QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLV, NUMBER 4
DECEMBER 1987, PAGES 669-678

DIFFUSION AND CONVECTION OF VORTICITY AT LOW REYNOLDS NUMBERS PRODUCED BY A ROTLET INTERIOR TO A CIRCULAR CYLINDER*

By

K. B. RANGER

University of Toronto, Toronto, Ontario, Canada

Abstract. The diffusion and convection of vorticity produced by a rotlet inside a circular cylinder is discussed at low Reynolds numbers by considering a nonlinear approximation to a complex form of the steady two-dimensional Navier–Stokes equations. An expression is found for the boundary vorticity and the modifications to the separation of streamlines are discussed as a function of the Reynolds number.

1. Introduction. The problem of steady convection and diffusion of vorticity in an incompressible viscous fluid is of long-standing interest in fluid dynamics. Vorticity is convected by a forced potential flow (e.g. a uniform stream) and in addition there is the self-convection of diffused vorticity which is expressed through the nonlinearity of the Navier–Stokes equations. For flow in the presence of a fixed boundary an exact analytic treatment represents a formidable task and either a numerical method has to be employed or it is necessary to introduce a simplifying approximation to the equations of motion in order to determine the desired physical information.

The present paper considers two-dimensional flow and the starting point is a complex form of the Navier–Stokes equations first given by R. Legendre [1]. This complex equation contains as dependent variables the Earnshaw stream function ψ and an auxiliary function ϕ, which Pillow [2] has described as an Airy flux function of momentum. An approximation is described for the complex equation of motion in which the Airy flux function of momentum, as it occurs in the convective terms of the equation, is replaced by a harmonic function. In turn this harmonic function is calculated by using the equations of motion, boundary conditions, and limiting cases. The main advantages of this new approximation are that the boundary vorticity can be found explicitly in a straightforward way and vorticity generated at the boundary is not convected through or around the boundary as in the cases of the linearizations of Oseen and Burgers. It follows that

*Received June 13, 1986.

when separation occurs the critical Reynolds number for which the phenomenon first commences can be predicted more accurately than the corresponding values found from the linearized Oseen and Burgers equations.

The approximation described here is applied to the specific model of a rotlet (producing constant torque) inside a cylinder. The resulting flow is qualitatively similar to the flow between two cylinders in which the inner is rotating with constant angular velocity and the outer is at rest. This particular flow has an added interest in that for certain positions of the rotlet there is separation of the streamlines in the Stokes flow and it is possible to predict how the flow is modified for increasing values of the Reynolds number.

The flow equations. The equations of motion governing the two-dimensional steady flow of an incompressible viscous fluid are:

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = -\operatorname{grad}(p/\rho) + \nu\nabla^2\mathbf{q}, \tag{1}$$

$$\operatorname{div}\mathbf{q} = 0, \tag{2}$$

where $\mathbf{q} = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$ is the fluid velocity, p the pressure, ρ the density, and ν the kinematic viscosity. The equation of continuity can be satisfied by introducing an Earnshaw stream function ψ by

$$u + iv = 2i\psi_{\bar z}, \qquad 2\frac{\partial}{\partial \bar z} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \tag{3}$$

and ψ is a solution of the complex Navier–Stokes equations, viz.

$$\phi_{\bar z\bar z} + i\psi_{\bar z\bar z} + \frac{1}{2\nu}\psi_{\bar z}^2 = f''(\bar z), \tag{4}$$

where $f(\bar z)$ is an arbitrary function of $\bar z$. The real function ϕ is defined by

$$-\nu\nabla^2\phi = p/\rho + 2\psi_z\psi_{\bar z}, \tag{5}$$

where the right-hand side is the Bernoulli function or total head of pressure. The complex equation (4) was first given by R. Legendre [1] who gave an application to boundary layer analysis. The author has given a derivation of the equation in [3] and Pillow [2] has described an interpretation for the function ϕ as an Airy flux function of momentum. It is noted that the complex conjugate of (4) is

$$\phi_{zz} - i\psi_{zz} + \frac{1}{2\nu}\psi_z^2 = \bar f''(z). \tag{6}$$

An elimination of ϕ yields the usual vorticity equation given in [3]. Also, an alternative way of displaying equation (4) is given by

$$\frac{\partial^2}{\partial \bar z^2}\left(e^{\psi/(2i\nu)}\right) = \frac{e^{\psi/(2i\nu)}}{2\nu}\left[\phi_{\bar z\bar z} - f''(\bar z)\right]. \tag{7}$$

