QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLV, NUMBER 2
JULY 1987, PAGES 223-238

ENERGY EQUIPARTITION AND FLUCTUATION-DISSIPATION THEOREMS FOR DAMPED FLEXIBLE STRUCTURES*

By

RONALD K. PEARSON (*E. I. du Pont de Nemours and Company, Wilmington, Delaware*)

AND

TIMOTHY L. JOHNSON (*General Electric Co., Schenectady, New York*)

Abstract. Dynamics and control of flexible mechanical structures has been the topic of much recent research. Here, we examine the energy distribution in finite-dimensional flexible structure models of the type obtained through finite element analysis. Modeling external disturbance forces as zero-mean white noise, we establish that symmetry of the damping matrix is a sufficient condition for the equipartition of potential and kinetic energy in the structure. In addition, we develop upper and lower bounds on the total energy stored in symmetrically damped structures in terms of the strength of the stochastic driving term and the Euclidean norms of the damping matrix and its inverse. In two special cases, explicit solutions for the total energy are obtained and may be viewed as fluctuation-dissipation theorems for the structure models. Convergence conditions for modal expansions of distributed parameter flexible structure models are then developed from these finite-dimensional results. These conditions are interpreted physically as interrelations between assumed damping mechanisms and disturbance force actuator models that must exist in formulating well-posed stochastic distributed-parameter flexible structure models.

1. Introduction. The dynamics and control of flexible mechanical structures has been the topic of much recent research [15, 21, 22]. One conclusion that can be drawn from the research done to date is that the character of the control problem is strongly influenced by the inherent damping mechanisms assumed in modeling these structures. One of the motivations for studying these control problems is that oscillations induced in the structure decay at an unacceptably slow rate, implying that this inherent damping is too weak. Thus, early efforts considered active control of undamped structures, which seemed a reasonable assumption in dealing with finite-dimensional systems (e.g., finite element

* Received October 1, 1985.

models). Later, however, it was shown that finite-dimensional feedback control laws implemented with realistic (i.e., finite size) measurement sensors and force actuators cannot impart a uniform exponential decay rate to the infinite-dimensional extensions of these systems [8]. Similarly, it was shown [1] that the existence of unmodeled modes in the closed loop system (a phenomenon referred to as "spillover") could lead to instability. Conversely, if a sufficiently "strong" damping model is assumed, it has been argued that the behavior of the system may be "modeled effectively by finite-dimensional systems" [3] and that spillover-induced instability is not a serious problem [19]. Specifically, while undamped flexible structures are modeled by hyperbolic partial differential equations and evolve as unitary groups of bounded linear operators, these "strong" damping mechanisms lead to structure models that evolve as analytic semigroups of bounded linear operators [3, 19].

Here, we consider the impact of various inherent damping assumptions on the character of flexible structure models. Specifically, we begin by considering an N-mode stochastic flexible structure model typical of those obtained by the finite element method. The stochastic formulation allows us to consider the steady-state total energy of the structure and assess the impact of both the damping assumptions and the way in which external forces excite the natural modes of the structure. From this, we obtain three main results. First, it is shown that the well-known equipartition of kinetic and potential energy that occurs in the scalar harmonic oscillator also occurs in N-mode flexible structure models if the damping matrix is symmetric. That is not surprising, but we also demonstrate by a simple counterexample that this energy equipartition result generally *does not hold* if the damping matrix is not symmetric. While most authors employ symmetric damping mechanisms, some nonsymmetric damping models have been investigated either as models for inherent damping of stationary structures [3] or as models for the dynamics of spinning structures [10]. Further, even if the structure is not spinning and the inherent damping operator is symmetric, damping enhancement through feedback control will generally yield asymmetric closed-loop damping operators unless the control system is designed specifically to preserve symmetry, as in the case of co-located velocity sensors and force actuators [2, 18]. Our second major result is the derivation of upper and lower bounds on the steady-state total energy of the structure. These bounds involve the damping matrix, the intensity of the driving noise source and the vector describing its influence on the modes of the structure. This motivates our third result which consists of algebraic expressions for the steady-state total energy of the structure in two special cases. Because they relate effects of the disturbance forces ("fluctuations") to the effects of the damping mechanisms ("dissipations") in determining the total energy, these results are analogous to the fluctuation-dissipation theorems developed in statistical mechanics [13, 14] to describe Brownian motion in viscous fluids or thermal noise in resistors.

Taking the distributed-parameter limit of these N-mode problems (i.e., $N \to \infty$), we obtain considerable insight into the modeling tradeoffs necessary to formulate a well-posed stochastic distributed-parameter flexible structure problem. For example, these results show that the viscous fluid damping model is inherently incompatible with external disturbances modeled as white noise acting at discrete points on the structure because the

total energy of the structure diverges in the distributed-parameter limit. To obtain convergence, the disturbance must either act over a finite portion of the structure or the damping mechanism must be "strong" enough to yield an asymptotically increasing damping ratio for the high frequency modes of the structure. Similarly, the results presented here suggest that it is possible to formulate well-posed stochastic problems for weakly damped structures with nonexponential decay rates, but only if the influence of the external disturbances on high frequency structural modes is sufficiently weak.

2. Energy Equipartition Theorem. The flexible structures considered here are described by the N-mode coupled harmonic oscillator model

$$d\dot{\mathbf{x}}(t) + \Delta\, d\mathbf{x}(t) + \Omega^2\mathbf{x}(t)\, dt = T dw(t) \tag{2.1}$$

where $\mathbf{x}(t) \in R^N$ represents the instantaneous deflections of the N oscillatory modes from equilibrium and $w(t)$ is a scalar Wiener process representing an external vibrational force acting on the structure. Here, $\Delta \in R^{N\times N}$ describes the inherent damping in the structure, $\Omega \in R^{N\times N}$ is the diagonal matrix of normal mode frequencies $\{\omega_i\}_{i=1}^N$ and $T \in R^N$ describes the influence of the disturbance force $w(t)$ on the natural modes of the structure. The covariance of $w(t)$ is given by

$$E\left\{(w(t) - w(s))^2\right\} = \sigma^2(t-s) \tag{2.2}$$

so $dw(t)$ may be regarded formally as a zero-mean Gaussian white noise process with variance σ^2. It should be noted that the results developed here are easily generalized to vector-valued Wiener processes, but the physical interpretations developed in section 6 are clearer for the scalar case.

