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DYNAMIC FRACTURE OF A BEAM UNDER COMPRESSION*

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1. Introduction. When strong fibers are used to reinforce a weaker matrix the resulting composite may be highly anisotropic. Stress analysis in anisotropic elasticity, especially for bodies with finite boundaries and undergoing dynamic fracture, is more difficult than that for corresponding isotropic problems. Simplified models for fiber-reinforced materials have therefore been developed. The model used here is known as the idealized theory; it is described by Pipkin [1] and will be discussed in Sec. 2.

This paper concerns the dynamic fracture of a beam of idealized material with its upper and lower faces under compression; the beam is described in Sec. 3. The equation of motion is found from momentum balance in Sec. 4; since this equation is to be solved on an initially unknown interval it is put in characteristic form.

To complete the formulation of the problem a fracture criterion must be specified. The commonly used energy release rate is calculated in the context of the present problem in Sec. 5. In Sec. 6, the fracture criterion is combined with the method of characteristics to obtain a differential equation for the crack speed in a special case; this differential equation is analyzed in Sec. 7. The results are discussed in Sec. 8.

2. The idealized theory. The idealized theory is based on the assumptions that the fibers are inextensible and continuously distributed. The material contains two orthogonal families of straight, parallel fibers and the axes of a system of Cartesian coordinates (x, y) coincide with the fiber directions. Suppose that the material undergoes small deformation under plane stress conditions. Fiber inextensibility then shows that

$$u = u(y,t), \qquad v = v(x,t), \tag{1}$$

where u and v are displacement components parallel to the x and y axes respectively. The shearing behaviour is linear elastic so that the shear stress σ_{xy} is

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}). \tag{2}$$

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In Eq. (2), μ is a shear modulus; a comma followed by a subscript denotes a partial derivative.

Equations of motion for displacement components are normally obtained by using stress-strain laws to eliminate the stresses from momentum equations such as

$$\sigma_{xx},_{x} + \sigma_{xy},_{y} = \rho v,_{tt}. \tag{3}$$

In the idealized material, however, the normal stresses σ_{xx} and σ_{yy} are reactions to the inextensibility contraints and are found after the deformation is known; equations of motion are therefore found directly from momentum balance.

For bodies containing cracks in static equilibrium, it has been shown ([1], [2]) that the idealized theory correctly predicts the overall features, including stress intensity factors, of more exact solutions within anisotropic elasticity; the same correspondence should hold in dynamic problems.

3. Beam geometry. An idealized beam of length L and height $2h$ contains a crack of time-dependent length $l(t)$ along its center line (Fig. 1.) The beam is initially undeformed and contains a crack of length l_0. The origin of coordinates is at the middle of the left-hand end. The edges at $y = \pm h$ are subjected to equal and opposite tractions $T(x, t)$. The beam is clamped at $x = L$ and the beam arms are given equal and opposite displacements at $x = 0$; symmetry allows us to concentrate on the upper half of the beam.

Since the end $x = L$ is fixed, symmetry and inextensibility together imply that u and v are both zero ahead of the crack tip. In what follows, only the situation in which the crack remains straight is considered; the only displacement component to be found is therefore $v(x, t)$ in $0 < x < l(t)$.

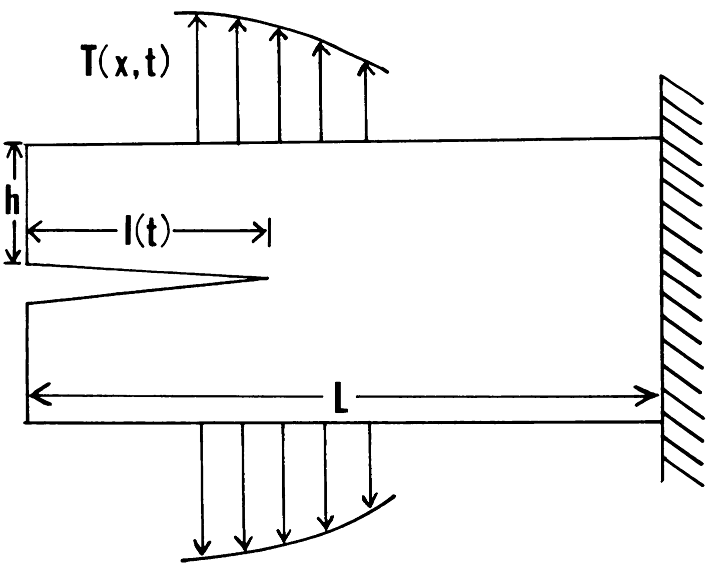

FIG. 1. Cracked beam under traction.

