QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLV, NUMBER 1
APRIL 1987, PAGES 155-164

ON OSCILLATION OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS*

By

M. R. S. KULENOVIĆ (*University of Rhode Island and University of Sarajevo, Jugoslavia*)

G. LADAS (*University of Rhode Island*)

A. MEIMARIDOU (*University of Rhode Island and Democritus University of Thrace, Xanthi, Greece*)

Abstract. Necessary, sufficient, and necessary and sufficient conditions are obtained for all solutions of the nonlinear differential equation

$$\frac{dy}{dt} + \sum_{j=1}^{n} q_j f\big(y(t-\tau_j)\big) = 0, \qquad t \geqslant 0, \tag{$*$}$$

to be oscillatory. These conditions are expressed in terms of the characteristic equation of the corresponding linear "variational" equation

$$\frac{dy}{dt} + \sum_{j=1}^{n} q_j y(t-\tau_j) = 0, \qquad t \geqslant 0. \tag{$**$}$$

Our results show that for a certain class of nonlinear functions f, $(*)$ oscillates if and only if $(**)$ oscillates. As an application of our results, we obtain simple sufficient and necessary and sufficient conditions for the oscillation of several nonlinear delay differential equations which appear in applications.

1. Introduction and preliminaries. Recently, Gopalsamy [3] obtained sufficient conditions for the oscillation of all solutions of the delay logistic equation with several delays

$$\dot{x}(t) + x(t)\left[\alpha - \sum_{j=1}^{n} \beta_j x(t-\tau_j)\right] = 0, \qquad t \geqslant 0, \tag{1}$$

about the steady state

$$x^* = \alpha / \sum_{j=1}^{n} \beta_j.$$

* Received December 4, 1985.

His result is an extension of a result of Kakutani and Markus [8] and E. M. Wright [12] for the delay logistic equation with one delay

$$\dot{N}(t) = rN(t)[1 - N(t-\tau)/K], \qquad t \geqslant 0, \tag{2}$$

which models the growth with time t of a population N with growth rate r and carrying capacity K.

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of a general delay differential equation; see Eq. (7) below, which includes the nonlinear equation (1) as well as some other nonlinear equations which have appeared in applications. And our conditions are much weaker than those in [3]. We also obtained a necessary and sufficient condition for the oscillation of all solutions of a nonlinear delay equation which does not include the logistic equation (1) but includes several other equations which have also appeared in applications.

As an interesting consequence of our results, we obtained a necessary and sufficient condition for the oscillation of all solutions of a linear delay equation with variable coefficients and several delays.

It is well known that when $\tau = 0$, the unique solution of Eq. (2) with $N(0) = N_0 > 0$ tends monotonically to K as $t \to \infty$. However, this is not true, in general, when $\tau > 0$. Indeed, when $r\tau e > 1$, the unique solution of Eq. (2) with initial function

$$N(t) = \phi(t), \qquad -\tau \leqslant t \leqslant 0,\ \phi \in C([-\tau, 0], \mathbf{R}), \text{ and } \phi(0) > 0, \tag{3}$$

oscillates about the steady state K of Eq. (2). That is, the function $N(t) - K$ has arbitrarily large zeros. This result is due to Kakutani and Markus [8] and Wright [12]. Gopalsamy [3] extended the above result for Eq. (2) to the more general Eq. (1). More precisely, he proved the following theorem.

THEOREM 1. Consider the delay logistic equation (1) where α, β_j, and τ_j, for $j = 1, \ldots, n$ are positive constants such that

$$ex^* \sum_{j=1}^{n} \beta_j \tau_j > 1, \tag{4}$$

where

$$x^* = \alpha / \sum_{j=1}^{n} \beta_j$$

is a positive steady state of Eq. (1). Let $\tau = \max_{1 \leqslant j \leqslant n} \tau_j$. Then the unique solution of Eq. (1) which satisfies the initial condition (3) oscillates about x^*.