This model is most conveniently cast in state-space form by defining the $2N$-dimensional state vector $\mathbf{z}(t) = [\Omega\mathbf{x}(t), \dot{\mathbf{x}}(t)]'$. Equation (2.1) then reduces to

$$d\mathbf{z}(t) = A\mathbf{z}(t)\, dt + D\, dw(t) \tag{2.3}$$

where the matrices A and D are defined as

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Delta \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ T \end{bmatrix}. \tag{2.4}$$

Steady-state potential, kinetic, and total energies for this structure model are defined as

$$\mathrm{PE} = \lim_{t\to\infty} E\left\{\|\Omega\mathbf{x}(t)\|^2\right\} \tag{2.5a}$$

$$\mathrm{KE} = \lim_{t\to\infty} E\left\{\|\dot{\mathbf{x}}(t)\|^2\right\} \tag{2.5b}$$

$$\mathrm{TE} = \mathrm{PE} + \mathrm{KE} = \lim_{t\to\infty} E\left\{\|\mathbf{z}(t)\|^2\right\}. \tag{2.5c}$$

These energies are most conveniently evaluated from the solution X of the algebraic Lyapunov equation

$$AX + XA' + \sigma^2 DD' = 0. \tag{2.6}$$

The $2N \times 2N$ matrix X is symmetric and nonnegative definite and may be partitioned into $N \times N$ submatrices as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{bmatrix} \tag{2.7}$$

where X_{11} and X_{22} are also symmetric and nonnegative definite. Potential, kinetic, and total energy are then given by

$$\mathrm{PE} = \mathrm{tr}[X_{11}] \tag{2.8a}$$

$$\mathrm{KE} = \mathrm{tr}[X_{22}] \tag{2.8b}$$

$$\mathrm{TE} = \mathrm{tr}[X]. \tag{2.8c}$$

To proceed, it is necessary to partition equation (2.6) into the three $N \times N$ matrix equations:

$$\Omega X_{12}' + X_{12}\Omega = 0 \tag{2.9a}$$

$$\Omega X_{22} - X_{11}\Omega - X_{12}\Delta' = 0 \tag{2.9b}$$

$$-\Omega X_{12} - X_{12}'\Omega - \Delta X_{22} - X_{22}\Delta' + V = 0 \tag{2.9c}$$

where $V = \sigma^2 TT'$ is the effective disturbance covariance matrix.

It is tempting to conjecture that $X_{12} = 0$ since it is clearly a solution of equation 2.9a. However, it is easy to show that this does not generally satisfy equations (2.9b) and (2.9c). Specifically, if $X_{12} = 0$, equation (2.9b) implies $X_{11} = \Omega X_{22}\Omega^{-1}$ but since both X_{11} and X_{22} are symmetric, this in turn implies $\Omega^2 X_{22} = X_{22}\Omega^2$. Because Ω is diagonal, however, this requires X_{22} to be the diagonal solution of

$$-\Delta X_{22} - X_{22}\Delta' + V = 0. \tag{2.10}$$

Clearly, this can only be true if very special (and generally unreasonable) relationships exist between the damping matrix and the effective disturbance covariance matrix V. For example, if Δ is diagonal (see section 6 for some specific examples), equation (2.10) cannot have a diagonal solution unless V is diagonal and this can only occur if the vector T is directed along one of the coordinate axes of R^N. Physically, this would mean that the noise source $w(t)$ only excited one of the N modes, say mode j, so that $[X_{22}]_{jj} = V_{jj}/2\Delta_{jj}$ with all other components of X_{22} being identically zero.

Oscillatory systems whose damping matrices are antisymmetric are referred to as gyroscopic systems [16]. Here, the term "completely nongyroscopic" will refer to oscillatory systems whose damping matrices have no antisymmetric part (i.e., $\Delta = \Delta'$). Most damping models considered to date are completely nongyroscopic, although the damping matrices required to describe spinning structures or the closed loop behavior of arbitrary controlled flexible structures will generally have gyroscopic components. Also, Chen and Russell have considered a class of asymmetric operators as candidates for modeling inherent damping in flexible structures [3]. The significance of such symmetry/asymmetry assumptions are summarized by the following result.

THEOREM 2.1. For completely nongyroscopic systems, PE = KE = $\frac{1}{2}$TE.

Proof. Pre-multiplying equation (2.9b) by Ω^{-1} and taking the trace yields

$$\operatorname{tr}[X_{22}] = \operatorname{tr}\left[\Omega^{-1}X_{11}\Omega + \Omega^{-1}X_{12}\Delta'\right] = \operatorname{tr}[X_{11}] + \operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right]$$
$$\Rightarrow \mathrm{KE} - \mathrm{PE} = \operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right].$$

Pre- and post-multiplying equation (2.9a) by Ω^{-1} yields

$$X_{12}'\Omega^{-1} = -\Omega^{-1}X_{12}$$
$$\begin{aligned}\Rightarrow \operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right] &= \tfrac{1}{2}\operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right] + \tfrac{1}{2}\operatorname{tr}\left[\Delta X_{12}'\Omega^{-1}\right]\\ &= \tfrac{1}{2}\operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right] - \tfrac{1}{2}\operatorname{tr}\left[\Delta\Omega^{-1}X_{12}\right]\\ &= \tfrac{1}{2}\operatorname{tr}\left[\Omega^{-1}X_{12}\Delta'\right] - \tfrac{1}{2}\operatorname{tr}\left[\Omega^{-1}X_{12}\Delta\right].\end{aligned}$$