4. Momentum balance and characteristics. In this section the equation of motion for $v(x, t)$ is found using momentum balance. Consider the part of the beam in $(x, x + \Delta x)$ where $x < l(t)$. The tractions on this material are the shear stresses on the vertical faces and $T(x, t)$ on $y = h$. The resultant force in the y-direction is found using Eq. (2) with $u = 0$ and integrating from $y = 0$ to $y = h$. Equating rate of momentum change with resultant force yields

$$\rho h \int_x^{x+\Delta x} v,_{tt}(\xi, t)\, d\xi = \mu h\left[v,_x(x + \Delta x, t) - v,_x(x, t)\right] + \int_x^{x+\Delta x} T(\xi, t)\, d\xi. \quad (4)$$

Dividing by $\rho h \Delta x$ and letting $\Delta x \to 0$ we obtain a nonhomogeneous wave equation

$$c^2 v,_{xx} - v,_{tt} = -\frac{1}{\rho h} T(x, t), \quad 0 < x < l(t), \quad (5)$$

where $c = (\mu/\rho)^{1/2}$ is the shear wave speed. In taking the limit to obtain Eq. (5) it is assumed that $T(x, t)$ contains no point forces. At the crack tip,

$$v(l(t), t) = 0 \quad (6)$$

while initial conditions are

$$v(x, 0) = v,_t(x, 0) = 0, \quad 0 < x < L. \quad (7)$$

Equation (5) is unusual in that the applied traction $T(x, t)$ appears in a position normally occupied by a body force term. This may be regarded as another consequence of fiber inextensibility—the effect of tractions at a fiber end is felt without attenuation at all interior points on that fiber. This is in line with the results of Everstine and Pipkin [3].

Since Eq. (5) is to be solved on $(0, l(t))$ it is now written in characteristic form. The beam slope and speed are denoted respectively by

$$q(x, t) = v,_x(x, t), \qquad s(x, t) = v,_t(x, t). \quad (8)$$

Then Eq. (5) becomes

$$c^2 q,_x - s,_t = -\frac{1}{\rho h} T(x, t), \quad (9)$$

while compatibility of derivatives in (8) gives

$$q,_t - s,_x = 0. \quad (10)$$

Combining (9) and (10) we find

$$\frac{d}{dt}(cq - s) = -\frac{1}{\rho h} T(x, t) \quad \text{on} \quad \frac{dx}{dt} = c, \quad (11)$$

$$\frac{d}{dt}(cq + s) = \frac{1}{\rho h} T(x, t) \quad \text{on} \quad \frac{dx}{dt} = -c. \quad (12)$$

When (6) is differentiated with respect to t the result in the present notation is

$$q(l(t), t)\dot{l}(t) + s(l(t), t) = 0, \quad (13)$$

where a superimposed dot denotes a time-derivative.

5. Energy release rate. The equation of motion and side conditions described so far provide exactly enough information to solve a problem on a fixed interval. To compensate for the extra unknown $l(t)$ a fracture criterion must be used, to specify the conditions under which a crack will propagate. A commonly used criterion is based on the energy release rate $\mathscr{G}$, where $\mathscr{G}$ is the energy released per unit length of new crack. The simple deformation in the present case makes it possible to calculate $\mathscr{G}$ directly.

Energy balance during fracture is expressed as

$$P = \dot{T} + \dot{W} + \mathscr{F}, \tag{14}$$

where T, W, and P are respectively the total kinetic and strain energy, P is the rate of working of applied tractions and $\mathscr{F}$ is the time rate of energy flow into the crack tip; all quantities are per unit thickness and include both beam arms. The expressions for T, W, and P are

$$T = \rho h \int_0^l s^2(x,t)\,dx, \qquad W = \mu h \int_0^l q^2(x,t)\,dx \tag{15}$$

and

$$P = 2\int_0^l s(x,t)T(x,t)\,dx - 2\mu h q(0,t)s(0,t). \tag{16}$$

Also, by definition of $\mathscr{F}$ and $\mathscr{G}$,

$$\mathscr{F} = \mathscr{G}\dot{l}. \tag{17}$$

Differentiating the integrals in Eq. (15) and combining gives

$$\dot{T} + \dot{W} = \mu h \dot{l}(1 + \dot{l}^2/c^2)q^2(l,t) + 2h\int_0^l (\mu q q_{,t} + \rho s s_{,t})\,dx, \tag{18}$$

where (13) has been used. When (10) is used for $q_{,t}$ and (9) for $s_{,t}$ in the integrand in (18) it may be written as $(qs)_{,x} + (1/h)T(x,t)$. Integrating, using (13) again and recalling (16) gives

