QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVI, NUMBER 1
MARCH 1988, PAGES 65-75

AN UNPERIODIC CONCENTRATED SONIC PULSE*

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Abstract. The purpose of this note is to define an unperiodic source which gives a highly concentrated pulse of scalar radiation. The source involves four constants, and concentration is obtained by giving small values to dimensionless combinations of those constants. The field is axially symmetric, and there is a focal point which travels along the x-axis with the speed of propagation; the focal field remains constant for some time, decaying after that. The method is that of the retarded potential, but presented in terms of equivalent Minkowskian geometry of space-time, the speed of propagation playing the part of the speed of light in relativity. The obvious application is to sonic communication in water, but no attempt is made to interpret the source in physical terms.

1. Introduction. We are concerned with the equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{tt} = -4\pi g(x, y, z, t), \tag{1.1}$$

where g is a given source-density in space-time. Suppose $g = 0$ outside the range $0 < t < t_1$. If we take the Cauchy conditions $\phi = \phi_t = 0$ for $t = 0$, then (1.1) has a unique solution, satisfying the wave equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{tt} = 0 \tag{1.2}$$

for $t > t_1$. This solution is given by the retarded-potential formula [1].

In choosing the function g, I have been guided by the Minkowskian geometry of space-time, and justify that choice by showing that, if certain constants are suitably chosen, (1.1) does in fact give a field with the desired pulselike character. As various notations are used in Minkowskian geometry, I shall now describe the one I shall use.

2. The Minkowskian approach. The metric form is

$$dx^2 + dy^2 + dz^2 - dt^2. \tag{2.1}$$

The scalar product of two 4-vectors is

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 - A_4 B_4, \tag{2.2}$$

in terms of their components. A 4-vector points into the future or the past according as its fourth component is positive or negative. A null vector $\mathbf{N}$ satisfies $\mathbf{N} \cdot \mathbf{N} = 0$.

*Received December 22, 1986.

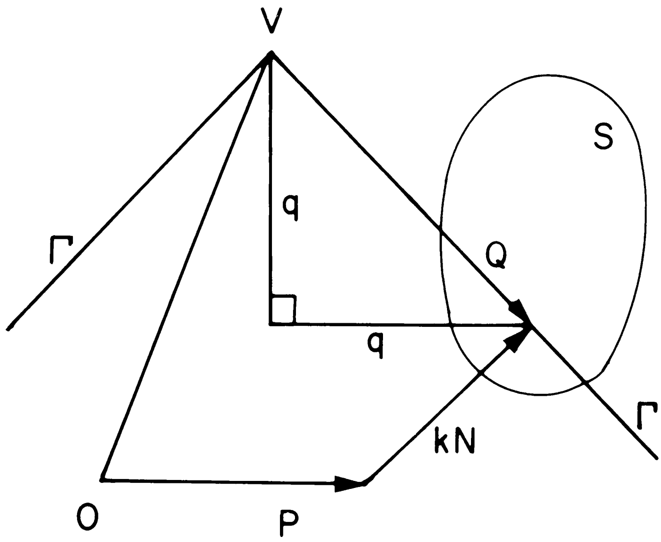

FIG. 1. Projection on plane showing the origin O, a 4-vector $\mathbf{V}$ drawn into the future, the null cone Γ having $\mathbf{V}$ for vertex, and a null vector $\mathbf{Q}$ drawn into the past from $\mathbf{V}$, such that $\mathbf{V}+\mathbf{Q}$ lies in the source S. Note the alternative paths, $\mathbf{V}+\mathbf{Q}=\mathbf{P}+k\mathbf{N}$.

Any event (or point) in space-time may be specified by giving its position 4-vector relative to the origin.

Any event is the vertex of two null cones composed of null lines. One cone goes into the future, the other into the past. I shall use only the latter. The position vector of a general event on the null cone Γ with vertex $\mathbf{V}$ may be written $\mathbf{V}+\mathbf{Q}$ where (see Fig. 1)

$$\mathbf{Q}\cdot\mathbf{Q}=0, \qquad Q_4<0. \tag{2.3}$$

It is convenient to define the positive number q as follows:

$$q=-Q_4=(Q_1^2+Q_2^2+Q_3^2)^{1/2}. \tag{2.4}$$

It has two important properties: q is the spatial distance between the events $\mathbf{V}$ and $\mathbf{V}+\mathbf{Q}$; q is the retardation of $\mathbf{V}$ relative to $\mathbf{V}+\mathbf{Q}$.