For a completely nongyroscopic system, $\Delta' = \Delta$, so KE − PE = 0. □

It is important to note that this energy equipartition result generally does not hold for systems whose damping matrix has a nonzero gyroscopic part $\Delta_g = \frac{1}{2}(\Delta - \Delta')$. Specifically, it follows from the proof of Theorem 2.1 that

$$\mathrm{PE} - \mathrm{KE} = \operatorname{tr}\left[\Omega^{-1}X_{12}\Delta_g\right] \tag{2.11}$$

and since both Δ_g and $\Omega^{-1}X_{12}$ are antisymmetric, for $N = 2$ we have

$$\Delta_g = \begin{bmatrix} 0 & \delta \\ -\delta & 0 \end{bmatrix}, \qquad \Omega^{-1}X_{12} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}. \tag{2.12}$$

Thus, PE − KE = $2\alpha\delta$ and since it has been argued that $X_{12} \neq 0$ in most realistic problems, α is nonzero and PE ≠ KE. It is clear from this result that energy equipartition is generally not to be expected unless the structure under consideration is completely nongyroscopic.

3. Energy bounds. Various bounds on norms and eigenvalues for the solutions of Lyapunov equations have been derived [7, 12, 17] and similar results could probably be developed for the traces of interest here. Tighter bounds may be obtained, however, by exploiting the structure of the partitioned Lyapunov equations (2.9). In particular, note that $\operatorname{tr}[\Omega X_{12}] = \operatorname{tr}[X_{12}\Omega] = 0$ since $X_{12}\Omega$ is antisymmetric by equation (2.9a). Thus, taking the trace of equation (2.9c) yields

$$\operatorname{tr}[\Delta X_{22}] = \tfrac{1}{2}\operatorname{tr}[V]. \tag{3.1}$$

This result, together with the following lemma, provides the basis for upper and lower bounds on the steady-state kinetic energy of the structure. In addition, if the structure is completely nongyroscopic, bounds on the steady-state total energy follow immediately from Theorem 2.1.

LEMMA 3.1. If A and B are symmetric $n \times n$ matrices with $A > 0$ and $B \geqslant 0$ then

$$\frac{\operatorname{tr}[B]}{\|A^{-1}\|} \leqslant \operatorname{tr}[AB] \leqslant \|A\| \operatorname{tr}[B].$$

Proof of Upper Bound. Since $A' = A$, it follows that $A = TDT'$ where $TT' = T'T = I$ and D is the diagonal matrix of the n real eigenvalues $\{\lambda_i\}$ of A. Thus,

$$\begin{aligned}\operatorname{tr}[AB] &= \operatorname{tr}[TDT'B] \\ &= \operatorname{tr}[DT'BT] \\ &= \sum_{i=1}^{n} \lambda_i [T'BT]_{ii}.\end{aligned}$$

Define $e_i = [\varepsilon_{1i}, \varepsilon_{2i}, \ldots, \varepsilon_{ni}]'$ where ε_{ij} is the Kronecker delta ($= 1$ if $i = j, 0$ otherwise) and note that

$$[T'BT]_{ii} = e_i'[T'BT]e_i = z_i'Bz_i$$

where $z_i = Te_i \Rightarrow [T'BT]_{ii} \geqslant 0$ since $B \geqslant 0$. Consequently, since $\lambda_i > 0$ for all i,

$$\begin{aligned}\operatorname{tr}[AB] &\leqslant \left(\max_{i \leqslant n} \lambda_i\right) \sum_{i=1}^{n} [T'BT]_{ii} \\ &= \|A\| \operatorname{tr}[T'BT] \\ &= \|A\| \operatorname{tr}[B].\end{aligned}$$

Proof of Lower Bound.

$$\begin{aligned}\operatorname{tr}[B] &= \operatorname{tr}[A^{-1}AB] \\ &= \operatorname{tr}[TD^{-1}T'AB] \\ &= \operatorname{tr}[D^{-1}T'ABT] \\ &= \sum_{i=1}^{n} \frac{1}{\lambda_i} [T'ABT]_{ii}.\end{aligned}$$

Note that

$$\begin{aligned}[T'ABT]_{ii} &= [T'TDT'BT]_{ii} \\ &= [DT'BT]_{ii} \\ &= \lambda_i [T'BT]_{ii} \\ &\geqslant 0\end{aligned}$$

$$\begin{aligned}\Rightarrow \operatorname{tr}[B] &\leqslant \left(\max_{i \leqslant n} \frac{1}{\lambda_i}\right) \sum_{i=1}^{n} [T'ABT]_{ii} \\ &= \|A^{-1}\| \operatorname{tr}[T'ABT] \\ &= \|A^{-1}\| \operatorname{tr}[AB].\end{aligned}$$

$$\therefore \operatorname{tr}[AB] \geqslant \frac{\operatorname{tr}[B]}{\|A^{-1}\|}. \qquad \square$$

THEOREM 3.2. Define the symmetric and gyroscopic parts of Δ as

$$\Delta_s = \tfrac{1}{2}(\Delta + \Delta'), \qquad \Delta_g = \tfrac{1}{2}(\Delta - \Delta').$$

For arbitrary Δ_g, if $\Delta_s > 0$, the kinetic energy satisfies

$$\frac{\frac{1}{2}\operatorname{tr}[V]}{\|\Delta_s\|} \leqslant \mathrm{KE} \leqslant \tfrac{1}{2}\|\Delta_s^{-1}\| \operatorname{tr}[V].$$