Now a complex stream function Ω can be written as $\Omega = \phi + i\psi$, in which case equation (4) can be put in the form

$$\Omega_{z\bar z} = \frac{1}{2\nu}(\Omega_{\bar z} - \phi_{\bar z})^2 + f''(\bar z). \tag{8}$$

It is customary to linearize or approximate the Navier–Stokes equations with respect to the stream function ψ or velocity field (e.g. Oseen, Burgers) for which the forced convecting field is essentially known through the prescribed boundary conditions. However, in the present analysis, the approximation is carried out with respect to the function ϕ, for which less is known explicitly, but some of its properties can be inferred through the equations of motion, boundary conditions, and limiting cases. The function $\phi_{\bar{z}}$, as it occurs explicitly on the right-hand side of (8), is replaced by $\chi_{\bar{z}}$ where $\chi = g(z) + \bar{g}(\bar{z})$, so that equation (8) is replaced by

$$\Omega_{z\bar{z}} = \frac{1}{2\nu}\left[\Omega_{\bar{z}} - \bar{g}'(\bar{z})\right]^2 + f''(\bar{z}). \tag{9}$$

Now $\phi = \chi$, $\psi = \psi_0 = F(z) + \bar{F}(\bar{z})$ is a solution of (9) if

$$\bar{g}''(\bar{z}) + i\bar{F}(\bar{z}) + \frac{1}{2\nu}(\bar{F}'(\bar{z}))^2 = f''(\bar{z}) \tag{10}$$

which also satisfies the exact Navier–Stokes equation expressed by (4). Now ψ_0 is a viscosity independent potential flow but does not in general represent explicitly the forced convection potential for the particular flow under consideration. The functions f, g, F and their relations to ϕ, ψ will be discussed in more detail at a later stage.

To solve (9) it is convenient to write $\omega = \Omega - \bar{g}(\bar{z})$ and also introduce a function $h(\bar{z})$ satisfying a Riccati type equation expressed by

$$f''(\bar{z}) - \bar{g}''(\bar{z}) = h''(\bar{z}) - \frac{1}{2\nu}(h'(\bar{z}))^2. \tag{11}$$

In terms of ω equations (9) and (11) give

$$\omega_{z\bar{z}} - \frac{1}{2\nu}\omega_{\bar{z}}^2 = h''(\bar{z}) - \frac{1}{2\nu}\left[h'(\bar{z})\right]^2, \tag{12}$$

for which a first integral is given by

$$\omega_{\bar{z}} - h'(\bar{z}) = L(z)e^{\frac{1}{2\nu}[\omega + h(\bar{z})]}, \tag{13}$$

where $L(z)$ is for the present arbitrary and is assumed regular analytic in the fluid domain. A second integration yields

$$-2\nu e^{-\frac{1}{2\nu}[\omega - h(\bar{z})]} = L(z)\int e^{h(\bar{z})/\nu}\, d\bar{z} + M(z) \tag{14}$$

where again $M(z)$ is regular analytic in the fluid region but otherwise arbitrary. Equation (14) is equivalent to

$$-2\nu e^{-\frac{1}{2\nu}[\Omega - h(\bar{z}) - \bar{g}(\bar{z})]} = H(\bar{z})L(z) + M(z) \tag{15}$$

with $H'(\bar{z}) = e^{h(\bar{z})/\nu}$. At this point it is necessary to consider a specific flow problem in which the boundary conditions can be formulated so as to determine the functions $L(z)$ and $M(z)$. The flow takes place interior to the unit circle, $z\bar{z} = 1$, and is forced or stirred by a rotlet at $z = c$, $0 < c < 1$. If the rotlet strength is K, a Reynolds number can be defined by $R = K/\nu$. The boundary conditions for ψ require

$$\psi = \psi_{\bar{z}} = 0, \quad \text{at } z\bar{z} = 1, \quad \psi \sim \log R_1$$
$$\text{as } R_1 = (r^2 + c^2 - 2rc\cos\theta)^{1/2} \to 0. \tag{16}$$

