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A BOUNDARY INTEGRAL EQUATION FOR DEFORMATIONS OF AN ELASTIC BODY WITH AN ARC CRACK*

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Abstract. A solution for a class of two-dimensional elasticity problems concerning an isotropic body with an arc crack in its interior is obtained in terms of an integral taken around the exterior boundary of the body. A numerical procedure for solving this integral equation is outlined and numerical results for a test problem are given.

1. Introduction. In recent years, the boundary integral equation method has been used as a numerical tool for solving various crack problems. For the case of a stress-free straight crack, the difficulties associated with modeling the crack may be effectively avoided by using this method as shown in Snyder and Cruse [1] and Clements and Haselgrove [2].

In this paper, we consider a class of two-dimensional elasticity problems concerning an isotropic body with an arc crack in its interior. The arc crack is assumed to become stress-free when the body is elastically deformed. A solution is obtained in terms of an integral taken around only the exterior boundary of the body. The crack surface is avoided in the integration by a suitable choice of Green's function. Such an integral equation is particularly useful for numerical calculation because it avoids the need to integrate around the boundary of the crack, where stress singularities can cause difficulties with any numerical integration schemes used. In addition, there is no need to approximate the arc crack by using a set of straight line segments. A simple and straightforward numerical procedure for solving this integral equation is outlined and numerical results for a particular test problem are given.

2. Statement of the problem. Referred to a Cartesian frame Ox_1x_2, consider a homogeneous isotropic elastic body R which satisfies either the conditions of plane strain or plane stress. The elastic behavior of the body is governed by the basic equations of two-dimen-

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sional elasticity, which take the form

$$(\lambda + \mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0, \tag{2.1}$$

where i, $j = 1, 2$, u_i denote the displacements, λ and μ are the appropriate constants, and the usual convention of summing over a repeated index is adopted.

Our task here is to find the displacements and stresses throughout the elastic body R, that is, we are required to find a solution to (2.1) which is valid in the region R in the Ox_1x_2 plane with boundary Γ and which satisfies specified displacements or tractions on Γ. The boundary Γ consists of two parts. The first part comprises the two faces of the crack L which lies along an arc of the circle $x_1^2 + x_2^2 = a^2$. The arc of the circle subtends an angle $-\phi < \theta < \phi$ where $\theta = \arctan(x_2/x_1)$. The second part of the boundary Γ is an arbitrary curve C which encloses a region containing the arc crack L (see Fig. 1). Displacements and/or tractions are specified on C in such a way that the crack surface becomes stress-free due to the elastic deformation.

3. Solution in terms of potentials. The partial differential equations given by the system (2.1) have the general solution (see, for example, Sokolnikoff [3])

$$\begin{aligned} 2\mu(u_1 + iu_2) &= \kappa\Omega(z) - z\,\overline{\Omega'(z)} - \overline{\omega(z)}, \\ \sigma_{22} - i\sigma_{12} &= \Omega'(z) + \overline{\Omega'(z)} + z\,\overline{\Omega''(z)} + \overline{\omega'(z)}, \\ \sigma_{11} + \sigma_{22} &= 2\left(\Omega'(z) + \overline{\Omega'(z)}\right), \end{aligned} \tag{3.1}$$