Proof. From equation (3.1),

$$\begin{aligned}\tfrac{1}{2}\mathrm{tr}[V] &= \mathrm{tr}[\Delta X_{22}]\\ &= \mathrm{tr}[\Delta_s X_{22}] + \mathrm{tr}[\Delta_g X_{22}].\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{tr}[\Delta_g X_{22}] &= \mathrm{tr}[(\Delta_g X_{22})']\\ &= \mathrm{tr}[X_{22}\Delta'_g]\\ &= -\mathrm{tr}[X_{22}\Delta_g] \ (\text{since } \Delta'_g = -\Delta_g)\\ &= -\mathrm{tr}[\Delta_g X_{22}] = 0\end{aligned}$$

$$\Rightarrow \tfrac{1}{2}\mathrm{tr}[V] = \mathrm{tr}[\Delta_s X_{22}].$$

Since $X_{22} \geqslant 0$, it follows from Lemma 3.1 that

$$\frac{\mathrm{tr}[X_{22}]}{\|\Delta_s^{-1}\|} \leqslant \mathrm{tr}[\Delta_s X_{22}] \leqslant \|\Delta_s\|\mathrm{tr}[X_{22}].$$

Thus,

$$\begin{aligned}\mathrm{KE} = \mathrm{tr}[X_{22}] &\geqslant \frac{\mathrm{tr}[\Delta_s X_{22}]}{\|\Delta_s\|}\\ &= \frac{\frac{1}{2}\mathrm{tr}[V]}{\|\Delta_s\|}\end{aligned}$$

and

$$\begin{aligned}\mathrm{KE} &\leqslant \|\Delta_s^{-1}\|\mathrm{tr}[\Delta_s X_{22}]\\ &= \tfrac{1}{2}\|\Delta_s^{-1}\|\mathrm{tr}[V].\end{aligned}$$

□

COROLLARY 3.3. For completely nongyroscopic structures, the total energy satisfies

$$\frac{\mathrm{tr}[V]}{\|\Delta\|} \leqslant \mathrm{TE} \leqslant \|\Delta^{-1}\|\mathrm{tr}[V].$$

Proof. For a completely nongyroscopic structure, $\Delta = \Delta_s$ and TE = 2KE by Theorem 2.1. □

To assess the tightness of these bounds, note that $\|\Delta_s\|\|\Delta_s^{-1}\|$ is the condition number of Δ_s. Thus, the tightest bounds will be obtained when the eigenvalues of Δ_s are clustered together, becoming exact when $\Delta_s = \delta I$ for some $\delta > 0$. Here, it follows immediately that $\mathrm{KE} = \mathrm{tr}[V]/2\delta$, and if $\Delta_g = 0$, $\mathrm{TE} = \mathrm{tr}[V]/\delta$.

4. Fluctuation-Dissipation Theorems. In statistical mechanics, various fluctuation-dissipation theorems have been developed that relate the zero-frequency power-spectral density of an external disturbance source (deemed responsible for "fluctuations" from equilibrium) to dissipation coefficients and the total equilibrium energy of the system under consideration [13, 14]. Typical applications include the description of Brownian

motion in viscous fluids [6, 14] and the description of thermal noise in resistors [14]. In [6], these ideas are extended to the Brownian motion of a simple harmonic oscillator and the hydrodynamical fluctuations of fluids described by linearized Navier–Stokes equations. This is accomplished by establishing the Lyapunov equation (2.6) as a generalized fluctuation-dissipation theorem and explicitly solving it for these two special cases.

Here, we establish two more general fluctuation-dissipation theorems by deriving analytic expressions relating total energy, damping matrix elements and driving noise covariance matrix elements for two general classes of problems. The first of these results is summarized in Theorem 4.1, which describes the energy distribution in a collection of N identical harmonic oscillators coupled by an arbitrary symmetric damping matrix. While this result is probably of little use in analyzing flexible mechanical structures of realistic complexity, it is of interest because it directly generalizes the single-mode fluctuation-dissipation theorem presented in [6] to N harmonic oscillators.

THEOREM 4.1. If all N modes are degenerate (i.e., $\omega_i = \omega$ for $i = 1, 2, \ldots, N$), and $\Delta = \Delta'$ then

$$\mathrm{TE} = \mathrm{tr}[\Delta^{-1}V].$$

Proof. Since $\Omega = \omega I$, it follows from equation (2.9a) that $X_{12}' = -X_{12}$. This reduces equation (2.9c) to

$$-\Delta X_{22} - X_{22}\Delta + V = 0$$

$$\Rightarrow X_{22} + \Delta^{-1}X_{22}\Delta = \Delta^{-1}V \Rightarrow \mathrm{tr}[X_{22}] + \mathrm{tr}\left[\Delta^{-1}X_{22}\Delta\right] = \mathrm{tr}[\Delta^{-1}V].$$

But $\mathrm{tr}[\Delta^{-1}X_{22}\Delta] = \mathrm{tr}[X_{22}\Delta\Delta^{-1}] = \mathrm{tr}[X_{22}]$ and $\mathrm{TE} = 2\,\mathrm{tr}[X_{22}] \Rightarrow \mathrm{TE} = \mathrm{tr}[\Delta^{-1}V]$. □

A more useful fluctuation-dissipation theorem can be established for the special case of a diagonal damping matrix. This result is summarized in Theorem 4.2 and is of greater interest for two reasons. First, many of the damping models considered to date for distributed parameter flexible structures yield modal damping matrices that are diagonal [3, 9]. While the complete modal decoupling they represent is undoubtedly not representative of real structures, these models do provide useful insight. Secondly, the explicit results presented in Theorem 4.2 provide a basis for the distributed parameter results presented in the next section.

