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AN APPLICATION OF THE MULTIVARIATE LAGRANGE-BÜRMANN EXPANSION IN MATHEMATICAL GEODESY*

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Abstract. In the simplified model of geodesy where the earth is conceived as a rotational ellipsoid, if the eccentricity of the ellipsoid is to be determined from gravity measurements, an equation of the form $y = x - zh(x)$ is to be solved for x, where y and z are small parameters whose values can be measured and h is a known function. We obtain the expansion of x in powers of y and z by means of the general Lagrange–Bürmann formula.

1. The problem. Using the standard notations of physical geodesy,

a = major axis of the earth ellipsoid,

GM = product of the earth's mass and the gravitational constant,

J_2 = a constant in the expansion of the normal gravity field in spherical harmonics, and

ω = angular velocity of the earth,

the equation satisfied by the eccentricity e of the ellipsoid may be stated as follows [1, 4]:

$$3J_2 = e^2 - \frac{4}{15}\frac{\omega^2 a^3}{GM}\frac{e^3}{2q_0}. \tag{1}$$

Here $2q_0$ is a known function of e,

$$2q_0 = \left(1 + 3/e'^2\right)\arctan e' - 3/e', \tag{2}$$

where

$$e' = e/\sqrt{1 - e^2} \tag{3}$$

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is the "second eccentricity." The constants a, GM, J_2, ω are either known or can be obtained accurately from gravity measurements. Equation (1) thus serves to obtain accurate values of e from gravity measurements. Our concern is with *solving* the equation and with exhibiting the dependence of the solution on the parameters.

The equation has the form

$$y = x - zh(x), \tag{4}$$

where

$$y = 3J_2, \qquad z = \frac{\omega^2 a^3}{GM},$$

are known and $x = e^2$ is to be determined. The function

$$h(x) = \frac{4}{15}\frac{x^{3/2}}{2q_0(\sqrt{x})} \tag{5}$$

is known. In the physical problem on hand, the numerical values of y and z are both of the order of 3×10^{-3}.

2. Numerical solution of the equation. This is discussed very thoroughly in [1], and values of e are obtained that are more accurate than those given in the literature. It follows from Eq. (4) of [1] that

$$\frac{1}{h(x)} = F\left(\frac{3}{2}, \frac{3}{2}; \frac{7}{2}; x\right), \tag{6}$$

where F is the hypergeometric function. Thus h is analytic not only for $0 < x < 1$ but also at $x = 0$. Moreover, since all coefficients in the series (6) are positive, as x increases from 0 to 1, $h(x)$ decreases from $h(0) = 1$ to $h(1) = 4/15\pi$. By writing (4) as a fixed point equation,

$$x = y + zh(x), \tag{7}$$

we see that for positive y and z such that $y + z < 1$ the equation has precisely one solution, which, if z satisfies the additional condition

$$|zh'(y+z)| < 1,$$

can be found as the limit of the iteration sequence defined by $x_0 = 0$,

$$x_{n+1} = y + zh(x_n), \qquad n = 0, 1, 2, \ldots.$$

The only numerical problem that arises is a considerable loss of accuracy, due to subtracting large numbers that are nearly equal, if h is evaluated by means of the defining relations (5) and (2). It is much preferable to compute h from the series expansion (6), which converges rapidly if x is small.

3. Analytical solution. Iteration does furnish a numerical solution of (4) for given y and z, but it does not show how this solution depends on the parameters. We therefore endeavor to find a series solution for (4). Our tool is the multidimensional Lagrange–Bürmann formula as discussed in [3]. We summarize these results briefly for convenience.

Let $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ be an admissible system of n power series in n indeterminates $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. ["Admissible" means that $P_j = c_j x_j$ + higher-order terms, where $c_j \neq 0$.] Let $\mathbf{Q}$ denote the inverse system of $\mathbf{P}$. ["Inverse" means that $\mathbf{Q}$ substituted into $\mathbf{P}$ yields $\mathbf{x}$.] Let R be an arbitrary (single) Laurent series in $\mathbf{x}$. Then the series obtained by substituting $\mathbf{Q}$ into R is given by

$$R \circ \mathbf{Q} = \sum_{\mathbf{k}} \operatorname{Res}(R\mathbf{P}^{-\mathbf{k}-\mathbf{e}}\mathbf{P}')\mathbf{x}^{\mathbf{k}}, \tag{8}$$

where the summation is with respect to all index vectors $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, and where

$$\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

$$\mathbf{e} = (1, 1, \ldots, 1),$$

$\mathbf{P}'$ is the Jacobian determinant of the system $\mathbf{P}$, and Res denotes the residue, that is, the coefficient of $\mathbf{x}^{-\mathbf{e}}$, in a Laurent series. The result (8) holds formally, that is, regardless of whether or not the series involved are convergent.

