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ON A NONLINEAR PROBLEM OF THE THEORY OF POTENTIAL FLOWS*

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1. Introduction. In [2], N. Geffen considered some nonlinear boundary-value problems for harmonic functions which are related to problems of hydrodynamics. There, in particular, the following question is discussed.

Let D be a two-dimensional simply-connected domain, the boundary ∂D of which is decomposed into two parts, ∂D_1 and ∂D_2. We are looking for a potential flow with velocity field $V = (u, v)$ in D whose absolute value $|V| = (u^2 + v^2)^{1/2}$ is a given function $f(t)$ on ∂D_1 and whose normal component V_n is a prescribed function $g(t)$ on ∂D_2 (t the arclength of the boundary).

Geffen conjectured that this problem has a solution at least for the case where

$$\int_{\partial D_2} V_n\, dt = \int_{\partial D_2} g\, dt = 0, \tag{1}$$

i.e., where there is no net flux through the boundary ∂D_2.

In the present paper we give an example which possesses no solution (in the class of functions which are square-summable on the boundary ∂D), although condition (1) is fulfilled.

2. Counterexample. We introduce the complex velocity $w = u - iv$, which is an analytic function in the domain D. Assume that D is the complex unit disk $D = \{z = x + iy: |z| < 1\}$. Further, let ∂D_1 be the lower and ∂D_2 the upper half of the boundary ∂D:

$$\partial D_1 = \{z = x + iy: |z| = 1,\ y < 0\}, \qquad \partial D_2 = \{z = x + iy: |z| = 1,\ y > 0\}.$$

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The boundary conditions we are concerned with read as follows:

$$|V|^2 = (u(t))^2 + (v(t))^2 = (f(t))^2 \quad \text{on } \partial D_1, \tag{2}$$
$$V_n = u(t)\cos t + v(t)\sin t = g(t) \quad \text{on } \partial D_2, \tag{3}$$

where t is the oriented angle between the point z of ∂D and the real axis ($0 < t < \pi$ on ∂D_2 and $\pi < t < 2\pi$ on ∂D_1).

The conditions (2) and (3) give rise to a boundary-value problem for the analytic function $w = u - iv$, a so-called Riemann–Hilbert problem. One can find a survey of the theory of linear Riemann–Hilbert problems for instance in the monographs of N. I. Muskhelishvili [5], F. D. Gakhov [1], and E. Meister [4].

To get an example which has no solution choose, for instance, $f(t) \equiv \varepsilon$, $g(t) = \sin 2t$, where ε is a sufficiently small positive constant to be specified later. Notice that g fulfils the hypothesis (1).

Assume that problem (2), (3) is solvable and $w_0 = u_0 - iv_0$ is a solution with a square-summable boundary function. Then replace the condition (2) by the linear relation

$$u(t)\cos t + v(t)\sin t = h(t) \quad \text{on } \partial D_1, \tag{4}$$

where $h(t) := u_0(t)\cos t + v_0(t)\sin t$ is the normal component of the velocity of that solution w_0. To study the properties of the solution determine the velocity field from its normal components on the whole boundary given by (3) and (4). This Riemann–Hilbert problem is linear and its index equals -1 (in the sense of [1], [5]; cf. also [4], but in this notation we have $\kappa = -2$). Thus there is one necessary and sufficient condition which ensures the existence of a solution:

$$\int_0^{\pi} g(t)\,dt + \int_{\pi}^{2\pi} h(t)\,dt = 0.$$

This means that the total flux through the boundary must be zero. Further, the solution is unique and can be given explicitly. We have

$$\begin{aligned} u(t) &= \cos t\, h_1(t) - \sin t\, Hh_1(t), \\ v(t) &= \sin t\, h_1(t) + \cos t\, Hh_1(t) \end{aligned} \tag{5}$$

on the boundary ∂D, where

$$h_1(t) := \begin{cases} g(t) \text{ if } 0 < t < \pi \\ h(t) \text{ if } \pi < t < 2\pi, \end{cases}$$

and H denotes the singular integral operator of Hilbert type:

$$Hh_1(t) = \frac{1}{2\pi}\int_0^{2\pi} h_1(s)\cot\frac{s-t}{2}\,ds.$$

From (5) we calculate the tangent component of the velocity field along the boundary:

$$V_t = -\sin t\, u(t) + \cos t\, v(t) = Hh_1(t).$$

Finally, we show that V_t cannot be small near the boundary ∂D_1, which contradicts the estimate

$$|V_t| \leqslant |V| = \varepsilon \quad \text{on} \quad \partial D_1. \tag{6}$$

We have

$$V_t(t) = Hh_1(t) = \frac{1}{2\pi}\int_0^{\pi} \sin 2s \cot\frac{s-t}{2}\, ds + \frac{1}{2\pi}\int_{\pi}^{2\pi} h(s)\cot\frac{s-t}{2}\, ds. \tag{7}$$

