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LYAPUNOV EXPONENTS AND SUBSPACE EVOLUTION*

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Abstract. Differential equations are derived which describe the evolution of area tensors and normals associated with the subspaces of an n-dimensional Euclidean phase space, E_n. These provide computational methods for determining the Lyapunov exponents of continuous dynamical systems.

1. Introduction. In [7] and [14] a method is given for the computation of the Lyapunov exponents. A subspace, E_m, of n-dimensional Euclidean phase space, E_n, is defined by a set of m orthogonal vectors which are evolved. These are repeatedly orthonormalized to prevent their collapse into a set of lower dimension.

In the following, subspaces are defined by the area tensor and its dual, the normal tensor. Sets of o.d.e. are derived which describe the evolution of these quantities, some of which have three-dimensional counterparts in the theory of continuum kinematics, [4], [5], and [9]. The Lyapunov exponents may be expressed as limits of functions of m-dimensional area elements, [7, 8]. Evaluation of the Lyapunov exponents was effected through the integration of the derived o.d.e. for area elements of dimension r, $1 < r < n$, along a trajectory. A numerical study indicates the utility of the derived algorithms.

Phase space. Consider a continuous dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{x}$ are spatial curvilinear coordinates in an n-dimensional Euclidean phase space, E_n. Under general conditions the solution of (1), [1],

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \tag{2}$$

is one-to-one, continuous together with its inverse and continuously differentiable with respect to $\mathbf{X}$ and t. $\mathbf{X}$ are material coordinates.

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A subspace, E_m, of dimension $m \leqslant n$ may be defined by the parametric equations $\mathbf{X} = \mathbf{X}(\mathbf{u})$ in material coordinates and, after the deformation (2), by $\mathbf{x} = \mathbf{x}(\mathbf{u})$ in spatial coordinates. $\mathbf{u}$ has m components. The corresponding area tensors $dA^{I_1 \cdots I_m}$ and $da^{i_1 \cdots i_m}$ are given by [2], [3],

$$da^{i_1 \cdots i_m} = \frac{\partial x^{i_1}}{\partial u_{j_1}} \cdots \frac{\partial x^{i_m}}{\partial u_{j_m}} du_{j_1} \wedge \cdots \wedge du_{j_m}, \tag{3}$$

and

$$dA^{I_1 \cdots I_m} = \frac{\partial X^{I_1}}{\partial u_{j_1}} \cdots \frac{\partial X^{I_m}}{\partial u_{j_m}} du_{j_1} \wedge \cdots \wedge du_{j_m}. \tag{4}$$

The summation convention is assumed to hold, $j_r = 1, m$. The range of all other indices is $1, n$.

The area element is given by, [2],

$$\left(da_{(m)}\right)^2 = (1/m!)\, da^{i_1 \cdots i_m} da_{i_1 \cdots i_m}. \tag{5}$$

It follows from (5) that $(1/da_{(m)})da^{i_1 \cdots i_m}$ is an area tensor of constant magnitude. The material derivative, D/Dt, [4], of (5) may be expressed as

$$\frac{D}{Dt} \ln da_{(m)} = (1/m!)\left[\frac{1}{da_{(m)}} \frac{D}{Dt}\left(da^{i_1 \cdots i_m}\right) \frac{1}{da_{(m)}} da_{i_1 \cdots i_m}\right] \tag{6}$$

where $Dg_{ij}/Dt = 0$. Since $\partial x^i/\partial u_j = (\partial x^i/\partial X^I)(\partial X^I/\partial u_j)$ it follows from (3) and (4) that, [4],

$$da^{i_1 \cdots i_m} = x^{i_1}_{,I_1} \cdots x^{i_m}_{,I_m}\, dA^{I_1 \cdots I_m} \tag{7}$$

where $(\ )_{,I} \equiv \partial(\ )/\partial X_I$ and $(\ )_{,i} \equiv \partial(\ )/\partial x_i$. Noting that $D(dA^{I_1 \cdots I_m})/Dt = 0$ and $D(x^i_{,I})/Dt = v^i_{;r} x^r_{,I}$, [5], the material derivative of (7) may be expressed as

$$\frac{1}{da_{(m)}} \frac{D}{Dt} da^{i_1 \cdots i_m} = v^{i_1}{}_{;r} \frac{1}{da_{(m)}} da^{r i_2 \cdots i_m} + v^{i_2}{}_{;r} \frac{1}{da_{(m)}} da^{i_1 r \cdots i_m} + \cdots + v^{i_m}{}_{;r} \frac{1}{da_{(m)}} da^{i_1 i_2 \cdots i_{m-1} r} \tag{8}$$