Since the functions $g(z)$ and $h(\bar{z})$ are arbitrary, it is possible to impose the additional boundary conditions

$$\phi - \chi = \phi_{\bar{z}} - \chi_{\bar{z}} = 0, \quad \text{on } z\bar{z} = 1. \tag{17}$$

It will be shown later that these conditions, in addition to partially determining the arbitrary functions $g(z)$ and $f(\bar{z})$, also prevent vorticity from being convected through or around the boundary. The boundary conditions (16), (17) can be combined in complex form as follows:

$$\Omega - g(z) - \bar{g}(\bar{z}) = \Omega_{\bar{z}} - \bar{g}'(\bar{z}) = 0 \quad \text{on } z\bar{z} = 1, \tag{18}$$

which also imply the condition

$$\Omega_z - g'(z) = 0 \quad \text{on } z\bar{z} = 1. \tag{19}$$

From equations (13) and (14) it follows that

$$L(z) = -e^{-\frac{1}{2}R[g(z)+h(\frac{1}{z})]}h'\left(\frac{1}{z}\right) \quad \text{on } z\bar{z} = 1, \tag{20}$$

$$H\left(\frac{1}{z}\right)L(z) + M(z) = -\frac{2}{R}e^{-\frac{1}{2}R[g(z)-h(\frac{1}{z})]} \quad \text{on } z\bar{z} = 1. \tag{21}$$

With $L(z)$ and $M(z)$ analytic on $z\bar{z} \leqslant 1$, the analytic continuations in the unit circle are provided by Poisson's integral formula [4] viz.,

$$L(z) = -\frac{1}{2\pi i}\oint e^{-\frac{R}{2}[g(\zeta)+h(\frac{1}{\zeta})]}h'\left(\frac{1}{\zeta}\right)\left\{\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{1}{\bar{z}}}\right\} d\zeta, \tag{22}$$

$$H\left(\frac{1}{z}\right)L(z) + M(z) = -\frac{1}{2\pi Ri}\oint e^{-\frac{R}{2}[g(\zeta)-h(\frac{1}{\zeta})]}\left\{\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{1}{\bar{z}}}\right\} d\zeta, \tag{23}$$

where the integration is taken round the circle $|\zeta| = 1$. However, in order to determine basic information concerning the flow picture, it is more straightforward to calculate the boundary vorticity, which is readily obtained from (13) by differentiation with respect to z, so that

$$\Omega_{z\bar{z}} = \left\{L'(z) + \frac{L(z)\omega_z R}{2}\right\}e^{\frac{1}{2}R[\omega + h(\bar{z})]} \tag{24}$$

and on the boundary $z\bar{z} = 1$,

$$\Omega_{z\bar{z}} = \left\{L'(z) + \frac{1}{2}L(z)g'(z)R\right\}e^{\frac{1}{2}R[g(z)+h(\frac{1}{z})]}. \tag{25}$$

Now from equation (20),

$$L'(z) = \left\{\frac{h''(\frac{1}{z})}{z^2} + \frac{1}{2}Rh'\left(\frac{1}{z}\right)\left[g'(z) - \frac{1}{z^2}h'\left(\frac{1}{z}\right)\right]\right\}e^{-\frac{1}{2}R[g(z)+h(\frac{1}{z})]}. \tag{26}$$

Hence from (25), (20), and (26)

$$\Omega_{z\bar{z}}|_{|z|=1} = \frac{1}{z^2}\left\{h''\left(\frac{1}{z}\right) - \frac{1}{2}R\left[h'\left(\frac{1}{z}\right)\right]^2\right\} = \frac{1}{z^2}\left\{f''\left(\frac{1}{z}\right) - \bar{g}''\left(\frac{1}{z}\right)\right\}$$

$$= \frac{1}{z^2}\left\{i\bar{F}''\left(\frac{1}{z}\right) + \frac{1}{2}R\left[\bar{F}'\left(\frac{1}{z}\right)\right]^2\right\}. \tag{27}$$

Now the limit $R \to 0$ is the Stokes solution which is described with some detail in the Appendix, and using these results it is found that

$$\Omega_{z\bar{z}}|_{|z|=1} = \frac{i}{z^2}\bar{F}''\left(\frac{1}{z}\right) = \frac{i}{2} - \frac{i}{(1-cz)^2}. \tag{28}$$