Since, by assumption, $|h(s)| = |V_n(s)| \leqslant \varepsilon$, on ∂D_1, the last integral is small; more precisely,

$$\left\| \frac{1}{2\pi}\int_{\pi}^{2\pi} h(s)\cot\frac{s-t}{2}\, ds \right\|_{L_2(\pi,2\pi)} \leqslant \|h(s)\|_{L_2(\pi,2\pi)} \leqslant \varepsilon\pi^{1/2}, \tag{8}$$

where $\|\cdot\|_{L_2}$ denotes the norm of a function in the space L_2 of square-summable functions. To get the estimate (8) one must take into account that the norm of H in $L_2(0,2\pi)$ is equal to one and must extend the function h to the interval $(0,2\pi)$ by zero.

From (7) and (8) we conclude that

$$\|V_t - V_t^0\|_{L_2(\pi,2\pi)} \leqslant \varepsilon\pi^{1/2} \tag{9}$$

where the "main part" V_t^0 of V_t is given by

$$V_t^0(t) = \frac{1}{2\pi}\int_0^{\pi} \sin 2s \cot\frac{s-t}{2}\, ds.$$

We check that the function V_t^0 is bounded from below on ∂D_1 by a positive constant. We have

$$V_t^0(t) = \frac{1}{4\pi}\int_0^{\pi/2} \sin s\left(\cot\frac{s-2t}{4} + \cot\frac{\pi-s-2t}{4} - \cot\frac{\pi+s-2t}{4} - \cot\frac{2\pi-s-2t}{4}\right) ds.$$

Since all the arguments of the cotangent functions belong to the interval $(-\pi,0)$ whenever $0 < s < \pi/2$ and $\pi < t < 2\pi$, we get, after a simple consideration,

$$\cot\frac{s-2t}{4} - \cot\frac{\pi+2-2t}{4} \geqslant 2\tan\frac{\pi}{8},$$

$$\cot\frac{\pi-s-2t}{4} - \cot\frac{2\pi-s-2t}{4} \geqslant 2\tan\frac{\pi}{8},$$

and thus

$$V_t^0(t) \geqslant \frac{1}{\pi}\tan\frac{\pi}{8} = c > 0 \quad \text{if } \pi < t < 2\pi. \tag{10}$$

If we choose $\varepsilon < c/2$, then (8), (9), and (10) lead to a contradiction:

$$\varepsilon\pi^{1/2} \geqslant \|V_t - V_t^0\|_{L_2(\pi,2\pi)} > \|\varepsilon - 2\varepsilon\|_{L_2(\pi,2\pi)} = \varepsilon\pi^{1/2}.$$

Hence a solution of (2), (3) cannot exist, provided that ε is sufficiently small.

The physical meaning of this counterexample is the following. If on one part of the boundary ∂D_2 there is a "significant" flux of the fluid into the interior of D, and on the other part of ∂D_2 the fluid streams out of D with equal flux, then it is impossible that the fluid is "nearly at rest" on ∂D_1.

3. Additional Remark. In [2], Geffen had reduced problem (2), (3) with $g(t) \equiv 0$ to some boundary-value problem for harmonic functions and had stated existence and uniqueness (in a sense) of the solution w. It is also possible to treat (2), (3) as a problem of Riemann–Hilbert type directly, as will be sketched here.

Following Geffen we assume that location and type of the stagnation points $z_1, \ldots, z_n$ of the flow are known a priori and factorize

$$w(z) = (z - z_1)^{\beta_1} \cdots (z - z_n)^{\beta_n} w_1(z).$$

Now, the transformation $W = \log w_1$ leads to the problem

$$\begin{aligned} \operatorname{Re} W &= f_1(t) \quad \text{on } \partial D_1, \\ \operatorname{Im} W &= g_1(t) \quad \text{on } \partial D_2, \end{aligned} \tag{11}$$

with given functions f_1 and g_1. This problem was examined by M. V. Keldysh and L. I. Sedov [3]. Moreover, they had studied the case where ∂D_1 and ∂D_2 consist of more than one connected component:

$$\partial D_1 = \partial D_{11} \cup \cdots \cup \partial D_{1m}, \qquad \partial D_2 = \partial D_{21} \cup \cdots \cup \partial D_{2m}.$$

Note that, in general, the solution W can be unbounded in a neighborhood of the points which are endpoints of both ∂D_1 and ∂D_2 if $m > 1$. Keldysh and Sedov had stated that a (unique) bounded solution exists if and only if $m - 1$ relations of the type

$$\int_{\partial D_1} f_1(s)\lambda_i(s)\, ds + \int_{\partial D_2} g_1(s)\mu_i(s)\, ds = 0, \quad i = 1, \ldots, m - 1,$$

(s the arclength on ∂D) are fulfilled. The real functions λ_i and μ_i depend on the geometry of D and on the partition of its boundary. For more details see also [1], Section 46.3.

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