where $v^i \equiv Dx^i/Dt$ and $(\ )_{;r} \equiv$ covariant derivative of ( ) with respect to x_r. The material derivative of the area tensor of constant magnitude may be expanded to give

$$\frac{D}{Dt}\left(\frac{1}{da_{(m)}} da^{i_1 \cdots i_m}\right) = \frac{1}{da_{(m)}} \frac{D}{Dt} da^{i_1 \cdots i_m} - \frac{1}{da_{(m)}} \frac{D}{Dt}\left(da_{(m)}\right) \frac{1}{da_{(m)}} da^{i_1 \cdots i_m} \tag{9}$$

Alternatively the area tensor $da^{i_1 \cdots i_m}$ determines its dual, [2], the normal tensor by

$$n_{r_1 \cdots r_{n-m}} = (1/m!)\, \varepsilon_{s_1 \cdots s_m r_1 \cdots r_{n-m}} da^{s_1 \cdots s_m} \tag{10}$$

where $\varepsilon_{i_1 \cdots i_m}$ is the alternating tensor. It follows from (10) that

$$da^{s_1 \cdots s_m} = (1/(n-m)!)\, \varepsilon^{s_1 \cdots s_m r_1 \cdots r_{n-m}} n_{r_1 \cdots r_{n-m}}. \tag{11}$$

Equation (11) may be used to express previously defined quantities in terms of the normal tensor. Equation (5) becomes

$$\left(da_{(m)}\right)^2 = (1/(n-m)!)n^{r_1 \cdots r_{n-m}} n_{r_1 \cdots r_{n-m}}. \tag{12}$$

Equation (12) implies that $n^{r_1 \cdots r_{n-m}}/da_{(m)}$ is a tensor of constant magnitude. Equation (6) becomes

$$\frac{D}{Dt}\ln da_{(m)} = (1/(n-m)!)\frac{1}{da_{(m)}}n^{q_1 \cdots q_{n-m}}\frac{1}{da_{(m)}}n_{r_1 \cdots r_{n-m}}v^k{}_{;r}\delta^{r r_1 \cdots r_{n-m}}_{k q_1 \cdots q_{n-m}} \tag{13}$$

where $\delta^{i \cdots j}_{k \cdots 1}$ is the generalized Kronecker delta, [6]. Equating $n - m = 1$ in (13) yields

$$\frac{D}{Dt}\ln da_{(m)} = v^k{}_{;k} - v_{k;r}\frac{1}{da_{(m)}}n^k\frac{1}{da_{(m)}}n^r \tag{14}$$

for the hypersurface. Substituting (11) into (8) gives

$$\frac{1}{da_{(m)}}\frac{D}{Dt}n_{q_1 \cdots q_{n-m}} = (1/(n-m)!)\delta^{r r_1 \cdots r_{n-m}}_{s q_1 \cdots q_{n-m}}v^s{}_{;r}\frac{1}{da_{(m)}}n_{r_1 \cdots r_{n-m}}. \tag{15}$$

Equation (9) becomes

$$\frac{D}{Dt}\left(\frac{1}{da_{(m)}}n^{q_1 \cdots q_{n-m}}\right) = \frac{1}{da_{(m)}}\frac{D}{Dt}n^{q_1 \cdots q_{n-m}} - \frac{1}{da_{(m)}}\frac{D}{Dt}\left(da_{(m)}\right)\frac{1}{da_{(m)}}n^{q_1 \cdots q_{n-m}}. \tag{16}$$

Lyapunov exponents. Let $da_{(m)}$ and $dA_{(m)}$ be spatial and material area elements of E_m respectively. In the literature these may also be referred to as volume elements. $E_m \subset E_n$. Let L.E. $\equiv$ Lyapunov exponent. The m-dimensional L.E. associated with a trajectory originating at $\mathbf{X}$ is defined as [7], [8],

$$\chi(\mathbf{X}, E_m) \equiv \lim_{t \to \infty}\left(\ln\left(da_{(m)}/dA_{(m)}\right)\right)/t. \tag{17}$$