The solution $x(t)$ of (1) and (3) can be found by the method of steps and satisfies the relation

$$x(t) = \phi(0) \exp\left(\int_0^t \left[\alpha - \sum_{j=1}^{n} \beta_j x(s - \tau_j) \right] ds \right), \qquad 0 \leqslant t \leqslant \tau. \tag{5}$$

From (5) we see that $x(t) > 0$ for $0 \leqslant t \leqslant \tau$ and, by induction, $x(t) > 0$ for $t \geqslant 0$. Set

$$x(t) = x^* \exp(y(t)), \qquad t \geqslant 0.$$

Then, for $t \geqslant \tau$, Eq. (1) reduces to

$$\dot{y}(t) + \sum_{j=1}^{n} x^* \beta_j [e^{y(t-\tau_j)} - 1] = 0, \qquad t \geqslant \tau, \tag{6}$$

with initial function

$$y(t) = \ln[x(t)/x^*], \qquad 0 \leqslant t \leqslant \tau.$$

Clearly, $x(t)$ oscillates about x^* if and only if $y(t)$ oscillates about zero, that is, if and only if $y(t)$ has arbitrarily large zeros.

Equation (6) is of the form

$$\dot{y}(t) + \sum_{j=1}^{n} q_j f\big(y(t - \tau_j)\big) = 0, \qquad t \geqslant t_0, \tag{7}$$

with $q_j = x^* \beta_j$ for $j = 1, \ldots, n$ and

$$f(u) = e^u - 1. \tag{8}$$

Equations of the form (7), usually with one delay, where the function f satisfies the hypotheses

$$f \in C(\mathbf{R}, \mathbf{R}) \quad \text{and} \quad uf(u) > 0 \quad \text{for} \quad u \neq 0 \tag{9}$$

and

$$\lim_{u \to 0} \frac{f(u)}{u} = 1, \tag{10}$$

have appeared in various studies in mathematical biology.

For example, the equation

$$\dot{y}(t) + \beta \frac{y(t - \tau)}{1 + |y(t - \tau)|^r} = 0, \tag{11}$$

where β, τ, and r are positive constants, has appeared in connection with physiological control theory. See Chapin and Nussbaum [2].

The equation

$$\dot{y}(t) + \alpha y(t) + \beta f(y(t - \tau)) = 0, \tag{12}$$

where α, β, and τ are positive constants and

$$f(u) = \begin{cases} -B, & \text{for } u \leqslant -B, \\ u, & \text{for } u > -B \end{cases},$$

for some positive constant B, has appeared in the Hartline–Ratliff-like nonstationary model for the vision process in the compound eye of the horseshoe crab *Limulus*. See Hadeler [4].

Finally, the equation

$$\dot{y}(t) + \alpha y(t) + \beta[1 - e^{-y(t-\tau)}] = 0, \tag{13}$$

where α, β, and τ are positive constants, was used by Lasota and Wazewska in modeling the adult red blood cell supply in an animal. See Kaplan and Yorke [9].

2. Main results. First, we will establish a sufficient condition for the oscillation of all solutions of the linear delay differential equation with variable coefficients

$$\dot{x}(t) + \sum_{j=1}^{n} Q_j(t) x(t - \tau_j) = 0, \qquad t \geqslant t_0, \tag{14}$$

where $Q_j \in C([t_0, \infty), \mathbf{R}^+)$, $\tau_j \geqslant 0$, and

$$\lim_{t\to\infty} Q_j(t) = q_j$$

for $j = 1, \ldots, n$. The condition is in terms of the characteristic equation

$$\lambda + \sum_{j=1}^{n} q_j e^{-\lambda\tau_j} = 0 \tag{15}$$

of the "asymptotic" equation

$$\dot{z}(t) + \sum_{j=1}^{n} q_j z(t-\tau_j) = 0, \tag{16}$$

corresponding to Eq. (14). Our proof is essentially based on the method of the proof given in [11] for equations with constant coefficients.

THEOREM 2. Assume that Eq. (15) has no real roots. Then every solution of Eq. (14) oscillates about zero.

Proof. First, we will make some observations which follow from the hypothesis that Eq. (15) has no real roots. Set

$$F(\lambda) \equiv \lambda + \sum_{j=1}^{n} q_j e^{-\lambda\tau_j}.$$

As $F(\infty) = \infty$, it follows that $F(\lambda) > 0$ for every $\lambda \in \mathbf{R}$. In particular, $F(0) = \sum_{j=1}^{n} q_j > 0$. Thus, at least one of the q_j's is positive. Also, for some positive q_{j_0} the corresponding delay τ_{j_0} should be positive since, otherwise, $\lambda = -\sum_{j=1}^{n} q_j$ would be a real root of Eq. (15). Thus $F(-\infty) = \infty$ and so $m = \min_{\lambda\in\mathbf{R}} F(\lambda)$ exists and is positive. Thus, $\lambda + \sum_{j=1}^{n} q_j e^{-\lambda\tau_j} \geqslant m$, $\lambda \in \mathbf{R}$, or equivalently,

$$\sum_{j=1}^{n} q_j e^{\lambda\tau_j} \geqslant \lambda + m, \qquad \lambda \in \mathbf{R}. \tag{17}$$