We shall need to integrate over a null cone, and this needs care. A 4-dimensional element of space-time has a 4-volume $dx\,dy\,dz\,dt$, and in general a 3-space immersed in space-time has a 3-volume. But although a null cone is a 3-space, its elements of 3-volume vanish, and one uses instead an element of 2-content defined as follows [2].

On the null cone considered above, take an element and project it on the hyperplane $t=0$. This gives an element of 3-volume which may be written $dQ_1\,dQ_2\,dQ_3$ or briefly $d_3\mathbf{Q}$. The corresponding element of 2-content on the null cone is defined as

$$d\omega=d_3\mathbf{Q}/q. \tag{2.5}$$

It is in fact Lorentz-invariant but that does not concern us here since we shall not use Lorentz transformations.

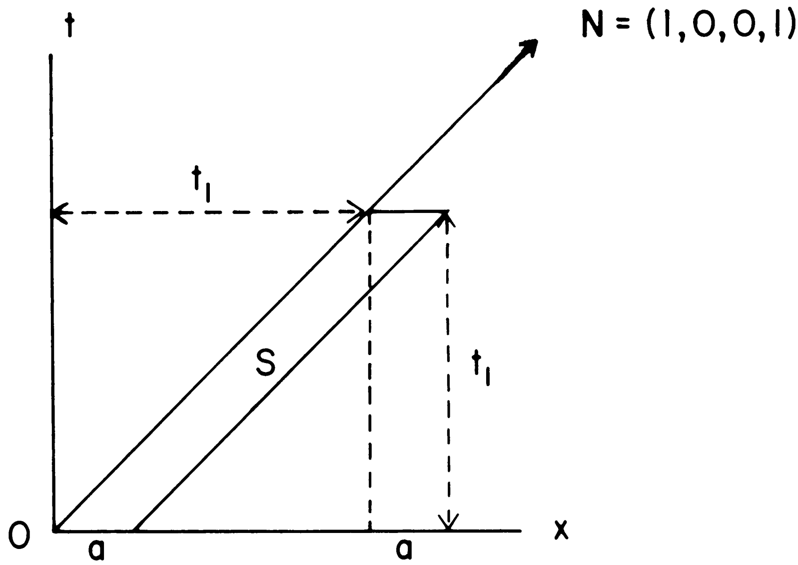

FIG. 2. Section of the source S by the (x, t)-plane.

Let us now return to the equation (1.1). Let $\mathbf{V}$ be the position 4-vector of any event, with V_4 positive. To find the field $\phi(\mathbf{V})$, draw from $\mathbf{V}$ the null cone Γ into the past, and write as earlier $\mathbf{V}+\mathbf{Q}$ for the general event on Γ. Then

$$\phi(\mathbf{V}) = \int_\Gamma g(\mathbf{V}+\mathbf{Q})\,d\omega = \int_\Gamma g(\mathbf{V}+\mathbf{Q})\,d_3\mathbf{Q}/q. \tag{2.6}$$

This may be proved directly [3], but it will be recognized as the retarded-potential formula slightly disguised; the substitution of $\mathbf{V}+\mathbf{Q}$ for $\mathbf{V}$ means retardation, and the denominator q stands for the customary r.

3. The source S**.** In the hyperplane $t=0$ take a circular disc D of thickness a and radius b; its interior satisfies

$$t=0, \qquad 0<x<a, \qquad y^2+z^2<b^2. \tag{3.1}$$

Let $\mathbf{N}$ be the null vector $(1, 0, 0, 1)$. Through the points of D draw null vectors $k\mathbf{N}$ with $0<k<t_1$. This generates a 4-dimensional domain in space-time: the source S is defined by specifying that $g(x, y, z, t)$ is a constant (g) in this domain and zero outside it. Note that the source is thus specified by four constants:

$$a, b, t_1, g. \tag{3.2}$$

The first three are dimensionally $[L]=[T]$; if we regard the field ϕ as dimensionless, then $[g]=[L^{-2}]=[T^{-2}]$.