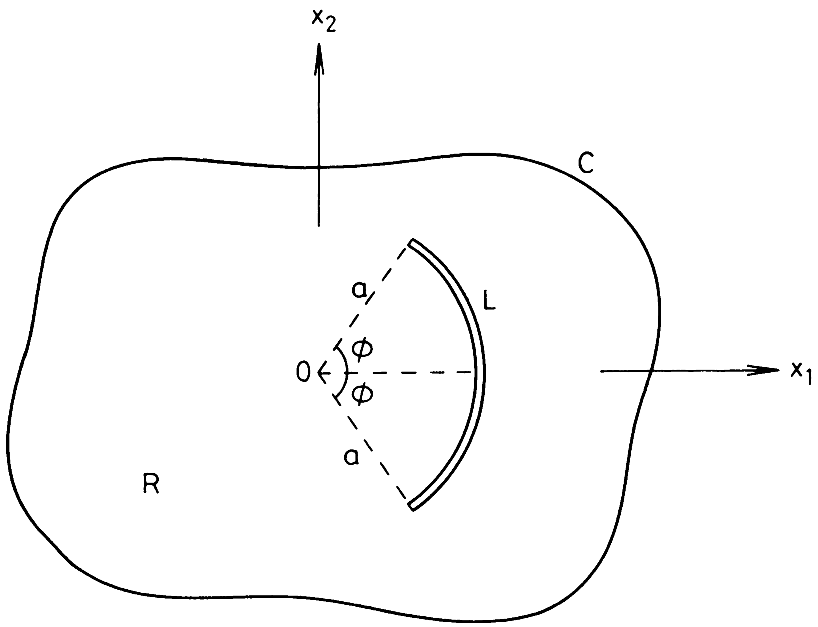

FIG. 1. An isotropic body with an arc crack in its interior.

where $\sigma_{ij}(i, j = 1,2)$ are the stresses in Cartesian coordinates, $i = \sqrt{-1}$, $z = x_1 + ix_2$, the potentials $\Omega(z)$ and $\omega(z)$ are analytic functions of z in the domain of interest, the primes denote derivatives with respect to z, and the bars denote the complex conjugates. The constant κ is given by

$$\kappa = (\lambda + 3\mu)/(\lambda + \mu).$$

In polar coordinates (r,θ) ($x_1 = r\cos\theta$ and $x_2 = r\sin\theta$), the displacements and the stresses are given by

$$\begin{gathered} 2\mu(u_r + iu_\theta) = \exp(-i\theta)\left(\kappa\Omega(z) - z\,\overline{\Omega'(z)} - \overline{\omega(z)}\right), \\ \sigma_{rr} + i\sigma_{r\theta} = \Omega'(z) + \overline{\Omega'(z)} - \left(\bar{z}\,\overline{\Omega''(z)} + \frac{\bar{z}}{z}\,\overline{\omega'(z)}\right), \\ \sigma_{rr} + i\sigma_{r\theta} = 2\left(\Omega'(z) + \overline{\Omega'(z)}\right). \end{gathered} \tag{3.2}$$

4. A fundamental singular solution. A fundamental singular solution of (2.1) is given by (3.1) or (3.2) with

$$\begin{gathered} \Omega(z) = -\psi\log(z - z_0), \\ \omega(z) = \kappa\bar{\psi}\log(z - z_0) + \psi\bar{z}_0/(z - z_0), \end{gathered} \tag{4.1}$$

where ψ is an arbitrary complex constant and $z_0 = \xi_1 + i\xi_2$ is a given point in the z complex plane. From (3.2) the stresses σ_{rr} and $\sigma_{r\theta}$ which correspond to the potentials given in (4.1) are given by

$$\sigma_{rr} + i\sigma_{r\theta} = \frac{-\psi}{z - z_0} - \frac{\bar{\psi}}{\bar{z} - \bar{z}_0} - \left(\frac{z\bar{\psi}}{(\bar{z} - \bar{z}_0)^2} + \frac{\bar{z}}{z}\left(\frac{\kappa\psi}{\bar{z} - \bar{z}_0} - \frac{\bar{\psi}z_0}{(\bar{z} - \bar{z}_0)^2}\right)\right). \tag{4.2}$$

On the crack surface L, (4.2) becomes

$$\begin{aligned} \sigma_{rr} + i\sigma_{r\theta} &= -\left(\frac{\psi}{t - z_0} - \frac{\bar{\psi}\eta t}{a^2(t - \eta)} + \frac{\eta^2\bar{\psi}t}{a^2(t - \eta)^2} - \frac{\kappa\eta\psi}{t(t - \eta)} - \frac{\eta^2\bar{\psi}z_0}{a^2(t - \eta)^2}\right) \\ &\stackrel{\text{def}}{=} -f(t), \end{aligned} \tag{4.3}$$

where $t \in L$ and $\eta = a^2/\bar{z}_0$.

5. Solution to a particular crack problem. Consider an infinite isotropic elastic body containing the arc crack L. We assume that the body is deformed by known stresses at infinity and the stresses on the inner and outer edges of the crack are given, respectively, by

$$(\sigma_{rr} + i\sigma_{r\theta})^+ = g_1(t), \qquad (\sigma_{rr} + i\sigma_{r\theta})^- = g_2(t), \tag{5.1}$$

where $t \in L$ and $g_1(t)$ and $g_2(t)$ are given functions of t.