Now assume, for the sake of contradiction, that Eq. (14) has an eventually positive solution $x(t)$. Then, from Eq. (14) it follows that $\dot{x}(t) < 0$ and so $x(t)$ is strictly decreasing. Furthermore

$$\dot{x}(t) + Q_{j_0}(t)x(t-\tau_{j_0}) \leqslant 0, \tag{18}$$

where the index j_0 is chosen in such a way that $q_{j_0} > 0$ and $\tau_{j_0} > 0$. Now define the set

$$\Lambda = \{\lambda \geqslant 0 \colon \dot{x}(t) + \lambda x(t) \leqslant 0\}.$$

Clearly $0 \in \Lambda$ and Λ is a subinterval of $\mathbf{R}^+$. The proof will be accomplished by showing that Λ has the following contradictory properties:

(P_1) Λ is bounded above.

(P_2) $\lambda \in \Lambda \Rightarrow (\lambda + m/2) \in \Lambda$, where m is the positive constant which satisfies (17).

To establish (P_1), observe that from (18) we have, for sufficiently large t,

$$\dot{x}(t) + \tfrac{1}{2} q_{j_0} x(t-\tau_{j_0}) \leqslant 0. \tag{19}$$

Applying the Lemma from [11] to the inequality (19), it follows that

$$x(t-\tau_{j_0}) < Bx(t), \tag{20}$$

where $B = 16/(q_{j_0}\tau_{j_0})^2$. Using (20) and the decreasing character of $x(t)$, it follows by iteration that there is a positive constant K such that

$$x(t-\tau_j) < Kx(t), \qquad j = 1, \dots, n.$$

But then eventually,

$$0 = \dot{x}(t) + \sum_{j=1}^{n} Q_j(t)x(t-\tau_j) \leqslant \dot{x}(t) + K\sum_{j=1}^{n} Q_j(t)x(t)$$

$$< \dot{x}(t) + \left(K\sum_{j=1}^{n} q_j + 1\right)x(t),$$

which proves that the positive constant $K\Sigma_{j=1}^{n}q_j + 1$ is an upper bound of Λ. To establish (P_2) set $\psi(t) = e^{\lambda t}x(t)$, where $\lambda \in \Lambda$. Then

$$\dot{\psi}(t) = e^{\lambda t}[\dot{x}(t) + \lambda x(t)] \leqslant 0,$$

which proves that $\psi(t)$ is decreasing. Now, choose an $\varepsilon > 0$ such that $Q_j(t) \geqslant q_j - \varepsilon > 0$ for $j = 1, \dots, n$ and t sufficiently large and $\varepsilon\Sigma_{j=1}^{n}\exp(\lambda\tau_j) < m/2$, which is possible because $\lambda \in \Lambda$ and Λ is bounded.

Then using (17) we have

$$\dot{x}(t) + \left(\lambda + \frac{m}{2}\right)x(t) = -\sum_{j=1}^{n} Q_j(t)x(t-\tau_j) + \left(\lambda + \frac{m}{2}\right)x(t)$$

$$= e^{-\lambda t}\left[-\sum_{j=1}^{n} Q_j(t)e^{\lambda\tau_j}\psi(t-\tau_j) + \left(\lambda + \frac{m}{2}\right)\psi(t)\right]$$

$$\leqslant e^{-\lambda t}\psi(t)\left[-\sum_{j=1}^{n}(q_j - \varepsilon)e^{\lambda\tau_j} + \lambda + \frac{m}{2}\right]$$

$$\leqslant e^{-\lambda t}\psi(t)\left[-\sum_{j=1}^{n} q_j e^{\lambda\tau_j} + \varepsilon\sum_{j=1}^{n} e^{\lambda\tau_j} + \lambda + \frac{m}{2}\right]$$

$$\leqslant e^{-\lambda t}\psi(t)k\left[-\lambda - m + \frac{m}{2} + \lambda + \frac{m}{2}\right] = 0,$$

which proves that $(\lambda + m/2) \in \Lambda$. The proof is complete.