The one-dimensional L.E. for the arc, ds, with tangent, $\mathbf{n}$, associated with a trajectory having initial conditions respectively of dS, $\mathbf{N}$, $\mathbf{X}$ is defined as, [7],

$$\chi(\mathbf{X}, \mathbf{n}) \equiv \lim_{t \to \infty}(\ln \lambda_{\mathbf{n}})/t \tag{18}$$

where $\lambda_{\mathbf{n}} \equiv ds/dS$, $(ds)^2 = g_{ij}dx^i dx^j$ and $(dS)^2 = G_{IJ}dX^I dX^J$. It is shown in [7], [8] that the one-dimensional L.E. for the arc takes at most n different values. They may therefore be ordered, $\chi_1 \geqslant \chi_2 \geqslant \cdots \geqslant \chi_n$. It follows, [7], [8] that for $E_m \subset E_n$,

$$\chi(\mathbf{X}, E_m) = \sum_{i=1}^{m}\chi_i(\mathbf{X}, \mathbf{n}_i). \tag{19}$$

The material derivative $D(\ )/Dt = d(\ )/dt$ along a given trajectory since $\mathbf{X}$, the initial conditions, are by definition constant in the differentiation, [9]. It follows that the o.d.e. (6, 8, 9, 13, 15, 16) describe the evolution of the area element, area tensor, and normal tensor along a trajectory. It is known, [7, 8], that if $\mathbf{N}$ and therefore dS is chosen at random in (18) then all corresponding $\mathbf{n}$ will evolve into the same vector. The $\mathbf{n}$ vectors satisfy the relationship

$$\chi(\mathbf{X}, \mathbf{n}) = \chi_1(\mathbf{X}). \tag{20}$$

A similar result applies to area and normal tensors. If the initial values of the components of the area tensor of constant magnitude, $da_{i_1 \cdots i_m}/da_{(m)}$, are chosen at random then all corresponding quantities will evolve into the same area tensor of constant magnitude. This result is also satisfied by the normal tensor of constant magnitude. The corresponding area elements, $da_{(m)}$, satisfy (17).

Computation of L.E. The Lorenz, Rossler, and Rossler Hyperchaos equations are subsequently used as test cases:
Lorenz equations, [10],

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) & a &= 16.0, \\ \dot{x}_2 &= x_1(b - x_3) - x_2 & b &= 45.92, \\ \dot{x}_3 &= x_1 x_2 - c x_3 & c &= 4.0; \end{aligned} \tag{21}$$

Rossler equations, [11],

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3) & a &= 0.15, \\ \dot{x}_2 &= x_1 + a x_2 & b &= 0.20, \\ \dot{x}_3 &= b + x_3(x_1 - c) & c &= 10.0; \end{aligned} \tag{22}$$

Rossler Hyperchaos equations, [12],

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3) & a &= 0.25, \\ \dot{x}_2 &= x_1 + a x_2 + x_4 & b &= 3.0, \\ \dot{x}_3 &= b + x_1 x_3 & c &= 0.05, \\ \dot{x}_4 &= c x_4 - d x_3 & d &= 0.50. \end{aligned} \tag{23}$$

For the given sets of parameters, (21, 22, 23) possess chaotic solutions.

The evolution of the area tensor of constant magnitude was computed from (9) in which the first term on the right-hand side is given by (8) and $(1/da)D(da)/Dt$, in the second term, by (6). Equations (6), (8), and (9) were solved simultaneously along a trajectory. Alternatively (6) may be substituted into (9) and the result solved simultaneously with (8) to give an area tensor of constant magnitude. Equation (6) may be integrated numerically to yield $\ln da_{(m)}$. The limit, (17), may then be found. $dA_{(m)} = 1$ in all subsequent computations and therefore $\ln dA_{(m)} = 0$ in (17). Similarly (13), (15), (16) determine the evolution of the normal tensor of constant magnitude, give $\ln da_{(m)}$ and through (17) yield $\chi(\mathbf{X}, E_m)$.

The sum of the first two L.E. for (21), (22), and (23) were found by evolving the E_2 area tensor. After 50,000 iterations converged values were 2.16, 0.128, and 0.19 respectively. Reference [13] reports values of 2.16, 0.13, and 0.19 respectively. The sum of the first three L.E. for (23) was found by evolving the E_3 area tensor. After 90,000 iterations the converged value was 0.19 which agrees with the result reported in [13]. In this case $\chi_3 = 0$. The evolution of the normal to the hypersurface, (14), (15), and (16) for (21) and (22) gave 2.15 and 0.126 for the sum of the first two L.E. after 50,000 iterations. These values are in close agreement with those found above through evolution of the E_2 tensor.

Conclusions. Numerical tests of the algorithms developed here, for the computation of the Lyapunov exponents, demonstrate their convergence to be at least as rapid as that of the method given in [7], [14] and discussed in [13]. The representation of subspaces by area and normal tensors and the form of the resulting differential equations describing their evolution have three-dimensional parallels in continuum kinematics, [4], [5], and [9].

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