If we view the source in space-time, it appears as what may be called a null-fibered skew cylinder (Fig. 2). Kinematically, we may think of it as the history of a piston D, suddenly created at $t=0$ and moving with unit speed along the x-axis until $t=t_1$, when it disappears (Fig. 3).

For the field (2.6) now gives

$$\phi(\mathbf{V}) = g\int_{(S,\Gamma)} d_3\mathbf{Q}/q, \tag{3.3}$$

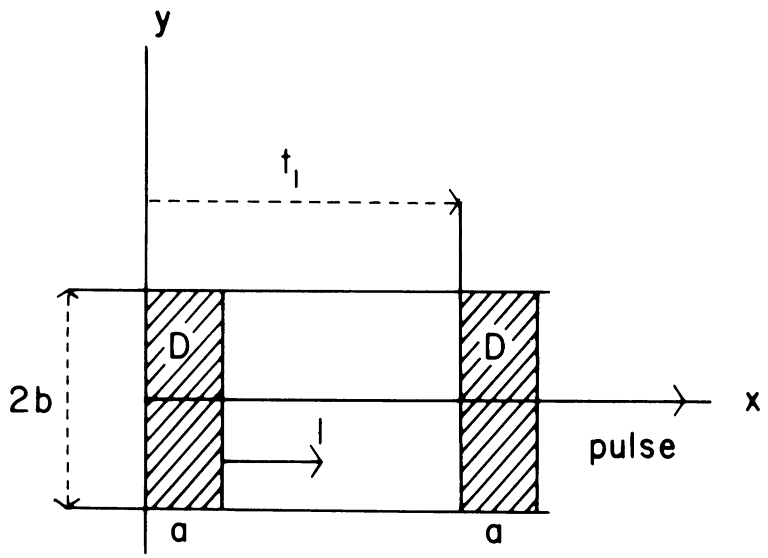

FIG. 3. Kinematical picture of the source S. The disc is created at $t = 0$, travels to the right with unit speed through a distance t_1, and then disappears, generating a pulse which travels out to the right.

the integral being taken over that part of the null cone which lies in the source S.

On account of the abruptness with which g vanishes on passing out of S, (3.3) is actually only a weak solution of the wave equation. The modification required to obtain a strong solution (in which second derivatives exist) is discussed in Sec. 10.

4. X-space: the disc D and the lens L. The integral (3.3) is to be taken over that portion of the null cone Γ which lies in the source S. There are two tricks in what follows. The first trick is to change the domain of integration into a portion of $t = 0$ by projection along the null vector $\mathbf{N}$.

For notational reasons let us denote by $\mathbf{P}$ the position vector of any point in $t = 0$, so that $P_4 = 0$. The projection in question is indicated in Fig. 1. The closure condition for a certain skew quadrilateral gives

$$\mathbf{V} + \mathbf{Q} = \mathbf{P} + k\mathbf{N}, \tag{4.1}$$

where k is some scalar. Hence

$$\begin{aligned} Q_1 &= P_1 + k - V_1, \\ Q_2 &= P_2 - V_2, \\ Q_3 &= P_3 - V_3, \\ Q_4 &= k - T, \end{aligned} \tag{4.2}$$

where I have put

$$V_4 = T, \tag{4.3}$$

the time at which the field is observed. By (2.4) $Q_4 = -q$ and so

$$k = T - q. \tag{4.4}$$

Hence

$$\begin{aligned} P_1 &= Q_1 + q - T + V_1, \\ P_2 &= Q_2 + V_2, \\ P_3 &= Q_3 + V_3. \end{aligned} \tag{4.5}$$