The potentials $\Omega(z)$ and $\omega(z)$ together with their necessary derivatives which provide a solution to (2.1) satisfying the conditions (5.1) may be obtained from (see England [4])

$$2\Omega'(z) - \Theta'(z) = \frac{P(z)}{2\pi i}\int_L \frac{g_1(t) + g_2(t)}{P^+(t)(t - z)}\,dt + \left(\frac{\beta_1}{z^2} + \frac{\Omega_1}{z} + D_0 + D_1 z\right)P(z), \tag{5.2}$$

$$\Theta'(z) = \frac{1}{2\pi i}\int_L \frac{g_1(t) - g_2(t)}{t - z}\,dt + A + iB - \overline{\Omega'(0)} + \frac{\kappa(X + iY)}{2\pi(1 + \kappa)z} + \frac{(C + iD)a^2}{z^2}, \tag{5.3}$$

$$\omega(z) = \overline{\Omega\left(\frac{a^2}{\bar{z}}\right)} - \frac{a^2}{z}\Omega'(z) - \overline{\Theta\left(\frac{a^2}{\bar{z}}\right)}, \tag{5.4}$$

where $P(z)$ is the Plemelj function defined by

$$P(z) = (z - a\exp(i\phi))^{-1/2}(z - a\exp(-i\phi))^{-1/2}, \tag{5.5}$$

$$\lim_{|z|\to\infty} zP(z) = 1, \tag{5.6}$$

and $P^+(t)$ is the Plemelj function over the inner edge of the crack. The constant $X + iY$ which appears in (5.3) is given by

$$X + iY = i\int_L (g_2(t) - g_1(t))\,dt. \tag{5.7}$$

By assuming that the rotation vanishes at infinity and using the known stresses at infinity, the constants $A + iB$ and $C + iD$ in (5.3) can be determined. The constants β_1 and β_2 are chosen in such a way that $\Omega'(z)$ in (5.2) is well defined as $z \to 0$, while D_0 and D_1 are obtained by examining the behaviors of (5.2) and (5.3) for large $|z|$.

The integral over the crack surface L which appears in (5.2) can be evaluated by using the results (see England [4])

$$\int_L \frac{h(t)}{P^+(t)(t - z)}\,dt = \frac{1}{2}\left(2\pi i S - \int_{C_\infty} \frac{h(\xi)}{P(\xi)(\xi - z)}\,d\xi\right), \tag{5.8}$$

where C_∞ is a circle of origin O and radius r, described anticlockwise, with $r \to \infty$ and S is the sum of the residues of the poles of $h(\xi)/P(\xi)(\xi - z)$.

6. A Green's function. In this section, a particular singular solution to (2.1) which is valid in the region R bounded by $\Gamma = C + L$ is sought. We require the solution to provide us with zero traction on the crack surface L. To find such solution, we write

$$\Omega(z) = \Omega_1(z) + \Omega_2(z), \qquad \omega(z) = \omega_1(z) + \omega_2(z), \tag{6.1}$$

where $\Omega_1(z)$ and $\omega_1(z)$ are given by $\Omega(z)$ and $\omega(z)$, respectively, in (4.1) and the potentials $\Omega_2(z)$ and $\omega_2(z)$ are to be determined.

Now, from (4.3) the stresses $\sigma_{rr}^{(1)}$ and $\sigma_{r\theta}^{(1)}$ which correspond to the potentials $\Omega_1(z)$ and $\omega_1(z)$ are given by

$$\left(\sigma_{rr}^{(1)} + i\sigma_{r\theta}^{(1)}\right)^+ = -f(t), \qquad \left(\sigma_{rr}^{(1)} + i\sigma_{r\theta}^{(1)}\right)^- = -f(t), \tag{6.2}$$

on the inner and outer edges of the crack L, respectively. Thus we require the stresses $\sigma_{rr}^{(2)}$ and $\sigma_{r\theta}^{(2)}$ which correspond to the potentials $\Omega_2(z)$ and $\omega_2(z)$ to be such that

$$\left(\sigma_{rr}^{(2)} + i\sigma_{r\theta}^{(2)}\right)^+ = f(t), \qquad \left(\sigma_{rr}^{(2)} + i\sigma_{r\theta}^{(2)}\right)^- = f(t), \tag{6.3}$$

on the inner and outer edges of the crack, respectively.