We require an application of (8), also given in [3]. Here we consider two systems of complex variables,

$$\mathbf{x} = (x_1, \ldots, x_p), \qquad \mathbf{y} = (y_1, \ldots, y_q),$$

and a system of p functions

$$f_i(\mathbf{x}, \mathbf{y}), \qquad i = 1, 2, \ldots, p,$$

analytic near $(\mathbf{0}, \mathbf{0})$. We write $\mathbf{f} = (f_1, \ldots, f_p)$, and we denote by $\mathbf{f}'$ the Jacobian determinant of this system with respect to the x_i, regarding the y_j as parameters. Assuming

$$\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}, \qquad \mathbf{f}'(\mathbf{0}, \mathbf{0}) \neq \mathbf{0},$$

the system of equations

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0} \tag{9}$$

for sufficiently small $|y_j|$ has precisely one solution $\mathbf{x}(\mathbf{y})$ which is analytic in $\mathbf{y}$ and which satisfies $\mathbf{x}(\mathbf{0}) = \mathbf{0}$. We wish to find the coefficients of the power series $\mathbf{x}(\mathbf{y})$ or, more generally, of $r(\mathbf{x}(\mathbf{y}), \mathbf{y})$, where r is a given analytic function.

For a solution by means of the Lagrange-Bürmann formula we assume, without loss of generality, that the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(\mathbf{0}, \mathbf{0}) \right), \qquad i, j = 1, \ldots, p,$$

is the identity. (This can be achieved by forming suitable linear combinations of the functions f_i and of the variables x_j.) In the power series expansion of $\mathbf{f}(\mathbf{x}, \mathbf{y})$, let $\mathbf{By}$ denote the terms that are linear in the y_j, that is,

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{By} + \text{terms of degree} \geqslant 2.$$

($\mathbf{B}$ is a matrix with p rows and q columns; we think of $\mathbf{y}$ as a column vector.) Consider the map of a $(p + q)$-dimensional neighborhood of $(\mathbf{0}, \mathbf{0})$ defined by

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{By} \\ \mathbf{y} \end{pmatrix}. \tag{10}$$

The system of $p + q$ power series representing this map near $(\mathbf{0}, \mathbf{0})$ is admissible; in fact, its Jacobian matrix at $(\mathbf{0}, \mathbf{0})$ is the identity. Hence the inverse system

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{x}(\mathbf{u}, \mathbf{v}) \\ \mathbf{y}(\mathbf{u}, \mathbf{v}) \end{pmatrix} \tag{11}$$

exists and can be represented by the Lagrange–Bürmann series. Letting

$$\mathbf{P} = \mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{B}\mathbf{y},$$

and noting that the Jacobian determinant of the whole system (10) is just $\mathbf{P}'$, the Jacobian with respect to $\mathbf{x}$, one obtains in view of $\mathbf{y} = \mathbf{v}$ for an arbitrary function r

$$r(\mathbf{x}(\mathbf{u}, \mathbf{v}), \mathbf{v}) = \sum_{\substack{\mathbf{k} \in Z^p \\ \mathbf{m} \in Z^q}} \operatorname{Res}\{ r(\mathbf{x}, \mathbf{y}) \mathbf{P}^{-\mathbf{k}-\mathbf{e}} \mathbf{y}^{-\mathbf{m}-\mathbf{e}} \mathbf{P}'(\mathbf{x}, \mathbf{y}) \} \mathbf{u}^{\mathbf{k}} \mathbf{v}^{\mathbf{m}}. \tag{12}$$

Now evidently $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ if and only if $\mathbf{u} = -\mathbf{B}\mathbf{v}$. Since $\mathbf{v} = \mathbf{y}$, the solution of (8) thus is

$$\mathbf{x}(\mathbf{y}) = \mathbf{x}(-\mathbf{B}\mathbf{y}, \mathbf{y}),$$

and from (12) we find the explicit series expansion

$$r(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \sum_{\substack{\mathbf{k} \in Z^p \\ \mathbf{m} \in Z^q}} \operatorname{Res}\{ \cdots \} (-\mathbf{B}\mathbf{y})^{\mathbf{k}} \mathbf{y}^{\mathbf{m}}, \tag{13}$$

where the residues are the same as in (12).