Integration shows that

$$F(z) = \log(z-c) - \frac{1}{2}\log z \tag{29}$$

and, for arbitrary R,

$$\Omega_{z\bar{z}}|_{|z|=1} = \frac{1}{z^2}\left\{\bar{F}''\left(\frac{1}{z}\right) + \frac{1}{2}R\left[\bar{F}'\left(\frac{1}{z}\right)\right]^2\right\}$$

$$= i\left[\frac{1}{2} - \frac{1}{(1-cz)^2}\right] + \frac{1}{2}R\left[\frac{1}{1-cz} - \frac{1}{2}\right]^2. \tag{30}$$

This expression is derived on the basis that the vorticity agrees with the exact solution of the Navier–Stokes equations in the limit $c \to 0$. The imaginary part of equation (27) expresses the boundary vorticity as

$$\zeta = \nabla^2\psi = \frac{4\left\{\frac{1}{2}(c^4-1) + 2c^2(1-c\cos\theta)\right\} + 2cR(1-c^2)\sin\theta}{(1+c^2-2c\cos\theta)^2}. \tag{31}$$

It is clear that the Reynolds number effect produces positive vorticity in the upper half-plane ($0 < \theta < \pi$) and negative vorticity in the lower half-plane ($\pi < \theta < 2\pi$). The Stokes flow separation, which first occurs at $\theta = \pi$, for $c > \sqrt{2} - 1$ (see Appendix), is distorted from the symmetric position by forming a larger region of separated flow in the upper half-plane than in the lower half-plane. Now the boundary vorticity from (31) may be written in the form

$$(1+c^2-2c\cos\theta)^2\zeta = 2(c^2+2)^2 - 10 + \left[64c^6 + 4c^2(1-c^2)R^2\right]^{1/2}\cos(\theta+\alpha) \tag{32}$$

where

$$\cos\alpha = \frac{8c^3}{\left[64c^6 + 4c^2(1-c^2)^2R^2\right]^{1/2}}, \qquad \sin\alpha = \frac{2c(1-c^2)R}{\left[64c^6 + 4c^2(1-c^2)R^2\right]^{1/2}}. \tag{33}$$

It follows from (32) that separation first occurs at $\theta = \pi - \alpha$ when the Reynolds number R exceeds R^* where

$$R^* = \frac{1}{2(1-c^2)}\left\{\frac{\left[10 - 2(c^2+2)^2\right]^2}{c^2} - 64c^4\right\}^{1/2}. \tag{34}$$

For $0 < c < \sqrt{2} - 1$, separation takes place entirely in the upper half-plane due to convective Reynolds number effects provided $R > R^*$. For this range of c, there is no separation in the lower half-plane (see Fig. 4). For $c > \sqrt{2} - 1$, there is Stokes flow separation which extends into the lower half-plane. For $R < R^*$ and $0 < c < \sqrt{2} - 1$, the streamlines are closed curves containing the rotlet (see Fig. 3). For $c = 0$, the streamlines are concentric circles.

Since the boundary vorticity essentially provides the required physical information, it remains to discuss the adequacy of the approximate equation (9), particularly in a vicinity of the boundary. On the boundary $z\bar{z} = 1$, equation (9) gives

$$\Omega_{\bar{z}\bar{z}} = \phi_{\bar{z}\bar{z}} + i\psi_{\bar{z}\bar{z}} = f''(\bar{z}) \tag{35}$$

which is consistent with the exact complex Navier–Stokes equation (4). Also, differentiation of (9) with respect to z yields

$$\Omega_{z\bar{z}\bar{z}} = \frac{1}{\nu}\left(\Omega_{\bar{z}} - \bar{g}'(\bar{z})\right)\Omega_{z\bar{z}} \tag{36}$$

so that, using the boundary condition (18), it follows that on $z\bar{z} = 1$,

$$\Omega_{z\bar{z}\bar{z}} = \phi_{z\bar{z}\bar{z}} + i\psi_{z\bar{z}\bar{z}} = 0. \tag{37}$$