By assuming that the stresses which correspond to $\Omega_2(z)$ and $\omega_2(z)$ vanish at infinity and using the results in Sec. 5, we obtain

$$2\Omega_2'(z) + \overline{\Omega_2'(0)} = P(z)\left(g(z) + \frac{\alpha}{z} + \beta + \gamma z\right), \tag{6.4}$$

where $\alpha = \kappa a \psi$, $\beta = -(\psi + [a\overline{\Omega_2'(0)} + \eta\bar{\psi}/a]\cos\phi)$, $\gamma = \overline{\Omega_2'(0)} + \eta\bar{\psi}/a^2$, and the function $g(z)$ is given by

$$g(z) = \frac{f(z)}{P(z)} - \frac{\psi}{P(z_0)(z - z_0)}$$
$$+ \frac{1}{P(\eta)}\left(\frac{\kappa\psi}{z-\eta} + \frac{\bar{\psi}\eta^2}{a^2(z-\eta)}\left[\frac{z_0}{z-\eta} - \frac{z_0 P'(\eta)}{P(\eta)} - \frac{\eta}{z-\eta} + \frac{\eta P'(\eta)}{P(\eta)}\right]\right). \tag{6.5}$$

The constant $\Omega_2'(0)$ can be evaluated by letting $z \to 0$ in (6.4).

Integrating (6.4), we obtain

$$2\Omega_2(z) = -\frac{\eta\bar{\psi}z}{a^2} + \frac{\bar{\psi}\eta^2}{a^2}\left(\frac{z_0-\eta}{z-\eta}\right)\left(1 - \frac{P(\eta)}{P(z)}\right) + \frac{1}{P(z)}\left(\overline{\Omega_2'(0)} + \frac{\bar{\psi}\eta}{a^2}\right) - \overline{\Omega_2'(0)}\,z$$
$$+\kappa\psi\log\left(\frac{P(\eta)(zP(z))^2}{2a\Upsilon(z,\eta)[(a - z\cos\phi)P(z) + 1]}\right)$$
$$+\psi\log\left(\frac{\Upsilon(z,z_0)(1 - P(z)(z - a\exp(i\phi))}{P(z)P(z_0)(1 + P(z)(z - a\exp(i\phi))}\right), \tag{6.6}$$

where

$$\Upsilon(z,\eta) = P(\eta) + P(z) + P(z)(z-\eta)(\eta - a\cos\phi)(P(\eta))^2. \tag{6.7}$$

The indefinite integrals which arise in the derivation of (6.6) can be found in Petit Bois [5] and the more difficult ones are listed in the Appendix. From (5.4) the potential $\omega_2(z)$ is given by

$$\omega_2(z) = \overline{\Omega_2(a^2/\bar{z})} - (a^2/z)\Omega_2'(z) + \overline{\Omega_2'(0)}\,z. \tag{6.8}$$

By substituting (6.4) and (6.8) into (3.2), it can be directly verified that (6.3) is satisfied. The displacements and stresses which satisfy the condition of zero traction on L can now be obtained from (3.1) by using the potentials $\Omega(z)$ and $\omega(z)$ in (6.1).

7. A boundary integral equation. For each given j ($j = 1, 2$), let the complex constant ψ which appears in the potentials in (6.1) be defined by

$$\psi = \psi_j = \begin{cases} -\mu/\kappa, & \text{if } j = 1; \\ -i\mu/\kappa, & \text{if } j = 2; \end{cases} \tag{7.1}$$

and let these potentials $\Omega(z)$ and $\omega(z)$ with constant ψ_j be denoted by $\Omega(z, j)$ and $\omega(z, j)$, respectively. Denote the displacements and the stresses which correspond to $\Omega(z, j)$ and $\omega(z, j)$ by U_{ij} and Ξ_{ikj}, respectively ($i, j, k = 1, 2$). That is, from (3.1)

$$\begin{aligned} 2\mu j(U_{1j} + iU_{2j}) &= \kappa\Omega(z, j) - z\overline{\Omega'(z, j)} - \overline{\omega(z, j)}, \\ \Xi_{22j} - i\Xi_{12j} &= \Omega'(z, j) + \overline{\Omega'(z, j)} + z\overline{\Omega''(z, j)} + \overline{\omega'(z, j)}, \\ \Xi_{11j} + \Xi_{22j} &= 2\left(\Omega'(z, j) + \overline{\Omega'(z, j)}\right). \end{aligned} \tag{7.2}$$

