistent of order p (sC_p) if all $p \times p$ nonzero subdeterminants are of the same sign (i.e., $\geqslant 0$ or $\leqslant 0$). The matrices are sC_3 and of rank 3.

Thus the criteria of the theorem are satisfied. We may easily show that they are also satisfied for the case of a rigid conducting bottom wall.

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