THEOREM 4.2. If $\Delta = \mathrm{diag}\{\delta_i\}$, then

$$\mathrm{TE} = \mathrm{tr}[\Delta^{-1}V] = \sum_{i=1}^{N} \frac{V_{ii}}{\delta_i}.$$

Proof. First, note that since $\Omega = \mathrm{diag}\{\omega_i\}$,

$$[\Omega X_{12}]_{ii} = \omega_i[X_{12}]_{ii} = [X_{12}\Omega]_{ii}.$$

However, by (2.9a), $X_{12}\Omega = -(X_{12}\Omega)' \Rightarrow [X_{12}\Omega]_{ii} = 0$. Also, note that $[\Delta X_{22}]_{ii} = \delta_i[X_{22}]_{ii}$. Thus, the ii element of equation (2.9c) is

$$2\delta_i[X_{22}]_{ii} = V_{ii} \Rightarrow \mathrm{TE} = 2\,\mathrm{tr}[X_{22}] = \sum_{i=1}^{N} \frac{V_{ii}}{\delta_i}.$$

Similarly, note that

$$[\Delta^{-1}V]_{ii} = \frac{1}{\delta_i}[V]_{ii} \Rightarrow \mathrm{TE} = \mathrm{tr}[\Delta^{-1}V]. \qquad \square$$

5. Extension to distributed parameter systems. The results of Theorem 4.2 provide considerable insight into the formulation of well-posed distributed parameter flexible structure models. To see this, we consider the distributed parameter structure model proposed in [9], i.e.,

$$\ddot{x}(t) + C_0\dot{x}(t) + A_0x(t) = 0 \tag{5.1}$$

where $x(t)$ is an element of the real infinite-dimensional Hilbert space H with inner product $\langle , \rangle$ and norm $\| \ \|$. Here, A_0: $D(A_0) \to H$ is an unbounded linear operator with domain dense in H and C_0: $D(C_0) \to H$ is a linear operator that may be either bounded or unbounded, but whose domain must contain $D(A_0)$ and is thus also dense in H. In addition, A_0 is assumed to be a nonnegative self-adjoint operator with compact inverse and satisfying the coercivity condition

$$\langle A_0x, x\rangle \geqslant m^2\|x\|^2 \tag{5.2}$$

for all $x \in D(A_0)$ and some $m > 0$. Similarly, C_0 is assumed to be symmetric, nonnegative, and relatively bounded with respect to A_0 [11], i.e., there exists $k > 0$ such that

$$\|C_0x\| \leqslant k^2\|A_0x\| \tag{5.3}$$

for all $x \in H$. Under these conditions, Gibson [9] has shown that equation (5.1) may be cast in evolution equation form as

$$\dot{z}(t) = Az(t) \tag{5.4}$$

where $z(t) = [x(t)\ \dot{x}(t)]'$ and A is the unique maximal dissipative extension of

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -C_0 \end{bmatrix}. \tag{5.5}$$

To apply the results developed here we add a stochastic disturbance term of the general type considered in [5, Ch. 5]. Specifically, we consider the stochastic evolution equation

$$dz(t) = Az(t)\,dt + D\,dw(t), \quad z(0) = 0 \tag{5.6}$$

where $w(t)$ is the same scalar Wiener Process considered in Sec. 2, and $D = [0, f]'$. Here, f is a constant element of the Hilbert space H. If $S(t)$ is the strongly continuous semigroup generated by the composite operator A, Eq. 5.6 has a unique strong solution given by

$$z(t) = \int_0^t S(t-s)f\,dw(s) \tag{5.7}$$

provided the following conditions are met [5, Theorem 5.35]:

$$S(t-s)f \in D(A) \quad \text{for all } t, s, \tag{5.8a}$$

$$\int_0^t \|AS(t-s)f\|^2\,ds < \infty. \tag{5.8b}$$

Note that if $f \in D(A)$, condition (5.8a) is satisfied and may thus be viewed as a spatial "smoothness condition" on the function f through which external disturbances act on the structure. Further, if condition (5.8a) is satisfied and the semigroup $S(t)$ is exponentially stable, then condition 5.8b is also satisfied. Gibson has shown that $S(t)$ is exponentially stable if the operator C_0 satisfies the coercivity condition [9]

$$\langle x, C_0 x \rangle \geqslant \alpha \| x \|^2 \tag{5.9}$$

for all $x \in H$ and some $\alpha > 0$. To proceed, note that the conditions imposed on A_0 guarantee that it has compact normal resolvent. Thus, its spectrum is a countably infinite sequence of isolated positive eigenvalues $\{\omega_n^2\}$, each with finite multiplicity [11, p. 187] and its eigenvectors $\{\phi_i\}$ form a complete orthonormal basis for H [11, p. 260]. Consequently, we can obtain finite-dimensional modal approximations of (5.1) that are in the form of (2.1) by defining

$$[\Omega^2]_{ij} = \langle \phi_i, A_0 \phi_j \rangle = \omega_i^2 \varepsilon_{ij}, \tag{5.10a}$$

$$[\Delta]_{ij} = \langle \phi_i, C_0 \phi_j \rangle, \tag{5.10b}$$

$$[T]_i = \langle \phi_i, f \rangle \tag{5.10c}$$

for $i,\ j \leqslant N$, where ε_{ij} is the Kronecker delta as before. Further, we can obtain the following convergence theorems by considering the results of Sec. 4 as $N \to \infty$.

THEOREM 5.1. If the eigenvalues of A_0 are distinct and C_0 commutes with A_0 and satisfies the conditions outlined above, then for all N

$$\frac{\sigma^2}{k^2} \sum_{i=1}^{N} \frac{f_n^2}{\omega_n^2} \leqslant \mathrm{TE}_N \leqslant \frac{\sigma^2}{\alpha} \sum_{n=1}^{N} f_n^2$$

where k and α are the constants defined in conditions 5.3 and 5.9, respectively, and $f_n = \langle \phi_n, f \rangle$.