The next result gives sufficient conditions for the oscillation of all solutions of Eq. (7) where f satisfies condition (9) and either condition (10) or

$$u[f(u) - u] \geqslant 0 \quad \text{in some neighborhood of zero.} \tag{21}$$

THEOREM 3. Consider Eq. (7) where $q_j > 0$ and $\tau_j \geqslant 0$ for $j = 1, \dots, n$. Assume that f satisfies (9) and either condition (10) or (21). Suppose that Eq. (15) has no real roots. Then every solution of Eq. (7) oscillates about zero.

Proof. Assume, for the sake of contradiction, that Eq. (7) has a nonoscillatory solution $y(t)$. We will assume that eventually $y(t) > 0$. The proof when $y(t)$ is eventually negative is similar.

In view of condition (9), it follows that eventually $f(y(t-\tau_j)) > 0$ for $j = 1, \dots, n$ and so $\dot{y}(t) < 0$. Therefore, $\lim_{t\to\infty} y(t) = l$ exists and $l \geqslant 0$. We claim that

$$\lim_{t\to\infty} y(t) = 0. \tag{22}$$

Otherwise $l > 0$ and so $f(l) > 0$, which implies that

$$\lim_{t \to \infty} \dot{y}(t) = -\left(\sum_{j=1}^{n} q_j \right) f(l). \tag{23}$$

But as a consequence of the hypothesis that Eq. (15) have no real roots, it follows (as in the proof of Theorem 2) that $\sum_{j=1}^{n} q_j > 0$. Then, from (23) we find that $\lim_{t \to \infty} y(t) = -\infty$, which is a contradiction. Thus (22) holds.

First assume that (21) holds. Then, in view of (22),

$$f\big(y(t - \tau_j)\big) \geqslant y(t - \tau_j)$$

and Eq. (7) yields the inequality

$$\dot{y}(t) + \sum_{j=1}^{n} q_j y(t - \tau_j) \leqslant 0 \tag{24}$$

for sufficiently large t. But, in view of Proposition 2 in [1], inequality (24) cannot have a positive solution. The proof in this case is complete.

Next, assume that (10) holds. Then, we can write Eq. (7) in the form

$$\dot{y}(t) + \sum_{j=1}^{n} Q_j(t) y(t - \tau_j) = 0, \tag{25}$$

where

$$Q_j(t) = q_j \frac{f\big(y(t - \tau_j)\big)}{y(t - \tau_j)} \geqslant 0$$

and

$$\lim_{t \to \infty} Q_j(t) = q_j \quad \text{for } j = 1, \ldots, n.$$

But then, from Theorem 2, it follows that every solution of Eq. (25) oscillates about zero. This contradicts the assumption that $y(t) > 0$ and completes the proof.

Theorem 3 is a substantial improvement on Gopalsamy's Theorem 1. On the one hand, we extended the class of functions to which Theorem 1 applies, and on the other hand, we improved condition (4). In fact, as observed by Hunt and Yorke [7], condition 4 implies that Eq. (15) with $q_j = x^* \beta_j$ has no real roots. Another, independent condition which implies that Eq. (15) has no real roots is, for example,

$$\left(\sum_{j=1}^{n} \tau_j \right) \left(\prod_{j=1}^{n} q_j \right)^{1/n} > \frac{1}{e},$$

which was used in [10]. Furthermore, in Theorem 3 the function f may be the "logistic" function (8), a linear function, any of the nonlinear functions in Eqs. (11), (12), and (13), and so forth.

The following result is a partial converse to both Theorems 2 and 3.

THEOREM 4. Consider the delay differential equation

$$\dot{x}(t) + \sum_{j=1}^{n} Q_j(t) f\big(x(t - \tau_j)\big) = 0, \qquad t \geqslant t_0, \tag{26}$$

where $Q_j \in C([t_0, \infty), \mathbf{R}^+)$, $\tau_j \geqslant 0$ for $j = 1, \ldots, n$, and f satisfies condition (9) and the condition

$$f(u) \leqslant u \qquad \text{for } u \geqslant 0. \tag{27}$$

Assume that there exist positive constants q_j such that

$$Q_j(t) \leqslant q_j \qquad \text{for } j = 1, \dots, n \tag{28}$$

and Eq. (15) has a real root. Then, for any initial point x_0, Eq. (26) has a nonoscillatory solution $x(t)$ such that $x(t_0) = x_0$.