In computing the integral (3.3) the 4-vector $\mathbf{V}$ is fixed—it is the event of observation. Thus (4.5) allows us to pass easily from an integral in the $\mathbf{Q}$'s to one in the $\mathbf{P}$'s. Since $\partial q/\partial Q_1 = Q_1/q$, the Jacobian is

$$J = \partial \mathbf{P}/\partial \mathbf{Q} = (Q_1 + q)/q, \tag{4.6}$$

and so

$$d_3\mathbf{P} = J d_3\mathbf{Q} = (P_1 + T - V_1) d_3\mathbf{Q}/q. \tag{4.7}$$

Thus (3.3) gives

$$\phi(\mathbf{V}) = g \int (P_1 + T - V_1)^{-1} d_3\mathbf{P}, \tag{4.8}$$

taken over that portion of $t = 0$ into which S projects, using $\mathbf{N}$ for projection. That is precisely the disc D of (3.1), now reading in present notation

$$0 < P_1 < a, \qquad P_2^2 + P_3^2 < b^2. \tag{4.9}$$

The second trick is to define $\mathbf{X}$-coordinates by

$$\begin{aligned} X_1 &= P_1 + T - V_1 = Q_1 + q, \\ X_2 &= P_2 - V_2 = Q_2, \\ X_3 &= P_3 - V_3 = Q_3, \end{aligned} \tag{4.10}$$

and regard them as Cartesian coordinates in a 3-space. Now (4.8) may be written

$$\phi(\mathbf{V}) = g\mathbf{I}, \qquad \mathbf{I} = \int d_3\mathbf{X}/X_1 \tag{4.11}$$

taken over the appropriate domain in $\mathbf{X}$-space.

To find that domain, we have to use (4.9) and they give

$$0 < X_1 - T + V_1 < a, \qquad (X_2 + V_2)^2 + (X_3 + V_3)^2 < b^2. \tag{4.12}$$

These inequalities confine the $\mathbf{X}$-point to a disc (which we may call D without confusion). It is a disc of thickness a and radius b, and its hub (i.e., the centre of its left-hand face) is at $\mathbf{X} = \mathbf{H}$ where

$$H_1 = T - V_1, \qquad H_2 = -V_2, \qquad H_3 = -V_3, \tag{4.13}$$

so that D is completely determined by the observation-event.

But there is a further restriction on the $\mathbf{X}$-point, due to the disappearance of the source at $t = t_1$. This means that $V_4 + Q_4$ has the range $(0, t_1)$; equivalently q has the range $(T - t_1, T)$. From the definition of q in (2.4), (4.10) give the algebraic relationship

$$X_1^2 - 2qX_1 + X_2^2 + X_3^2 = 0. \tag{4.14}$$

Thus, as q covers its range, the X-point fills a *lens* L bounded by two spheres

$$\begin{gathered} X_1^2 - 2TX_1 + X_2^2 + X_3^2 = 0, \\ X_1^2 - 2T'X_1 + X_2^2 + X_3^2 = 0, \\ T' = T - t_1. \end{gathered} \tag{4.15}$$

Hence we have the following

THEOREM: Given a source S as described in Sec. 3, involving the four constants a, b, t_1, g, the field $\phi(\mathbf{V})$ at the event $\mathbf{V}$ (where $V_4 = T$) is given by

$$\phi(\mathbf{V}) = g\mathbf{I}(\mathbf{V}), \qquad \mathbf{I}(\mathbf{V}) = \int_{(D,L)} d_3X/X_1, \tag{4.16}$$

where (D, L) indicates the domain in $\mathbf{X}$-space common to the disc D with hub as in (4.13) and the lens L bounded by the spheres (4.15).

Note that D is determined by the three variables $(T - V_1, V_2, V_3)$; L is determined by T alone.

5. The X-diagram and the kinematical picture. $\mathbf{X}$-space is 3-dimensional. For working purposes we use the plane $0X_1X_2$ (Fig. 4). Although the hub H of the disc D will not in general lie in that plane, we know that the field has rotational symmetry round the X_1-axis, and so, to calculate integrals as in (4.16), it is permissible to take H in the plane of the diagram. The disc D then appears as a rectangle: two cases are shown. F indicates the *focal disc* with H at the origin, so that the position and time of the observation are connected by

$$V_1 = T, \qquad V_2 = 0, \qquad V_3 = 0. \tag{5.1}$$

The lens L appears as the domain between the two circles Σ, Σ' which touch the X_2-axis at O; their radii are T and $T' = T - t_1$.