$$\dot{T} + \dot{W} = \mu h(1 - \dot{l}^2/c^2)q^2(l,t) + P. \tag{19}$$

Solving for $\mathscr{G}$ from Eqs. (14), (17), and (19) we find

$$\mathscr{G} = \mu h(1 - \dot{l}^2/c^2)q^2(l,t). \tag{20}$$

If $\mathscr{G}_c$ is the critical value of $\mathscr{G}$ required for crack propagation, i.e., $\mathscr{G}_c$ is the fracture toughness, the condition for fracture may be written as

$$c^2\mathscr{G}_c/(\mu h) = (c^2 - \dot{l}^2)q^2(l,t). \tag{21}$$

By setting $\dot{l} = 0$ in (21) the critical magnitude of the tip slope required for crack motion is

$$|q|_c = \left(\frac{\mathscr{G}_c}{\mu h}\right)^{1/2}. \tag{22}$$

6. Fracture under uniform compression. Crack motion is now discussed in the case where the initially cracked portion of the beam is free of traction on $y = \pm h$ and uniform compression of magnitude T_0 is applied for $x > l_0$; this is expressed by

$$T(x,t) = -T_0 H(x - l_0), \quad 0 < x < L, \tag{23}$$

where $T_0 > 0$ and $H(\ )$ is the unit step function. Beginning at $t = 0$ the beam arms are wedged apart at constant speed V so that

$$s(0, t) = V, \quad t > 0. \tag{24}$$

While the crack tip remains stationary a simple use of the characteristics shows that the size of the tip slope jumps by $2V/c$ after each time interval of length l_0/c; in particular,

$$q(l_0, l_0/c+) = -2V/c, \qquad s(x, l_0/c+) = V, \quad 0 < x < l_0. \tag{25}$$

Let $|q|$ in $(25)_1$ be large enough to cause crack propagation; by (22) this requires

$$V > \frac{c}{2}\left(\frac{\mathcal{G}_c}{\mu h}\right)^{1/2}. \tag{26}$$

A differential equation for $l(t)$ may be found using Fig. 2. From Fig. 2, the first expression found for $l(t)$ will be valid at most on the part AC of the tip path, i.e., until the first shear wave generated by crack motion at A overtakes the moving tip at C after reflection at $x = 0$. In this time interval, PQ, QS, and JK are typical characteristic segments and S is $(l(t), t)$.

At $t = l_0/c$, $q_P = -V/c$ and $s_P = V$ so that using (12) and (23) on PQ gives $q_Q = -V/c$. Then, using (11) and (23) on QS gives

$$(cq - s)_S = \left(\frac{T_0}{\rho h}\right)(t - t_R) - 2V. \tag{27}$$

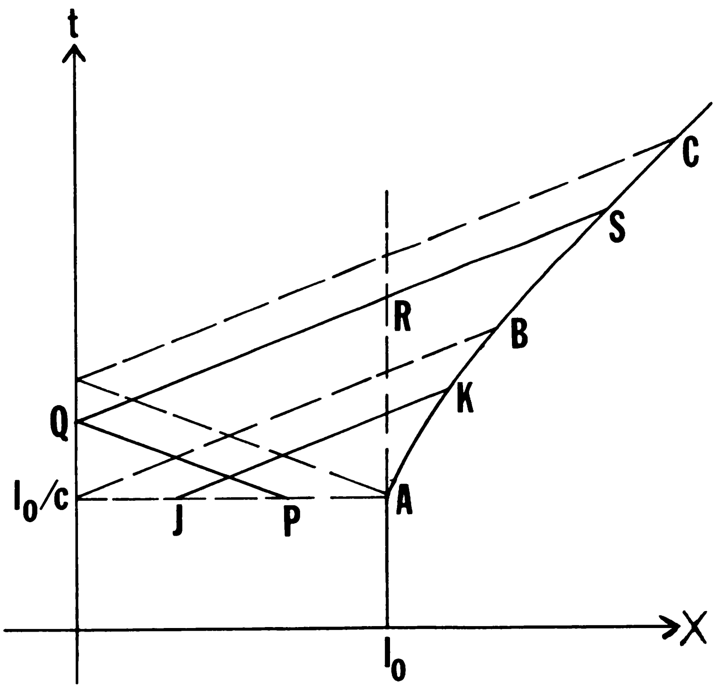

FIG. 2. Characteristic diagram for the crack speed.