4. Application to the geodesic equation. To apply (13) to the solution of (4), we let $p = 1$, $q = 2$,

$$\mathbf{x} = (x), \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}.$$

The equation to be solved is $f(\mathbf{x}, \mathbf{y}) = 0$, where

$$f(\mathbf{x}, \mathbf{y}) = x - y - zh(x),$$

which in order to isolate first-order terms we write in the form

$$f(\mathbf{x}, \mathbf{y}) = x - y - z - zxg(x),$$

where

$$g(x) = \frac{1}{x}(h(x) - 1) = O(1).$$

We see that

$$\mathbf{B}\mathbf{y} = -y - z.$$

The map (10) in our case is thus

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} P \\ y \\ z \end{pmatrix}, \qquad P = x(1 - zg(x)).$$

If $r(\mathbf{x},\mathbf{y}) = x$, (13) now yields

$$x(y,z) = \sum_{\substack{k>0 \\ (m,n)\in Z^2}} \operatorname{Res}\{xP^{-k-1}P'y^{-m-1}z^{-n-1}\}(y+z)^k y^m z^n, \tag{14}$$

and it only remains to evaluate the residues.

Since P does not depend on y, we need $m = 0$ to obtain a residue in y. Using

$$P^{-k-1}P' = -\frac{1}{k}(P^{-k})',$$

we thus get

$$x(y,z) = -\sum_{\substack{k>0 \\ n\geqq 0}} \frac{1}{k} \operatorname{Res}_{x,z}\{x(P^{-k})'z^{-n-1}\}(y+z)^k z^n,$$

where the residue now is taken only with respect to the variables x and z. In view of

$$P = x(1 - zg(x)),$$

we may use the binomial series to obtain

$$\begin{aligned} p^{-k} &= x^{-k}(1-zg(x))^{-k} \\ &= x^{-k}\sum_{l=0}^{\infty}(-1)^l\binom{-k}{l}z^l g^l, \end{aligned}$$

where $\binom{-k}{l}$ is a binomial coefficient. Now for given $k > 0$ and $n \geqq 0$,

$$\begin{aligned} \operatorname{Res}_{x,z}\{x(P^{-k})'z^{-n-1}\} &= \operatorname{Res}_x \text{ of coefficient of } z^n \text{ in } x(P^{-k})' \\ &= -\operatorname{Res}_x \text{ of coefficient of } z^n \text{ in } P^{-k} \\ &= -\text{coefficient of } x^{k-1} \text{ in } (-1)^n\binom{-k}{n}g^n \\ &= (-1)^{n+1}\binom{-k}{n}g_{k-1}^{(n)}, \end{aligned}$$

where Res_x denotes the residue with respect to the single variable x, and where the coefficients $g_k^{(n)}$ are defined by

$$[g(x)]^n = \sum_{k=0}^{\infty} g_k^{(n)}x^k.$$

We thus finally let

$$\begin{aligned} x(y,z) &= \sum_{\substack{k>0 \\ n\geqq 0}}^{\infty} \frac{(-1)^n}{k}\binom{-k}{n}g_{k-1}^{(n)}(y+z)^k z^n \\ &= y + z + \sum_{\substack{k>0 \\ n>0}} \frac{(-1)^n}{k}\binom{-k}{n}g_{k-1}^{(n)}(y+z)^k z^n. \end{aligned} \tag{15}$$

5. Truncation error. In numerical computation, the series (15) will have to be truncated, for instance, by neglecting the terms where $k + n \geqq p$ for some positive integer p. We therefore estimate the truncation error

$$t_p(y, z) = \sum_{\substack{k+n=p \\ k>0, n>0}} \frac{(-1)^n}{k} \binom{-k}{n} g_{k-1}^{(n)} (y+z)^k z^n.$$

From (6), the coefficients a_n in

$$[h(x)]^{-1} = \sum_{n=0}^{\infty} a_n x^n$$

are easily seen to satisfy $|a_n| \leqslant 1$. In view of $a_0 = 1$ we therefore have for $|x| \leqslant \rho$, $\rho < 1$,

$$\left|[h(x)]^{-1}\right| \geqq 1 - \rho - \rho^2 - \cdots = (1 - 2\rho)/(1 - \rho),$$

and thus, if $0 \leqq \rho < \frac{1}{2}$,

$$|h(x)| \leqq (1 - \rho)/(1 - 2\rho).$$

Using the principle of the maximum, there follows for $|x| \leqq \rho < \frac{1}{2}$,

$$|g(x)| \leqslant \left| \frac{1}{\rho} \left(\frac{1 - \rho}{1 - 2\rho} - 1 \right) \right| = \frac{1}{1 - 2\rho}.$$