Proof. First, note that if C_0 commutes with A_0, then for any eigenvector ϕ_i of A_0,

$$A_0 C_0 \phi_i = C_0 A_0 \phi_i = \omega_i^2 C_0 \phi_i.$$

Thus, $C_0 \phi_i$ is an eigenvector of A_0 with eigenvalue ω_i^2

$$\Rightarrow C_0 \phi_i = \delta_i \phi_i \quad \text{for some } \delta_i$$

$$\Rightarrow \Delta_N = \operatorname{diag}\{\delta_i\}_{i=1}^{N}.$$

From Theorem 4.2, for any fixed N, we have

$$\mathrm{TE}_N = \sum_{n=1}^{N} \frac{V_{nn}}{\delta_n} = \sigma^2 \sum_{n=1}^{N} \frac{f_n^2}{\delta_n}$$

since $V_{nn} = \sigma^2 [T_N T_N']_{nn} = \sigma^2 f_n^2$. Next, note that $\alpha \leqslant \delta_n \leqslant k^2 \omega_n^2$ for all n since otherwise we would violate either the relative boundedness condition (5.3), i.e.,

$$\delta_n > k^2 \omega_n \Rightarrow \| C_0 \phi_n \| = \delta_n \| \phi_n \| > k^2 \omega_n^2 \| \phi_n \| = k^2 \| A_0 \phi_n \|$$

or the coercivity condition (5.9), i.e.,

$$\delta_n < \alpha \Rightarrow \langle \phi_n, C_0\phi_n \rangle = \delta_n \|\phi_n\|^2 < \alpha \|\phi_n\|^2.$$

Thus, we have

$$\frac{f_n^2}{k^2\omega_n^2} \leqslant \frac{f_n^2}{\delta_n} \leqslant \frac{f_n^2}{\alpha}$$

and the result of the theorem follows immediately. □

Corollary 5.2. If $f \in H$ and the conditions of Theorem 5.1 are satisfied then

$$\mathrm{TE} = \sigma^2 \sum_{n=1}^{\infty} \frac{f_n^2}{\delta_n}.$$

Proof. From Theorem 5.1,

$$\mathrm{TE}_n = \sigma^2 \sum_{n=1}^{N} \frac{f_n^2}{\delta_n} \leqslant \frac{\sigma^2}{\alpha} \sum_{n=1}^{N} f_n^2.$$

Since $\delta_n > 0$ for all n, the sequence $\{\mathrm{TE}_N\}$ is monotonically increasing and therefore converges to a limit TE if it is bounded above.

Thus, since $f \in H$,

$$\|f\| = \sum_{n=1}^{\infty} f_n^2 \geqslant \sum_{n=1}^{N} f_n^2$$

for all N so $\{\mathrm{TE}_N\}$ converges to the limit

$$\mathrm{TE} = \sigma^2 \sum_{n=1}^{\infty} \frac{f_n^2}{\delta_n}.$$ □

Corollary 5.3. If $f \in H$ and the conditions of Theorem 5.1 are satisfied, then

$$\frac{\sigma^2}{k^2} \|A_0^{-1/2} f\| \leqslant \mathrm{TE} \leqslant \frac{\sigma^2}{\alpha} \|f\|^2$$

where $A_0^{-1/2}$ is the self-adjoint square root of the compact operator A_0^{-1}, as defined in [3].

Proof. Since the eigenfunctions of A_0 span H, we can expand $f \in H$ as

$$f = \sum_{n=1}^{\infty} \langle \phi_n, f \rangle \phi_n = \sum_{n=1}^{\infty} f_n \phi_n \Rightarrow A_0^{-1/2} f = \sum_{n=1}^{\infty} f_n A_0^{-1/2} \phi_n = \sum_{n=1}^{\infty} (f_n/\omega_n) \phi_n$$

$$\Rightarrow \|A_0^{-1/2} f\|^2 = \sum_{n=1}^{\infty} \frac{f_n^2}{\omega_n^2}.$$

Thus, as $N \to \infty$,

$$\frac{\sigma^2}{k^2} \sum_{n=1}^{N} \frac{f_n^2}{\omega_n^2} \to \frac{\sigma^2}{k^2} \|A_0^{-1/2} f\|^2,$$

$$\mathrm{TE}_N \to \mathrm{TE},$$

and

$$\frac{\sigma^2}{\alpha}\sum_{n=1}^{N} f_n^2 \to \frac{\sigma^2}{\alpha}\|f\|^2,$$

and the result follows immediately. □

6. Physical interpretations and conclusions. The physical significance of these results can be illustrated by the following simple example. Small amplitude vibrations of a thin stretched string are described by the wave equation

$$\frac{\partial^2 x(z,t)}{\partial t^2} + C_0\frac{\partial x(z,t)}{\partial t} - \frac{\partial^2 x(z,t)}{\partial z^2} = f(z)\eta(t). \tag{6.1}$$

Here, $x(z,t)$ is defined on the interval $[0,\pi]$ of the real z-axis and subject to the boundary conditions

$$x(0,t) = x(\pi,t) = 0 \tag{6.2}$$

for all $t \geqslant 0$, $f(z) \in L_2[0,\pi]$, and $\eta(t)$ is a zero-mean scalar white noise process with variance σ^2. This corresponds to the model (5.1) with $A_0 = -\partial^2/\partial z^2$ defined on the domain

$$D(A_0) = \left\{ x(z) \in L_2[0,\pi] \,\middle|\, \frac{\partial x}{\partial z}, \frac{\partial^2 x}{\partial z^2} \text{ absolutely continuous and } x(0) = x(\pi) = 0 \right\}. \tag{6.3}$$

Here the eigenvalues of A_0 are $\omega_n^2 = n^2$ and the normalized eigenfunctions are $\phi_n(z) = \sqrt{2/\pi}\sin nz$. Physically, $f(z)$ describes the spatial profile over which the external force $\eta(t)$ acts on the structure; the influence of $\eta(t)$ on each mode is defined by the Fourier series coefficients

$$f_n = \sqrt{2/\pi}\int_0^{\pi} f(z)\sin nz\,dz. \tag{6.4}$$

To apply the results developed here, we consider the following five damping models:

$$C_0 = \delta \Rightarrow \Delta_{ij} = \delta\varepsilon_{ij}, \tag{6.5a}$$

$$C_0 = 2\alpha A_0 \Rightarrow \Delta_{ij} = 2\alpha\omega_i^2\varepsilon_{ij}, \tag{6.5b}$$

$$C_0 = 2\zeta A_0^{1/2} \Rightarrow \Delta_{ij} = 2\zeta\omega_i\varepsilon_{ij}, \tag{6.5c}$$

$$C_0 = 2\zeta A_0^{1/2} + CA_0^{1/2} \Rightarrow \Delta_{ij} = 2\zeta\omega_i\varepsilon_{ij} + \omega_j\langle\phi_i, C\phi_j\rangle, \tag{6.5d}$$

$$C_0 x = m\langle g, x\rangle g \Rightarrow \Delta_{ij} = m\langle\phi_i, g\rangle\langle\phi_j, g\rangle. \tag{6.5e}$$