Using the analysis in Rizzo [6], we obtain

$$u_j(\underline{x}_0) = \tau\nu\int_{\Gamma=C+L} \left[u_i(\underline{x})T_{ij}(\underline{x}, \underline{x}_0) - t_i(\underline{x})U_{ij}(\underline{x}, \underline{x}_0)\right] dS(\underline{x}), \tag{7.3}$$

where $\underline{x} = (x_1, x_2)$, $\underline{x}_0 = (\xi_1, \xi_2)$, $\tau = 1$ if $\underline{x}_0 \in R$, $\tau = 2$ if $\underline{x}_0 \in \Gamma$ (assuming that Γ has a continuously turning tangent), u_i and t_i are, respectively, the displacements and tractions for the problem described in Sec. 2, the constant ν is given by

$$\nu = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)},$$

and T_{ij} are given by

$$T_{ij} = \Xi_{ikj} n_k, \tag{7.4}$$

where n_k are the components of the unit normal vector to R on Γ.

Since $T_{ij} = 0$ over the crack surface L and we require our desired solution to the problem in Sec. 2 to be such that $t_i = 0$ on L, Eq. (7.3) reduces to

$$u_j(\underline{x}_0) = \tau\nu\int_C \left[u_i(\underline{x})T_{ij}(\underline{x}, \underline{x}_0) - t_i(\underline{x})U_{ij}(\underline{x}, \underline{x}_0)\right] dS(\underline{x}). \tag{7.5}$$

Equation (7.5) provides an integral equation for the numerical solution of the problem described in Sec. 2. If t_i are specified on C, then the integral can be discretized to yield a set of linear algebraic equations to determine u_i at various points on the boundary C. On the other hand, if u_i are specified on C, then the integral can be discretized to solve numerically for t_i at various points on C. Once t_i and u_i are known on the boundary C, u_i at any point in R can be computed numerically by use of (7.5) with $\tau = 1$.

8. A numerical example. Consider

$$\begin{aligned} \Omega(z) &= \frac{1}{8}\left(z + (z^2 + 1)^{1/2}\right) + \frac{1}{4}\left(\frac{1}{z} + \frac{1}{z}(z^2 + 1)^{1/2}\right), \\ \omega(z) &= \overline{\Omega\left(\frac{1}{\bar{z}}\right)} - \frac{1}{z}\Omega'(z) - \frac{1}{4}\left(\frac{1}{z} + 2z\right). \end{aligned} \tag{8.1}$$

It can easily be verified that the potentials in (8.1) provide an analytic solution to (2.1) for the domain R and the solution satisfies the condition $t_i = 0$ on a semicircular crack ($\phi = \pi/2$) which lies along the circle $x_1^2 + x_2^2 = 1$ (see, for example, England [4]). Note that the Plemelj function for this particular case is given by

$$P(z) = (z^2 + 1)^{-1/2}. \tag{8.2}$$

For a test problem, we use the potentials in (8.1) to generate displacements u_i on the boundary C, which is taken to be the square $ABCD$ in Fig. 2, and solve for t_i on $ABCD$ by using Eq. (7.5).