Proof. Denote by μ any real root of Eq. (15). Clearly $\mu < 0$. Set $\tau = \max_{1 \leqslant j \leqslant n} \tau_j$. Let M be the Banach space of real bounded and continuous functions on $[t_0 - \tau, \infty)$ with the uniform norm. Let X be the subset of M consisting of those functions $x(t)$ that satisfy the following properties

(i) $x(t)$ is nonincreasing for $t \geqslant t_0$ and $x(t) \equiv x_0 \exp(\mu(t - t_0))$ for $t \in [t_0 - \tau, t_0]$;

(ii) $x_0 \exp(\mu(t - t_0)) \leqslant x(t) \leqslant x_0 \exp(-\mu\tau)$ for $t \geqslant t_0$; and

(iii) $x(t - \tau_j) \leqslant \exp(-\mu\tau_j)x(t)$ for every $j = 1, \dots, n$ and $t \geqslant t_0$.

Consider the mapping F on X defined as follows:

$$(Fx)(t) = \begin{cases} x_0 e^{\mu(t-t_0)}, & t_0 - \tau \leqslant t \leqslant t_0 \\ x_0 \exp\left(-\sum_{j=1}^{n} \int_{t_0}^{t} Q_j(s) \dfrac{f(x(s - \tau_j))}{x(s)}\, ds \right), & t > t_0. \end{cases}$$

Obviously $(Fx)(t)$ is a nonincreasing continuous function and

$$(Fx)(t) \leqslant x_0 \exp(-\mu\tau).$$

Next we will prove that $(Fx)(t) \geqslant x_0 \exp(\mu(t - t_0))$. In fact, using conditions (27) and (28), we have

$$\begin{aligned}(Fx)(t) &= x_0 \exp\left(-\sum_{j=1}^{n} \int_{t_0}^{t} Q_j(s) \frac{f(x(s - \tau_j))}{x(s)}\, ds \right) \geqslant x_0 \exp\left(-\sum_{j=1}^{n} q_j \int_{t_0}^{t} \frac{x(s - \tau_j)}{x(s)}\, ds \right) \\ &\geqslant x_0 \exp\left(-\sum_{j=1}^{n} q_j e^{-\mu\tau_j} \int_{t_0}^{t} ds \right) = x_0 \exp\left(-\sum_{j=1}^{n} q_j e^{-\mu\tau_j}(t - t_0) \right) = x_0 \exp(\mu(t - t_0)),\end{aligned}$$

where in the last step we used the fact that μ is a root of the characteristic equation.

Again, using conditions (27) and (28) and the characteristic equation (15), we get for every $k = 1, \dots, n$ and $t \geqslant t_0$

$$\begin{aligned}(Fx)(t - \tau_k) &= x_0 \exp\left(-\sum_{j=1}^{n} \int_{t_0}^{t-\tau_k} Q_j(s) \frac{f(x(s - \tau_j))}{x(s)}\, ds \right) \\ &= (Fx)(t) \exp\left(\sum_{j=1}^{n} \int_{t-\tau_k}^{t} Q_j(s) \frac{f(x(s - \tau_j))}{x(s)}\, ds \right) \\ &\leqslant (Fx)(t) \exp\left(\sum_{j=1}^{n} q_j \int_{t-\tau_k}^{t} \frac{x(s - \tau_j)}{x(s)}\, ds \right) \\ &\leqslant (Fx)(t) \exp\left(\sum_{j=1}^{n} q_j e^{-\mu\tau_j} \int_{t-\tau_k}^{t} ds \right) = (Fx)(t) \exp\left(\sum_{j=1}^{n} q_j e^{-\mu\tau_j}\tau_k \right) = (Fx)(t) e^{-\mu\tau_k}.\end{aligned}$$

Thus, we have proved that $FX \subseteq X$.

The set X is obviously nonempty (because $\exp(\mu(t-t_0)) \in X$), closed, and convex.

It is easy to see that the operator F is continuous.