The first three of these models have been widely studied because of their mathematical simplicity, the fourth is a more complicated variation of the third examined by Chen and Russell [3], and the last model describes a simple active damping strategy. Specifically, damping model (6.5a) corresponds to damping of the structure by a viscous fluid that generates a restoring force linearly proportional to velocity. In this case, the operator C_0 is bounded and it follows from perturbation results for semigroups [11, p. 497] that the time

evolution of the damped system is described by a unitary group just as the undamped system is. Further, this damping model satisfies the conditions of Corollary 5.2 so the total energy of the structure is given by

$$\mathrm{TE}_{\mathrm{a}} = \sigma^2 \sum_{n=1}^{\infty} \frac{f_n^2}{\delta} = \sigma^2 \| f \|^2 / \delta. \tag{6.6}$$

Damping model (6.5b) also satisfies the conditions of Corollary 5.2, yielding a total energy of

$$\mathrm{TE}_{\mathrm{b}} = \frac{\sigma^2}{2\alpha} \sum_{n=1}^{\infty} f_n^2 / n^2. \tag{6.7}$$

Note, however, that this damping mechanism is much stronger in the sense that the damped system is now described by a holomorphic semigroup [3, 19] rather than a unitary group. Physically, this damping mechanism can be motivated for the Euler–Bernoulli beam with an internal damping stress proportional to the instantaneous strain rate in the beam [4, p. 301].

Intermediate in strength between models (6.5a) and (6.5b), damping model (6.5c) represents a constant damping ratio for all modes of the structure. Although it has no obvious mechanistic basis, this model is an intuitively pleasing extension of the linearly damped simple harmonic oscillator. Like model (6.5b), this damping model also yields a system whose dynamics are described by a holomorphic semigroup [3]. By Corollary 5.2, the total energy of this structure is

$$\mathrm{TE}_{\mathrm{c}} = \frac{\sigma^2}{2\zeta} \sum_{n=1}^{\infty} f_n^2 / n. \tag{6.8}$$

Note that if $\alpha = \zeta = \delta/2$, it follows from equations (6.6)–(6.8) that $\mathrm{TE}_{\mathrm{b}} < \mathrm{TE}_{\mathrm{c}} < \mathrm{TE}_{\mathrm{a}}$ in agreement with our notion of the relative strength of these damping models.

Damping model (6.5d) is a perturbation of model (6.5c) proposed by Chen and Russell in which C is an appropriately defined bounded linear operator [3]. If $\|C\|$ is sufficiently small, it can be shown that this damping model also leads to a system described by a holomorphic semigroup. Here, however, the damping matrix is generally not symmetric, so from the discussion at the end of Sec. 2, we should not expect equipartition of kinetic and potential energy to hold for this damping model. Similarly, the other results derived from equipartition are not applicable. Finally, damping model (6.5e) represents the effects of active damping applied to an initially undamped structure. Specifically, this damping model represents a velocity sensor $v(t) = \langle g(z), x(z,t) \rangle$ co-located with a force actuator $f(z,t) = g(z)h(t)$ and coupled with the feedback law $h(t) = -mv(t)$. This control scheme has been widely discussed [2, 18] because of its desirable stability properties, but it has been shown that if $g(z) \in L_2[0, \pi]$, the damped structure will not exhibit a uniform exponential decay rate because the operator C_0 is compact [9, 18]. Further, since Δ_N is a dyadic product for any N, it has rank 1 and Δ_N^{-1} does not exist. Thus, while equipartition of kinetic and potential energy follow from Theorem 2.1, the upper bound on total energy defined by Corollary 3.3 does not exist. The lower bound remains valid, however, and may

be simplified by noting that, since Δ_N has rank 1, $\|\Delta_N\| = \mathrm{tr}[\Delta_N]$ so that

$$\mathrm{TE}_N \geqslant \frac{\mathrm{tr}[V_N]}{\|\Delta_N\|} = \frac{\sigma^2 \Sigma_{n=1}^N f_n^2}{m \Sigma_{n=1}^N g_n^2} \tag{6.9}$$

where $g_n = \langle \phi_n, g \rangle$. Taking the limit of (6.9) as $N \to \infty$ yields the result that if the distributed parameter total energy is finite, it is bounded below by

$$\mathrm{TE} \geqslant \frac{\sigma^2 \|f\|^2}{m \|g\|^2}. \tag{6.10}$$

Unfortunately, this is all we can conclude in general for this problem since it is easy to concoct specific examples for which the total energy either converges (e.g., $f = g = \phi_j$ for fixed $j, m > 0$) or diverges (e.g., $f = \phi_i$, $g = \phi_j$, $i \neq j$, $m \neq 0$).

These results suggest two important conclusions regarding the formulation of well-posed stochastic, distributed-parameter, flexible structure models. The first is that while uniform exponential stability is desirable, it may not be necessary for well-posedness. For example, consider the damping model

$$C_0 = \gamma A_0^{-1/2}. \tag{6.11}$$

While this model does not correspond to any physical damping mechanism of which the authors are aware, it does yield a stable system with a nonexponential decay rate since C_0 is a compact operator [8].

