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RESPONSE BOUNDS FOR HYSTERETIC SECOND ORDER SYSTEMS*

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The behavior of many engineering systems is governed by the second order differential equation

$$\ddot{U}(t) + F[U(t)] = \ddot{G}(t), \tag{1}$$

where $\ddot{G}(t)$ is a specified oscillatory function of time t, a dot denotes differentiation with respect to t, $F(U)$ is a nonlinear restoring function representing the system hysteresis, as shown in Fig. 1, and $U(0) = \dot{U}(0) = 0$. In a recent paper [1], it has been shown that

$$f = F(u) \leqslant (1/\alpha)\ddot{g}, \tag{2}$$

where $f = \sup|F(U)|$, $u = \sup|U(t)|$, $\ddot{g} = \sup|\ddot{G}(t)|$, and $0 < \alpha \leqslant 1$ is given by

$$\alpha = \frac{A}{4fu}, \tag{3}$$

where A is the area of the hysteresis loop.

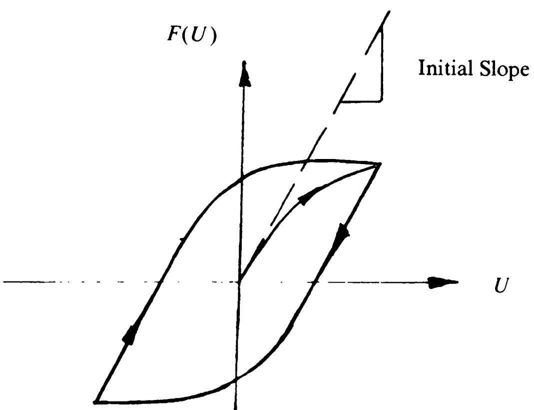

FIG. 1. Representation of $F(U)$

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The quality of the upper bound on $U(t)$, as given by inequality (2), deteriorates with the reduction in the initial slope of $F(U)$. That is, the estimate provided by inequality (2) is good for stiff systems. For soft systems (i.e., systems for which the initial slope of $F(U)$ is small), this estimate may be much higher than the actual values of $U(t)$.

To find a bound on $U(t)$ suitable for soft systems, both sides of Eq. (1) are multiplied by $\dot{U}(t)$, and the resulting expression is integrated over the interval $[t_i, t_{i+1}]$ to yield

$$\int_{t_i}^{t_{i+1}} F(U)\dot{U}(t)\,dt = \int_{t_i}^{t_{i+1}} \ddot{G}(t)\dot{U}(t)\,dt, \tag{4}$$

where t_i and t_{i+1} are two consecutive times of zero crossing of $\dot{U}(t)$. If both sides of Eq. (1) are multiplied by $\dot{G}(t)$ and the resulting expression is integrated over the same interval $[t_i, t_{i+1}]$, it yields

$$\int_{t_i}^{t_{i+1}} \ddot{U}(t)\dot{G}(t)\,dt + \int_{t_i}^{t_{i+1}} F(U)\dot{G}(t)\,dt = \tfrac{1}{2}\dot{G}^2(t)\Big]_{t_i}^{t_{i+1}}. \tag{5}$$

Since t_i and t_{i+1} are the times of zero crossing for $\dot{U}(t)$, integration by parts of the first term on the left-hand side of the above expression yields

$$\int_{t_i}^{t_{i+1}} \ddot{U}(t)\dot{G}(t)\,dt = -\int_{t_i}^{t_{i+1}} \dot{U}(t)\ddot{G}(t)\,dt. \tag{6}$$

Comparing Eqs. (4) and (6), one obtains

$$\int_{t_i}^{t_{i+1}} \ddot{U}(t)\dot{G}(t)\,dt = -\int_{t_i}^{t_{i+1}} F(U)\dot{U}(t)\,dt. \tag{7}$$

Substitution from Eq. (7) into Eq. (5) results in

$$\int_{t_i}^{t_{i+1}} F(U)\dot{U}(t)\,dt + \tfrac{1}{2}\dot{G}^2(t_{i+1}) = \int_{t_i}^{t_{i+1}} F(U)\dot{G}(t)\,dt + \tfrac{1}{2}\dot{G}^2(t_i). \tag{8}$$

Since $F(U)$ has at most one zero crossing in the time interval $[t_i, t_{i+1}]$, then

$$\left|\int_{t_i}^{t_{i+1}} F(U)\dot{G}(t)\,dt\right| = \left|\int_{G(t_1)}^{G(t_{i+1})} F(U)\,dG\right| \leqslant 2fg, \tag{9}$$

where $g = \sup|G(t)|$. Also,

$$\left|\int_{t_i}^{t_{i+1}} F(U)\dot{U}(t)\,dt\right| = \left|\int_{U(t_i)}^{U(t_{i+1})} F(U)\,dU\right| = \alpha(2fu), \tag{10}$$

where α is the reduction factor which makes the equality satisfied, and its value is given by relation (3). Comparisons of equalities (8) and (10) and inequality (9) yield

$$\alpha fu \leqslant fg + \left(\tfrac{1}{4}\right)\dot{g}^2, \tag{11}$$

where $\dot{g} = \sup|\dot{G}(t)|$. The simultaneous occurrence of f and u is an implicit assumption in both inequalities (2) and (11). Equivalently, inequality (11) can be written as

$$\alpha uF(u) \leqslant gF(u) + \left(\tfrac{1}{4}\right)\dot{g}^2. \tag{12}$$

This inequality yields an upper bound on u with α as a parameter suitable for soft systems.

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References

[1] N. Mostaghel, *A response bound for hysteretic second order systems*, Quart. Appl. Math. **XLIII**, 199–200 (1985)