In terms of the pressure and vorticity distributions, (37) is equivalent to

$$\frac{\partial p}{\partial x} = -\mu\frac{\partial}{\partial y}\nabla^2\psi, \qquad \frac{\partial p}{\partial y} = \mu\frac{\partial}{\partial x}\nabla^2\psi \tag{38}$$

where μ is the coefficient of viscosity. Equations (38) are again consistent with exact Navier–Stokes equations evaluated at the boundary. Vorticity is not convected through or around the boundary as it is in Oseen or Burgers flow. However, there is an additional requirement from the exact vorticity equation which is $\nabla^4\psi|_{r=1} = 0$. In the present case this requires either $\nabla^2\phi|_{r=1} = 0$ or $\nabla^2\psi|_{r=1} = 0$, which in general is not satisfied and may be regarded as a deficiency of the analysis in approximating the boundary vorticity. Sketches of the streamlines for various values of the parameters are given below.

Appendix—The Stokes flow. Consider the two-dimensional flow interior to the unit circle $r = 1$, which is stirred by a rotlet at $r = c$, $\theta = 0$, $0 \leqslant c < 1$. From (9), the limit $R \to 0$ gives the equation

$$\phi_{\bar{z}\bar{z}} + i\psi_{\bar{z}\bar{z}} = f''(\bar{z}). \tag{A1}$$

Elimination of ϕ yields the biharmonic equation

$$\nabla^4\psi = 0, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{A2}$$

The boundary conditions are

$$\psi \sim \log R_1 \quad \text{as } R_1 = \left(r^2 + c^2 - 2rc\cos\theta\right)^{1/2} \to 0, \tag{A3}$$

and the conditions of no slip are

$$\psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{at } r = 1. \tag{A4}$$

The stream function is readily found to be

$$\psi = \log R_1 - \log(R_2 c) + (r^2 - 1)\left[\frac{1}{c}\frac{(r\cos\theta - \frac{1}{c})}{R_2^2} + \frac{1}{2}\right], \tag{A5}$$

and $R_2 = (r^2 + \frac{1}{c^2} - \frac{2r}{c}\cos\theta)^{1/2}$ is measured from the inverse point $(\frac{1}{c}, 0)$. With $z = re^{i\theta}$, ψ can be represented in the complex form

$$2\psi = \log(z - c) + \log(\bar{z} - c) - \log\left(z - \frac{1}{c}\right) - \log\left(\bar{z} - \frac{1}{c}\right)$$
$$-2\log c + (z\bar{z} - 1)\left[\frac{1}{cz - 1} + \frac{1}{c\bar{z} - 1} + 1\right] \tag{A6}$$

from which it follows that

$$2\psi_{z\bar{z}} = 1 - \frac{1}{(cz - 1)^2} - \frac{1}{(c\bar{z} - 1)^2} \tag{A7}$$

and

$$\phi_{z\bar{z}} + i\psi_{z\bar{z}} = \frac{i}{2} - \frac{i}{(cz - 1)^2}. \tag{A8}$$

It is noted that, in the case $c \to 0$, the solution for ψ is

$$\psi = \log r - \frac{1}{2}r^2, \tag{A9}$$

which is an exact solution of the Navier–Stokes equations with constant vorticity $\nabla^2\psi = -2$, and the streamlines are concentric circles. The vorticity on the boundary is

$$\zeta = \nabla^2\psi = \frac{2(c^4 - 1) + 8c^2(1 - c\cos\theta)}{(1 + c^2 - 2c\cos\theta)^2}. \tag{A10}$$

This expression has been discussed in [5] and, for $0 < c < \sqrt{2} - 1$, the flow consists of closed streamlines containing the rotlet and is qualitatively similar to the potential flow as shown in Fig. 1. For $c > \sqrt{2} - 1$, separation first occurs at $\theta = \pi$ and spreads symmetrically about this point in both upper and lower half-planes (see Fig. 2). The torque on the circle $r = 1$ is constant and independent of the rotlet position. As the rotlet moves towards the boundary at $r = 1$, $\theta = 0$, the shear stress on the boundary is increased and the contribution to the torque in this region exceeds the constant value on the circle. Consequently the torque changes sign which in turn produces the onset of separation at $\theta = \pi$.