Cauchy's estimate now yields

$$\left|g_k^{(n)}\right| \leqq \frac{1}{(1 - 2\rho)^n} \frac{1}{\rho^k}, \qquad 0 < \rho < \frac{1}{2}.$$

Now let $|y + z| \leqq \rho_1$, $|z| \leqq \rho_2$. In view of

$$\frac{(-1)^n}{k} \binom{-k}{n} = \frac{1}{k} \binom{k+n-1}{n} = \frac{1}{k+n} \binom{k+n}{n},$$

there follows

$$\left| \sum_{\substack{k+n=q \\ k>0, n>0}} \frac{(-1)^n}{k} \binom{-k}{n} g_{k-1}^{(n)} (y+z)^k z^n \right| \leqq \frac{1}{q} \sum_{k+n=q} \binom{q}{n} (1 - 2\rho)^{-n} \rho^{-k+1} \rho_1^k \rho_2^n$$

$$= \frac{\rho}{q} \left(\frac{\rho_2}{1 - 2\rho} + \frac{\rho_1}{\rho} \right)^q.$$

Therefore, if

$$\sigma = \frac{\rho_2}{1 - 2\rho} + \frac{\rho_1}{\rho} < 1, \tag{16}$$

we find the truncation error estimate

$$\left|t_p(y, z)\right| \leqq \frac{\rho}{p} \frac{\sigma^p}{1 - \sigma}. \tag{17}$$

Choosing, for instance, $\rho = \frac{1}{3}$, there results the simple formula

$$|t_p(y,z)| \leqq \frac{1}{3p} \frac{(3\rho_1 + 3\rho_2)^p}{1 - (3\rho_1 + 3\rho_2)}. \tag{18}$$

6. Numerical values. It remains to compute the coefficients $g_k^{(n)}$. This is a routine computation which is best performed with a symbolic manipulator. Using the MAPLE program of the University of Waterloo [2] we computed the $g_k^{(n)}$ as well as the coefficients

$$a_k^{(n)} = \frac{(-1)^n}{k}\binom{-k}{n} g_k^{(n)}$$

of the series (15) in rational arithmetic for $1 \leqq k \leqq 10$, $1 \leqq n \leqq 10$. Complete tables of these values are available from the authors on request. Here we give only the values that are required to write the terms of the series for $k + n < 5$:

$$\begin{aligned} h(x) &= 1 - \frac{9}{14}x - \frac{13}{392}x^2 - \frac{4189}{181104}x^3 - \cdots, \\ g(x) &= -\frac{9}{14} - \frac{13}{392}x - \frac{4189}{181104}x^2 - \cdots, \\ [g(x)]^2 &= \frac{81}{196} + \frac{117}{2744}x + \cdots, \\ [g(x)]^3 &= -\frac{729}{2744} - \cdots. \end{aligned}$$

This results in

$$\begin{aligned} x(y,z) = {} & (y+z)\left\{1 - \frac{9}{14}z + \frac{81}{196}z^2 - \frac{729}{2744}z^3 + \cdots\right\} \\ & + (y+z)^2\left\{-\frac{13}{392}z + \frac{351}{5488}z^2 + \cdots\right\} \\ & + (y+z)^3\left\{-\frac{4189}{181104}z + \cdots\right\} \\ & + \cdots. \end{aligned} \tag{19}$$

From the values of the parameters given in [1] we have

$$y = 3.247890 \times 10^{-3}, \qquad z = 3.461391 \times 10^{-3}.$$

Substituting these into (19) we get

$$x = 6.694379 \times 10^{-3}$$

with a truncation error $t_5(y,z)$, which by (18) is less than

$$\frac{1}{15} \frac{[3 \times 6.709281 \times 10^{-3}]^5}{0.979872} = 2.25 \times 10^{-10},$$

and which thus is less than the error in x due to rounding or measuring errors in y and z.

REFERENCES

[1] J. C. Alexander, *The numerics of computing geodetic ellipsoids*, SIAM Rev. **27**, 241–247 (1985)

[2] B. W. Char, K. O. Geddes, G. H. Gonnet, and S. M. Watt, *Maple user's manual*, 3rd ed., Univ. of Waterloo Res. Rep. CS-83-41 (1983)

[3] P. Henrici, *Die Lagrange-Bürmannsche Formel bei formalen Potenzreihen*, Jber. Deutsch. Math.-Verein. **86**, 115–134 (1984)

[4] W. A. Hieskanen and H. Moritz, *Physical geodesy*, Freeman, San Francisco, 1966