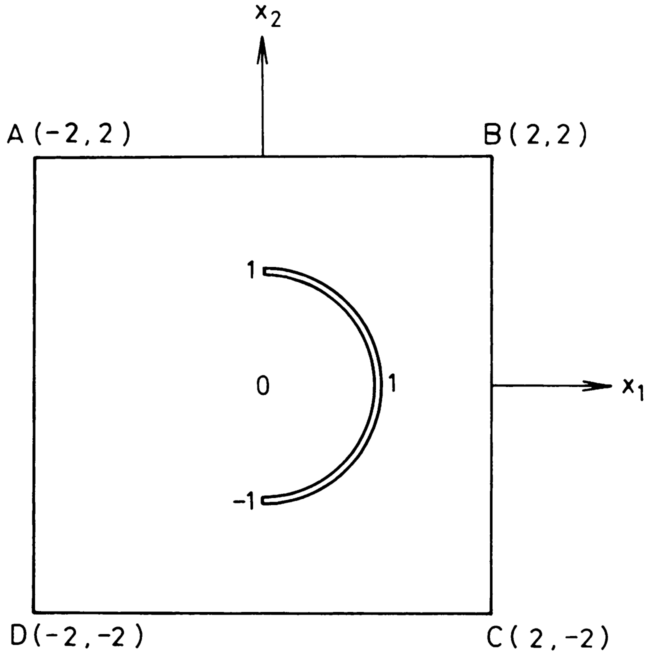

FIG. 2. Square boundary C and semicircular crack.

A simple and straightforward procedure is used to solve (7.5) numerically for t_i on C. The boundary C is discretized into N segments between the points $\underline{q}_{m-1}$ and $\underline{q}_m$, $m = 1, 2, \ldots, N$, with $\underline{q}_o = \underline{q}_N$. We assume that the unknown tractions t_i are constant on each of these segments and take the value t_i^m on the segment having the endpoints $\underline{q}_{m-1}$ and $\underline{q}_m$. By taking $\underline{x}_0$ in (7.5) to be each of the N midpoints of the segments in turn, (7.5) yields the approximation

$$\sum_{m=1}^{N} t_i^m \int_{\underline{q}_{m-1}}^{\underline{q}_m} U_{ij}(\underline{x}, \underline{p}_k)\, dS(\underline{x}) = -\frac{1}{2\nu} u_j(\underline{p}_k) + \sum_{m=1}^{N} \int_{\underline{q}_{m-1}}^{\underline{q}_m} u_i(x) T_{ij}(\underline{x}, \underline{p}_k)\, dS(\underline{x}), \tag{8.3}$$

for $j = 1, 2$, $k = 1, 2, \ldots, N$, and $\underline{p}_k$ is the midpoint of the segment having the endpoints $\underline{q}_{k-1}$ and $\underline{q}$. The system (8.3) provides us with a set of $2N$ linear algebraic equations in the unknowns t_i^m.

The integral on the right-hand side of (8.3) is evaluated by using Simpson's 3/8 rule (see Abramowitz and Stegun [7]). From (7.2), the function $U_{ij}(\underline{x}, \underline{x}_0)$ can be rewritten as

$$U_{ij}(\underline{x}, \underline{x}_0) = \delta_{ij} \log R(\underline{x}, \underline{x}_0) - \frac{1}{\kappa} R_{,i}(\underline{x}, \underline{x}_0) R_{,j}(\underline{x}, \underline{x}_0) + U_{ij}^*(\underline{x}, \underline{x}_0), \tag{8.4}$$

where $R(\underline{x}, \underline{x}_0) = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}$, δ_{ij} is the Kronecker delta, $R_{,i}$ is the first partial derivative of R with respect to x_i, and $U_{ij}^*(\underline{x}, \underline{x}_0)$ are the displacements which correspond to the potentials $\Omega_2(z)$ and $\omega_2(z)$ in (6.6) and (6.8), respectively, with $\psi = \psi_j$.

Thus the integral on the left-hand side of (8.3) can be written as

$$\int_{q_{m-1}}^{q_m} U_{ij}(\underline{x}, \underline{p}_k)\, dS(\underline{x}) = \delta_{ij} \int_{q_{m-1}}^{q_m} \log R(\underline{x}, \underline{p}_k)\, dS(\underline{x})$$

$$- \frac{1}{\kappa} \int_{q_{m-1}}^{q_m} R_{,i}(\underline{x}, \underline{p}_k) R_{,j}(\underline{x}, \underline{p}_k)\, dS(\underline{x}) + \int_{q_{m-1}}^{q_m} U_{ij}^*(\underline{x}, \underline{p}_k)\, dS(\underline{x}). \tag{8.5}$$

All the integrals on the right-hand side of (8.5) are evaluated by using a Simpson approximation except for the case $k = m$, for which the first two integrals on the right-hand side are evaluated differently. For $k = m$, by treating the kth segment as being straight, the first integral is given by

$$\int_{q_{m-1}}^{q_m} \log R(\underline{x}, \underline{p}_m)\, dS(\underline{x}) = R(\underline{q}_m, \underline{p}_m)\left[\log R(\underline{q}_m, \underline{p}_m) - 1\right]$$

$$+ R(\underline{q}_{m-1}, \underline{p}_m)\left[\log R(\underline{q}_{m-1}, \underline{p}_m) - 1\right],$$

and the second one is evaluated by using a trapezoidal approximation.