In order to prove that FX is relatively compact in M it is enough to prove that $d/dt[(Fx)(t)]\,dt$ is uniformly bounded. In fact,

$$\frac{d}{dt}[(Fx)(t)] = -\sum_{j=1}^{n} Q_j(t)\frac{f(x(t-\tau_j))}{x(t)}(Fx)(t),$$

and so

$$\left|\frac{d}{dt}[(Fx)(t)]\right| = \sum_{j=1}^{n} Q_j(t)\frac{f(x(t-\tau_j))}{x(t)}(Fx)(t) \leqslant \sum_{j=1}^{n} q_j\frac{x(t-\tau_j)}{x(t)}e^{-\mu\tau}$$

$$\leqslant e^{-\mu\tau}\sum_{j=1}^{n} q_j e^{-\mu\tau_j} = -\mu e^{-\mu\tau},$$

where we have used the fact that μ is a root of the characteristic equation (15). Thus all the conditions of Schauder's fixed-point theorem are satisfied and the mapping F has a fixed point x such that $Fx = x$. Obviously x is a nonoscillatory solution of (26) such that $x(t_0) = x_0$. The proof of the theorem is complete.

It is worth noting that the above result holds without any monotonicity restriction on the function f.

REMARK 1. It should be mentioned that if condition (27) of Theorem 4 is replaced by $f(u) \leqslant Mu$ with $M > 1$, while every other condition of the theorem is satisfied, the conclusion of Theorem 4 will be false. To see this, take $f(u) \equiv Mu$ with $M > 1$ and consider the equations

$$\dot{y}(t) + Mqy(t-\tau) = 0 \tag{29}$$

and

$$\dot{z}(t) + qz(t-\tau) = 0. \tag{30}$$

Assume that $1/Me < q\tau < 1/e$. Then Eq. (15) has a real root. But every solution of Eq. (29) oscillates.

The above remark also shows that condition (10) in Theorem 3 is sharp in the sense that it cannot be replaced by a condition of the form $\lim_{u\to 0}[f(u)/u] = M$, where $M > 1$.

REMARK 2. Condition (27) implies that

$$\int_0^1 \frac{du}{f(u)} = \infty.$$

It should be mentioned that when f satisfies condition (9), $f(u)$ is nondecreasing in $|u| \leqslant 1$,

$$\int_0^1 \frac{du}{f(u)} < \infty, \quad \text{and} \quad \int_0^{-1} \frac{du}{f(u)} < \infty,$$

then Eq. (7) is oscillatory. See [6].

Combining Theorems 2 and 4 we obtain the following necessary and sufficient condition for the oscillation of all solutions of a linear delay equation with variable coefficients.

THEOREM 5. Consider Eq. (14) and assume that $Q_j \in C([t_0, \infty), \mathbf{R}^+)$, $Q_j(t) \leqslant q_j$, and $\lim_{t \to \infty} Q_j(t) = q_j$ for $j = 1, \ldots, n$. Then Eq. (14) is oscillatory if and only if the characteristic equation (15) has no real roots. Or equivalently, Eq. (14) is oscillatory if and only if the "asymptotic" equation (16) is.

REMARK 3. Combining Theorems 3 and 4 we obtain the following characterization of the oscillation of the nonlinear equation (7).

THEOREM 6. Consider Eq. (7) subject to conditions (9), (10), and (27). Then every solution of Eq. (7) oscillates if and only if the characteristic equation (15) of the corresponding linearized variational problem has no real roots.

REMARK 4. It is interesting to note that any of the nonlinear functions in Eqs. (11)–(13) satisfy conditions (9), (10), and (27) of Theorem 6. Furthermore, our results can be extended to more general equations of the form

$$\dot{y}(t) + \sum_{j=1}^{n} q_j f_j\big(y(t - \tau_j)\big) = 0,$$

which involve different functions f_j each of which satisfies conditions (9), (10), and (27).

3. Applications. In this section we will first apply our results to Eqs. (11)–(13) and obtain efficient necessary and sufficient conditions, expressed in terms of α, β, and τ, for the oscillation of all solutions of these equations. Note that each of these equations is of the form

$$\dot{y}(t) + \alpha y(t) + \beta f(y(t - \tau)) = 0, \tag{31}$$

where α, β, and τ are positive constants and where f satisfies conditions (9), (10), and (27).

According to Theorem 6 and Remark 4, Eq. (31) is oscillatory if and only if the characteristic equation

$$F(\lambda) \equiv \lambda + \alpha + \beta e^{-\lambda\tau} = 0 \tag{32}$$

has no real roots. It is easy to see that

$$\min_{\lambda \in \mathbf{R}} \big(\lambda + \alpha + \beta e^{-\lambda\tau}\big) = \frac{\ln(e\beta\tau e^{\alpha\tau})}{\tau}. \tag{33}$$

As $F(\pm\infty) = \infty$, we conclude that the characteristic equation (32) has no real roots if and only if $\min_{\lambda \in \mathbf{R}} F(\lambda) > 0$, or, equivalently, if and only if

$$e\beta\tau e^{\alpha\tau} > 1. \tag{34}$$

Hence, (34) is a necessary and sufficient condition for the oscillation of all solutions of Eq. (31).