Finally, it is noted that the Burgers approximation of the problem is expressed by

$$\frac{R}{r}\frac{\partial(\psi_0, \nabla^2\psi)}{\partial(r, \theta)} = \nabla^4\psi \tag{A11}$$

where, in this case, ψ_0 is the potential flow for a rotlet inside the circle $r = 1$ and is given by

$$\psi_0 = \log R_1 - \log(R_2 c). \tag{A12}$$

The boundary conditions of no slip require

$$\psi = \frac{\partial \psi}{\partial r} = 0, \quad \text{at } r = 1 \tag{A13}$$

and, as $R_1 \to 0$,

$$\psi \sim \log R_1. \tag{A14}$$

An analytic solution for this boundary value problem is virtually impossible and, in any case, vorticity is convected around the boundary which reduces the accuracy in obtaining the boundary vorticity.

Sketches of the streamlines for a rotlet interior to the unit circle

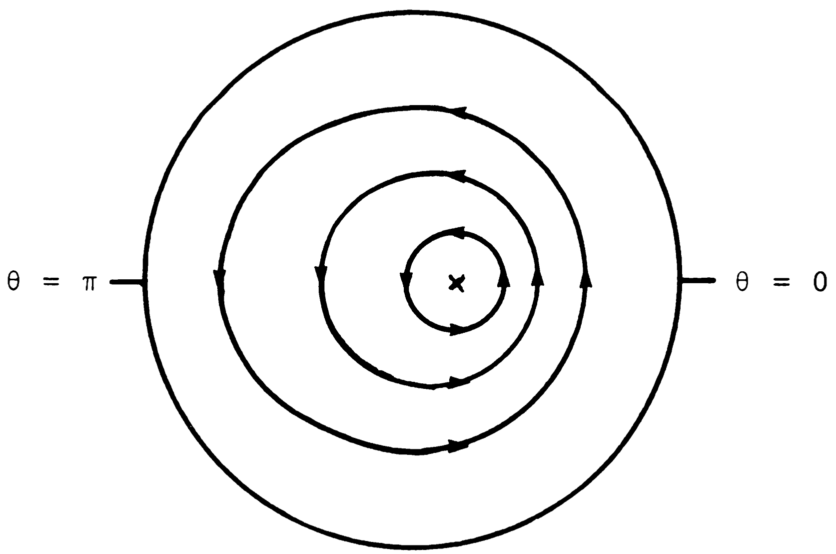

FIG. 1. The Stokes flow for $0 \leqslant c < \sqrt{2} - 1$. The streamlines are closed curves containing the rotlet and do not differ qualitatively from the irrotational flow.

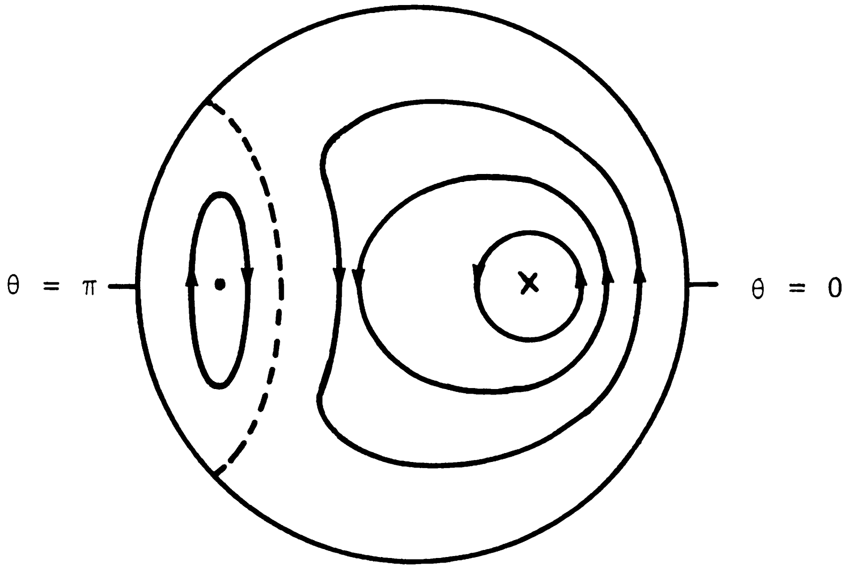

FIG. 2. The Stokes flow for $1 > c \geqslant \sqrt{2} - 1$. Separation occurs in both upper and lower half-planes.