Since R and S lie on a line with $dx/dt = c$, $t - t_R = (l(t) - l_0)/c$; using this in (27) together with (13) and dropping the subscript S gives

$$q(l, t) = \left[\left(\frac{T_0}{\rho ch}\right)(l(t) - l_0) - 2V\right](c + \dot{l})^{-1}. \tag{28}$$

From Fig. 2, Eq. (28) holds, provided $\dot{l} > 0$, until $t = t_C$, i.e., until $l(t) = ct - 2l_0$. (The argument above shows only that (28) holds on BC; use of Eq. (11) on JK shows that it holds on AB.) Squaring in (28) and combining with (21) yields

$$c^2\frac{\mathcal{G}_c}{\mu h} = \left[\left(\frac{T_0}{\rho ch}\right)(l(t) - l_0) - 2V\right](c - \dot{l})(c + \dot{l})^{-1}. \tag{29}$$

While solving for $\dot{l}$ in (29), make the substitutions

$$L(t) = l(t) - l_0 - (2\rho ch/T_0), \qquad k = (\mu h\mathcal{G}_c)^{1/2}/T_0. \tag{30}$$

With this notation, (29) gives

$$dL/dt = c(L^2 - k^2)/(L^2 + k^2), \tag{31}$$

where, by Eq. $(30)_1$, $dl/dt = dL/dt$. The behaviour of $\dot{l}$ may be extracted from (31).

7. Analysis of the crack speed. Letting $l(t) = l_0$ in $(30)_1$, Eq. (31) gives the initial speed $\dot{l}(l_0/c +)$ as U, where

$$U = c\left(1 - \frac{\mathcal{G}_c}{4\rho hV^2}\right)\Big/\left(1 + \frac{\mathcal{G}_c}{4\rho hV^2}\right). \tag{32}$$

Note that (26) guarantees that $U > 0$; also, U is independent of T_0. If T_0 is set equal to zero in (29) the same speed U is obtained but in that case it holds on all of AC in Fig. 2. The model therefore predicts that the crack speed will be piecewise constant if the beam is traction free on $y = \pm h$. In connection with the last result see the review by Burridge and Keller [4].

Any realistic model should predict that a sufficiently large compression will arrest fracture. From (31), $\dot{l} = 0$ iff $L^2 = k^2$ or $L = -k$, where the minus sign is chosen as it corresponds to the earliest stopping time; from (30) this gives

$$l(t) = l_0 + \left[2\rho chV - (\mu h\mathcal{G}_c)^{1/2}\right]/T_0. \tag{33}$$

The tip acceleration is found by differentiating (31); the result is

$$d^2l/dt^2 = \left[4ck^2/(L^2 + k^2)^2\right]L\dot{l}. \tag{34}$$

Equation (34) shows that the tip decelerates as long as $L < 0$, i.e., until $l(t) = l_0 + 2\rho chV/T_0$, which occurs later than (33). Combining (33) and (34) we have that, given T_0 large enough, the crack decelerates from its initial speed U to rest, all within the part AC of the tip path in Fig. 2. (If and when $\dot{l} = 0$, however, the analysis based on Fig. 2 becomes invalid and the problem must be reexamined.)

Finally, (31) can actually be integrated to give $l(t)$ implicitly in terms of t. Inverting both sides of (31) gives

$$t - l_0/c = (1/c)\int_{L(l_0/c)}^{L(t)} (\xi^2 + k^2)(\xi^2 - k^2)^{-1}\,d\xi. \tag{35}$$

On integrating and simplifying using (30) we obtain

$$t = (1/c)\left[l(t) + k\ln|K(L(t) - k)/(L(t) + k)|\right], \tag{36}$$

where

$$K = \left[(\mu h\mathcal{G}_c)^{1/2} - 2\rho chV\right]/\left[(\mu h\mathcal{G}_c)^{1/2} + 2\rho chV\right]. \tag{37}$$

8. Discussion. The solution of the present problem in orthotropic elasticity involves two coupled wave equations complicated further by the moving crack. The idealized theory reduces the problem to a single wave equation which, in conjunction with the fracture criterion, leads to an ordinary differential equation for the crack speed.

Other fracture criteria may be used with the model. For example, the discontinuous shear across the "singular fiber" at $x = l(t)$ is balanced by a finite concentrated force in that fiber. The magnitude of that force at the crack tip is calculated in [5] as $G(t) = \mu h(1 - \dot{l}^2/c^2)q(l, t)$. If fracture is regarded as successive breaking of fibers this suggests a critical force criterion, with fracture occurring while the tip force is maintained at a critical level G_c. This criterion may be used to eliminate $q(l, t)$ from (28) and the analysis parallels that for the energy criterion.

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