Finally, we will apply our results to the equation

$$\dot{x}(t) + \beta x(t-1)\big[1 - x^2(t)\big] = 0, \qquad t \geqslant t_0, \tag{35}$$

where $\beta > 0$ with initial function

$$x(t) = \phi(t), \qquad -1 \leqslant t \leqslant 0, \text{ such that } |\phi(0)| < 1, \tag{36}$$

where $\phi \in C([-1,0],\mathbf{R})$. This equation has been investigated by several authors. See, for example, Hale [5, p. 260], where it is stated that the solution of (35) and (36) oscillates if $\beta > 1$. Applying Theorem 6, we will show that this is true if and only if $\beta > 1/e$. But we need a little preparation. It is easily seen that the solution of (35) and (36) exists for $t \geqslant 0$ and satisfies the condition $-1 < x(t) < 1$. Set

$$x(t) = \frac{e^{2y(t)} - 1}{e^{2y(t)} + 1}, \qquad t \geqslant 0$$

and observe that $x(t)$ oscillates if and only if $y(t)$ oscillates. Then Eq. (35) reduces to

$$\dot{y}(t) + \beta \frac{e^{2y(t-1)} - 1}{e^{2y(t-1)} + 1} = 0, \tag{37}$$

which is of the form of Eq. (7) with $n = 1$, $q_1 = \beta$, $\tau_1 = 1$, and

$$f(u) = \frac{e^{2u} - 1}{e^{2u} + 1}.$$

Clearly, this f satisfies conditions (9), (10), and (27) and so, by Theorem 6, every solution of Eq. (37) oscillates if and only if the associated characteristic equation

$$\lambda + \beta e^{-\lambda} = 0$$

has no real roots. Our claim follows from the observation that

$$\min_{\lambda \in \mathbf{R}} (\lambda + \beta e^{-\lambda}) = \ln(\beta e).$$

References

[1] O. Arino, G. Ladas, and Y. G. Sficas, *On oscillations of some retarded differential equations*, SIAM J. Math. Anal. (to appear)

[2] S. A. Chapin and R. D. Nussbaum, *Asymptotic estimates for the periods of periodic solutions of differential-delay equations*, Mich. Math. J. **31**, 215–229 (1984)

[3] K. Gopalsamy, *Oscillations in a delay-logistic equation*, Quart. Appl. Math. **44**, 447–461 (1986)

[4] K. P. Hadeler, *Delay equations in biology*, in *Functional differential equations and approximation of fixed points*, pp. 136–156, Lecture Notes in Math., 730, Springer, Berlin, 1979

[5] J. K. Hale, *Theory of functional-differential equations*, Springer-Verlag, New York, 1977

[6] A. F. Ivanov and V. N. Shevelo, *On the oscillation and asymptotic behavior of solutions of first order functional differential equations* (Russian), Uk. Mat. Zh. **33**, 745–751 (1981)

[7] B. R. Hunt and J. A. Yorke, *When all solutions of* $\dot{x}(t) = -\sum_{i=1}^{n} q_i(t)x(t - T_i(t))$ *oscillate*, J. Differential Equations **53**, 139–145 (1984)

[8] S. Kakutani and L. Markus, *On the non-linear difference-differential equation* $\dot{y}(t) = [A - By(t - \tau)]y(t)$, in *Contribution to the theory of nonlinear oscillations*, Vol. 4, pp. 1–18, Princeton Univ. Press, 1958

[9] J. L. Kaplan and J. A. Yorke, *On the nonlinear differential delay equation* $\dot{x}(t) = -f(x(t), x(t-1))$, J. Differential Equations **23**, 293–314 (1977)

[10] G. Ladas and I. P. Stavroulakis, *Oscillations caused by several retarded and advanced arguments*, J. Differential Equations **44**, 134–152 (1982)

[11] G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, *Necessary and sufficient conditions for oscillations*, Amer. Math. Monthly **90**, 637–640 (1983)

[12] E. M. Wright, *A nonlinear difference-differential equation*, J. Reine Angew. Math. **194**, 66–87 (1955)