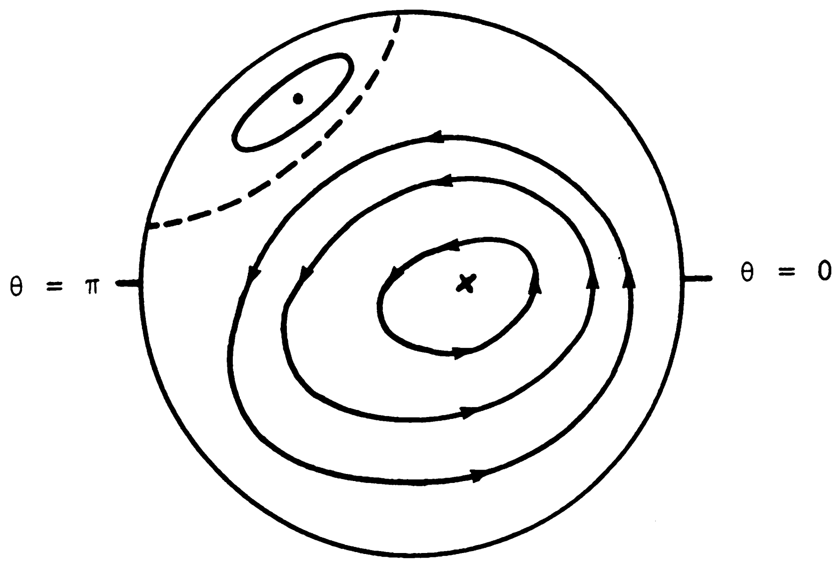

FIG. 3. Sketch of the streamlines for $R > R^*$ and $0 < c < \sqrt{2} - 1$. The separation occurs entirely in the upper half-plane.

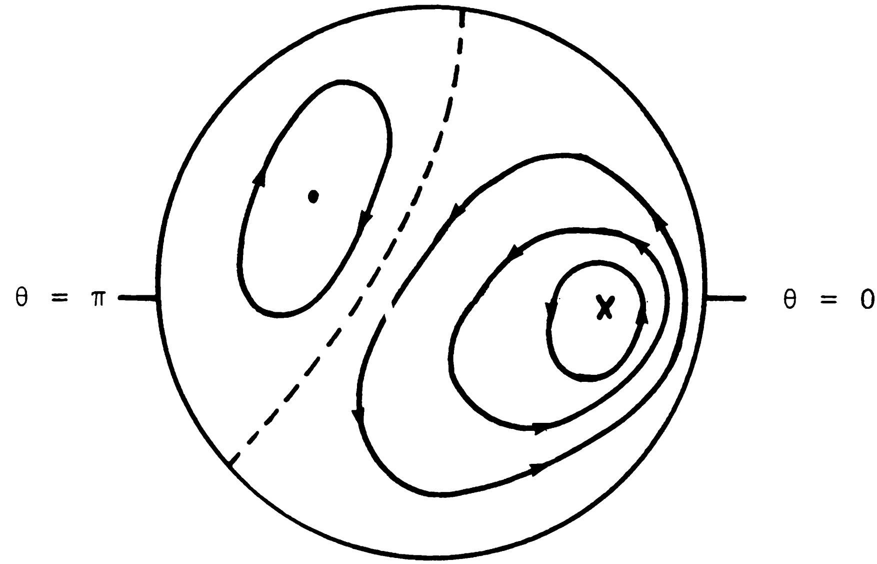

FIG. 4. The streamlines for $R > R^*$ and $c > \sqrt{2} - 1$. In this case separation occurs in the lower half-plane but this is not related to Reynolds number effects.

REFERENCES

[1] R. Legendre, *Solution plus complète du problème Blasius*, Comptes Rendus, Tom **228**, 2008–2010 (June 1949)

[2] A. F. Pillow, Private Communication, Department of Mathematics, University of Queensland, St. Lucia, Qld 4067, Australia

[3] K. B. Ranger, *The critical separation Reynolds number for streaming flow past a circular cylinder*, International Journal of Multiphase Flow **10**, No. 2, 159–171 (1984)

[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part 1, p. 373, McGraw-Hill, New York, 1953

[5] K. B. Ranger, *Eddies in two dimensional Stokes flow*, International Journal of Engineering Science **18**, 181–190 (1980)