Two facts are important. The domain common to D and L is essentially 3-dimensional and cannot be read off from this plane diagram, which merely serves as a guide. Second, the diagram is merely a snapshot of a moving picture. The co-ordinates $\mathbf{X}$ are in a sense comoving: the disc D is fixed for an observer who moves with unit speed parallel to the x-axis (the axis of the source). But as T increases, the lens L changes. Fig. 4 shows the case where both the bounding spheres cut the right-hand face of the focal disc F. This very interesting case will be discussed later.

Although the $\mathbf{X}$-diagram contains all the information required, it is not easy to see what it means physically. To get a more satisfactory view, consider Fig. 5. Here we have the plane OV_1V_2 (or Oxy) in physical space, viewed at arbitrary time T. The source is indicated by the segment Ot_1, on the V_1 axis, about which the field has rotational symmetry. F is the *focus*, satisfying (5.1); it is the point reached at time T if we start from O at $T = 0$. The plane FP is the *focal plane*. There is no field to the right of FP since no disturbance can travel faster than unit speed.

In order to justify the claim that the source gives a compact pulse, we need to examine the field on and behind the focal plane FP; the spread of the field on FP may be called *lateral spread* and the field behind the *wake*.

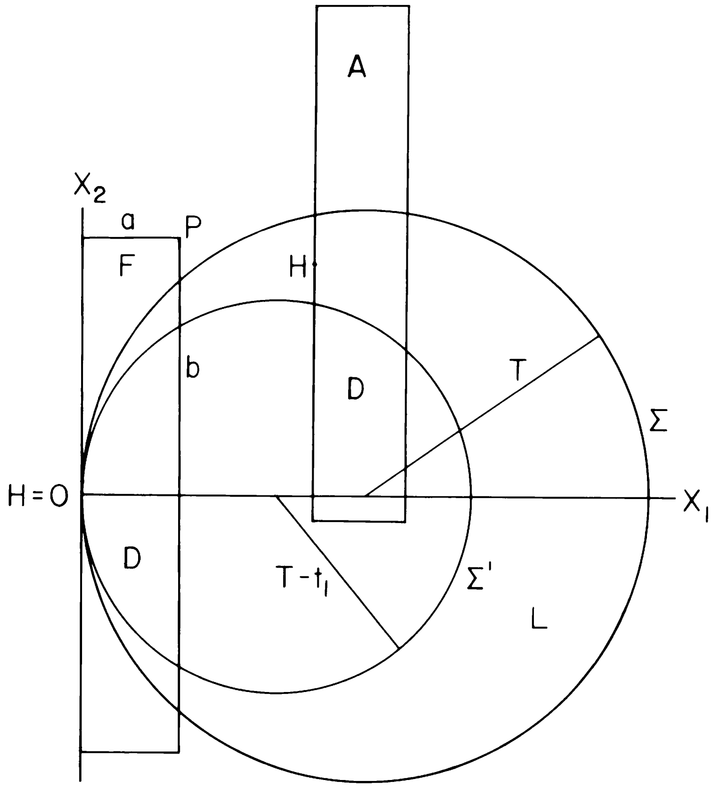

FIG. 4. The plane $X_3 = 0$ in **X**-space. The rectangles marked D are projections of the disc D on this plane: the one marked F is the focal disc. The lens L is represented by the region between the two circles, Σ of radius T and Σ' of radius $T - t_1$.

6. The focal field. Although the domains of integration are easy to describe (intersection of a disc D with a lens L), explicit integration as required in (4.16) is not in general feasible. I shall accordingly pick out certain special events of observation for which the integration is simple, and thus establish the general pattern of the field.

Let us start with the focal field, for which (5.1) hold and the disc D appears as that marked F in Fig. 4. We should consider the lenses which correspond to T in the range (t_1, ∞). But instead of taking that whole range, I shall first concentrate on those values of T for which the face $X_1 = a$ of the disc is cut by both the spheres of the lens, as indicated in Fig. 4. We should start to find the field as soon as the source expires, that is for $T = t_1$. At that instant the inner surface of the lens is a mere point, and this sphere does not reach $X_1 = a$, as required in Fig. 4, until its diameter is equal to a, the thickness of the disc. This occurs at time $T = t_1 + a/2$. But it is also required that the outer sphere of the lens should not have reached the corner (a, b); we see from (4.15) that this will occur when $T = (a^2 + b^2)/(2a)$. Thus the configuration shown will occur for the time-range