Choosing $\lambda = \mu = 1$, and discretizing the boundary C into n equal-length segments, we solve (8.3) for t_i^m. The results for $N = 16$ and $N = 80$ are given in Table 1 for various points on the boundary C. The numerical values obtained are in good agreement with the exact ones. It is clear that when the number of segments used is increased there is significant improvement in the accuracy of the numerical approximation.

9. Conclusion. A solution to a class of two-dimensional elasticity problems concerning an isotropic body with an arc crack in its interior is obtained in terms of an integral taken around only the exterior boundary of the body. Such an integral equation provides us with a good numerical procedure for calculating the displacements and tractions throughout the body because the need to model the crack surface and to integrate over it is not present. In addition, the procedure can readily accommodate a wide variety of exterior boundaries. The numerical results for the particular test problem give a good indication of the degree of accuracy which can be achieved by using this numerical procedure.

TABLE 1
Results for the test problem.

Points	$N = 16$		$N = 80$		Exact values	
(x_1, x_2)	t_1	t_2	t_1	t_2	t_1	t_2
(−1.50, 2.00)	0.03625	−0.07523	0.07658	−0.07443	0.07457	−0.07477
(−0.50, 2.00)	−0.02671	−0.01318	−0.03249	−0.02669	−0.03225	−0.02621
(0.50, 2.00)	0.00330	0.18365	0.02324	0.18381	0.02347	0.18256
(1.50, 2.00)	0.06585	−0.04418	−0.00984	−0.04677	−0.00578	−0.04703
(2.00, 1.50)	1.01329	−0.15384	1.05384	−0.12370	1.05399	−0.12719
(2.00, 0.50)	0.64091	−0.16823	0.64866	−0.17964	0.64975	−0.17903
(2.00, −0.50)	0.64102	0.16819	0.64869	0.17963	0.64975	0.17903
(2.00, −1.50)	1.01329	0.15366	1.05385	0.12370	1.05399	0.12719
(−2.00, −1.50)	−0.94234	0.12312	−0.96008	0.10702	−0.96013	0.10886
(−2.00, −0.50)	−0.78826	0.06708	−0.79023	0.07162	−0.79051	0.07162
(−2.00, 0.50)	−0.78835	−0.06677	−0.79025	−0.07154	−0.79051	−0.07162
(−2.00, 1.50)	−0.94223	−0.12249	−0.96007	−0.10689	−0.96013	−0.10886

Appendix. The following indefinite integrals arise in the derivation of $\Omega_2(z)$ in (6.6):

$$\int P(z)\,dx = \log\left(\frac{1 + P(z)(z - a\exp(i\phi))}{1 - P(z)(z - a\exp(i\phi))}\right),$$

$$\int zP(z)\,dz = \frac{1}{P(z)} + a\cos\phi \int P(z)\,dz,$$

$$\int \frac{P(z)}{z}\,dz = -\frac{1}{a}\log\left(\frac{2a([a - z\cos\phi]P(z) + 1)}{zP(z)}\right),$$

$$\int \frac{P(z)}{z-\eta}\,dz = -P(\eta)\log\left(\frac{P(\eta) + P(z) + (P(\eta))^2 P(z)[\eta - a\cos\phi]}{P(\eta)P(z)(z-\eta)}\right),$$

$$\int \frac{P(z)}{(z-\eta)^2}\,dz = (P(\eta))^2\left\{-\frac{1}{P(z)(z-\eta)} - (\eta - a\cos\phi)\int \frac{P(z)}{z-\eta}\,dz\right\}.$$

($P(z)$ is the Plemelj function defined in (5.5).)

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