In analogy with Corollary 5.2, we would expect that

$$\mathrm{TE} = \sigma^2 \sum_{n=1}^{\infty} \frac{f_n^2}{\delta_n} = \frac{\sigma^2}{\gamma} \sum_{n=1}^{\infty} n f_n^2 \tag{6.12}$$

provided the sequence $\{f_n\}$ decays sufficiently rapidly for the last sum to converge. Physically, this corresponds to a smoothness condition on $f(z)$ [20, Sec. 4.42].

The second modeling conclusion drawn here is the converse of the first. That is, even if the inherent damping is sufficiently strong to guarantee a uniform exponential decay rate, the actuator influence function $f(z)$ must still be sufficiently smooth to guarantee convergence of the modal energy sum in Corollary 5.2. Thus, since $C_0 = \delta$ has a bounded inverse, damping model (6.5a) will yield a uniform exponential decay rate for the damped structure [18]. Here, the total energy is well-defined for any $f(z) \in L_2[0, \pi]$ by Eq. (6.6), but the results cannot be extended to disturbances acting at discrete points. While such disturbances can be modeled by introducing the Dirac delta distribution $f(z) = \delta(z - z_0)$ [5], its Fourier series components are $f_n = \sqrt{2/\pi} \sin n z_0$ and the resulting total energy sum does not converge. It may be possible to overcome this problem by considering the stronger damping mechanism (6.5c) for which the corresponding modal energy sum does converge, i.e.,

$$\mathrm{TE} = \sigma^2 \sqrt{2/\pi} \sum_{n=1}^{\infty} \frac{\sin n z_0}{n} = \frac{\sigma^2 (\pi - z_0)}{\sqrt{2\pi}}, \tag{6.13}$$

although it should be noted that this convergence is very slow. The situation is even worse if point torquers are considered, since they must be modeled by the derivative of the Dirac delta distribution, $f(z) = \delta'(z - z_0)$. Here, $f_n = \sqrt{2/\pi}\, n \cos nz_0$ so it follows from the lower bound of Theorem 5.1 that the total energy diverges for any relatively bounded damping operator C_0.

In summary, the results presented here demonstrate that both damping models and disturbance models must be carefully chosen in formulating realistic flexible structure control problems. While recent results have demonstrated the feasibility of designing finite-dimensional controllers for such structures from truncated modal approximations [9, 19], the results presented here illustrate that the validity of this conclusion is a strong function of the damping and disturbance models employed. Even more important in practical terms, however, the fluctuation-dissipation results presented in Sec. 4 provide a means of assessing how many modes must be retained in such an approximate model to obtain reasonable results. Specifically, given C_0 and $f(z)$ satisfying the conditions of Corollary 5.2, the number of modes retained can be determined by requiring N sufficiently large that the total energy computed in Theorem 4.2 is a given fraction (say 90%) of the total energy computed in Corollary 5.2. This suggests the use of the strongest damping mechanisms and the smoothest disturbance force profiles justifiable in order to minimize the finite-dimensional model size required.

References

[1] M. J. Balas, *Modal control of certain flexible dynamic systems*, SIAM J. Control Optim. **16**, 450–462 (1978)

[2] M. J. Balas, *Direct velocity feedback control of large space structures*, AIAA J. Guidance and Control **2**, 252–253 (1979)

[3] G. Chen and D. L. Russell, *A mathematical model for linear elastic systems with structural damping*, Quart. Appl. Math. **39**, 433–454 (1978)

[4] R. W. Clough and J. Penzien, *Dynamics of structures*, McGraw-Hill, New York, 1975

[5] R. F. Curtain and A. J. Pritchard, *Infinite dimensional linear systems theory*, Springer-Verlag, New York, 1978

[6] R. F. Fox and G. E. Uhlenbeck, *Contributions to non-equilibrium thermodynamics. I. Theory of hydrodynamical fluctuations*, Phys. Fluids **13**, 1893–1902 (1970)

[7] J. C. Geromel and J. Bernussou, *On bounds of Lyapunov's matrix equation*, IEEE Trans. Automat. Control **24**, 482–487 (1979)

[8] J. S. Gibson, *A note on stabilization of infinite dimensional oscillators by compact linear feedback*, SIAM J. Control Optim. **18**, 311–316 (1980)

[9] J. S. Gibson, *An analysis of optimal modal regulation: convergence and stability*, SIAM J. Control Optim. **19**, 686–707 (1981)

[10] P. C. Hughes and R. E. Skelton, *Controllability and observability for flexible spacecraft*, J. Guidance and Control **3**, 452–460 (1980)

[11] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York, 1980

[12] W. H. Kwon and A. E. Pearson, *A note on the algebraic matrix Riccati equation*, IEEE Trans. Automat. Control **22**, 143–144 (1977)

[13] L. D. Landau and E. M. Lifschitz, *Statistical physics*, Pergamon Press, New York, 1980

[14] B. G. Levich, *Theoretical physics, vol. 2: Statistical physics, electromagnetic processes in matter*, John Wiley & Sons, New York, 1971

[15] *Industry workshop on large space structures: Compilation of company presentations and responses to a key issue questionnaire*, NASA CR-144997, McDonnell-Douglas Astronautics Co., May 1976

[16] A. H. Nayfeh and D. T. Mook, *Nonlinear oscillations*, Wiley, New York, 1979

[17] R. V. Patel and M. Toda, *On norm bounds for algebraic Riccatti and Lyapunov equations*, IEEE Trans. Automat. Control **23**, 87–88 (1978)

[18] D. L. Russell, *Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods*, J. Differential Equations **19**, 344–370 (1975)

[19] Y. Sakawa, *Feedback control of second order evolution equations with damping*, SIAM J. Control Optim. **22**, 343–361 (1984)

[20] G. Shilov, *Elementary functional analysis*, MIT Press, Cambridge, Massachusetts, 1974

[21] *Proceedings of VPI & SU/AIAA symposium on dynamics and control of large flexible spacecraft*, Blacksburg, Virginia, June 1979

[22] *Dynamics and control of large flexible spacecraft: proceedings of the third VPI & SU/AIAA symposium*, Blacksburg, Virginia, June 1981