$$(t_1 + a/2) < T < (a^2 + b^2)/(2a); \tag{6.1}$$

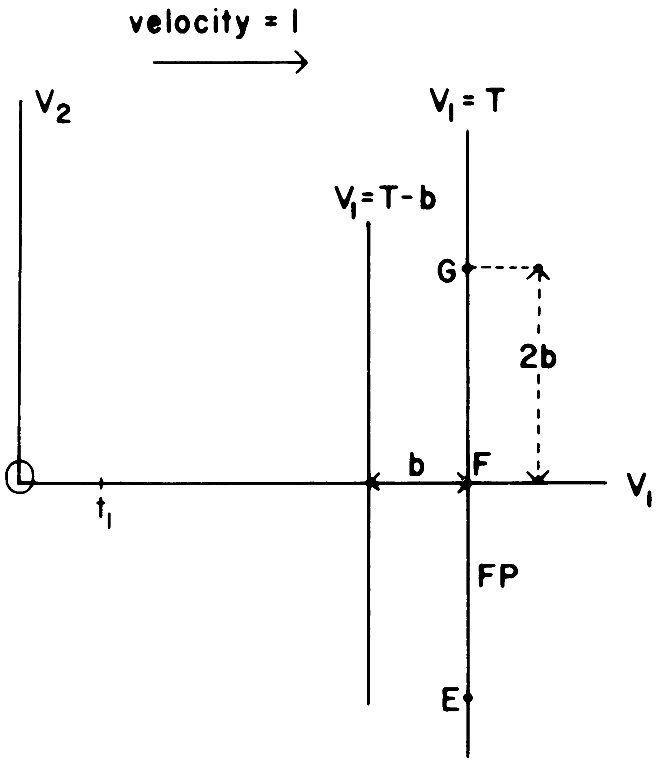

FIG. 5. Kinematical picture in physical space. The focus F and the focal plane FP are moving to the right at unit speed. There is no field in front of FP; the wake follows it.

this requires

$$t_1 < b^2/(2a). \tag{6.2}$$

That is the only inequality we shall so far impose on the constants which define the source.

It is now very simple to perform the integration (4.16). The trick is to slice the disc by planes $X_1 =$ const. and recognize that, by (4.14) and (4.15), this section is a ring bounded by concentric circles with squared radii

$$R^2 = 2TX_1 - X_1^2, \qquad R'^2 = 2T'X_1 - X_1^2. \tag{6.3}$$

The area between them is $2\pi t_1 X_1$. Thus the denominator in (4.16) cancels out and we get the remarkably simple exact result: *the focal field has the constant value*

$$\phi = gI = 2\pi g a t_1, \tag{6.4}$$

for the time range (6.1), *the constants having been subjected to the inequality* (6.2) *but none other.*

There is no difficulty in dealing completely and exactly with the focal field. There are three stages:

Stage I: Both circles Σ, Σ' emerge from the box F across the right-hand side, as shown in Fig. 4. This occurs during the time interval

$$t_1 + a/2 < T < (a^2 + b^2)/(2a), \tag{6.5}$$

and the focal field is the constant

$$\phi = 2\pi g a t_1. \tag{6.6}$$

Stage II: The circle Σ emerges from the box F through the top ($X_2 = b$) while Σ' still emerges through the right-hand side. This occurs during the time interval

$$(a^2 + b^2)/(2a) < T < (a^2 + b^2)/(2a) + t_1, \tag{6.7}$$

so that it lasts a time t_1. The focal field decays, but the formula is not interesting.

Stage III: Both circles Σ, Σ' emerge from the box F through the top. This stage lasts from the upper limit in (6.7) to $T = \infty$. Again the field is easy to calculate. It tends to zero as T tends to infinity.

All these statements are mathematically accurate, the only inequality assumed being (6.2).

7. Lateral spread. Take the hub of the disc at $X_1 = 0$, $X_2 = 2b$, $X_3 = 0$. This is equivalent to pushing the box of Fig. 4 a distance $2b$ up the page. The corresponding disc D will then have no overlap with the lens L as depicted in Fig. 4. This means that during Stage I the field vanishes in the focal plane outside a circle of radius $2b$.

Viewed kinematically as in Fig. 5, there is no field on the focal plane FP outside the circle drawn perpendicular to the paper with diameter EG. This holds throughout Stage I.

8. The wake. Displace the hub of the disc F of Fig. 4 a distance greater than b to the right, combined with any displacement in the X_2-direction. The displaced position might be that marked A in Fig. 4. Then throughout the displaced disc we have $X_1 > b$. Hence by (4.16) the integral I is less than b^{-1} multiplied by the volume common to the disc and the lens. But this volume is less than the volume of the disc, $\pi a b^2$. Hence if $T - V_1 > b$, the field satisfies the inequality

$$\phi < \pi g a b. \tag{8.1}$$

To interpret this in the kinematical picture of Fig. 5, we draw the line $V_1 = T - b$. The result (8.1) tells us that, for all time T, the field in the wake to the left of that line is bounded as indicated.

9. The pulse. The four constants a, b, t_1, g define the source. By subjecting them to the inequality

$$t_1 < b^2/(2a) \tag{9.1}$$

as in (6.2), we find that for a certain range of time, as in (6.7), the focal field remains constant at

$$\phi = 2\pi g a t_1. \tag{9.2}$$

This in itself is of some interest, but it does not imply that we have a concentrated pulse of radiation. We have not compared the focal field with the field elsewhere, except in the matter of lateral spread: the argument of Sec. 7 indicates that there is no field in the focal plane outside a circle of radius $2b$. But we cannot assert that this circle is *small* since its radius is a dimensional quantity.

Let us write

$$a/b = p, \qquad b/t_1 = q, \tag{9.3}$$

so that p and q are dimensionless; let us choose them to be small. To satisfy (9.1), we require simply

$$p < q/2. \tag{9.4}$$

If we denote by T^* the time at which Stage I ends, and the focal field starts to decay, then as in (6.7)

$$T^*/t_1 = 1 + pq/2 + q/(2p). \tag{9.5}$$

We can make this dimensionless ratio as large as we please by making p/q small.

As a measure of the angular lateral spread, we take the ratio

$$2b/T^* = 4p, \tag{9.6}$$

approximately, for p/q small. As for the wake, where we have the upper bound (8.1), the ratio of this to the focal field is

$$b/(2t_1) = q/2. \tag{9.7}$$

To sum up, we get a compact pulse at the focus by subjecting the constants a, b, t_1 to the conditions (9.3) with p and q small dimensionless numbers, the ratio p/q being small. If we regard the source as a gun of length t_1, its bore $2b$ is relatively small, while the "bullet"—the disc—is extremely short.

10. Smoothing the solution. The integral (4.16) gives us only a weak solution of the wave equation. This integral is taken over the domain in **X**-space common to the disc D and the lens L. It is obvious that this gives a continuous field ϕ, but even the first derivatives do not exist everywhere.

Let us now review the argument, modifying it to give a smooth solution to the wave equation. The essential modification occurs in the definition of the source in Sec. 3: the formula (2.6) is merely the retarded potential formula expressed in a Minkowskian setting.

In defining the source S we modify the argument of Sec. 3. We accept the disc as in (3.1) in $t = 0$ and generate from it the null cylinder S as indicated in Fig. 2. But instead of taking $g = \text{const.}$ in S, with $g = 0$ outside S, we now give to g in the disc (3.1) values which go to zero very smoothly at the boundary of the disc. These values of g are propagated unchanged along the null lines which compose S in Fig. 2.

Reverting to the notation **P** (with $P_4 = 0$) for a positive vector in $t = 0$, we now have $g(\mathbf{P})$, a given function, and, in applying (2.6), we may write $g(\mathbf{V} + \mathbf{Q}) = g(\mathbf{P})$, so that we have

$$\phi(\mathbf{V}) = \int_\Gamma g(\mathbf{P}) d_3\mathbf{Q}/q, \tag{10.1}$$

where Γ as earlier indicates the null cone with vertex $\mathbf{V}$. This formula replaces (3.3).

As the argument proceeds through Sec. 5, we are to modify (4.8) by deleting the g outside the integration and putting $g(\mathbf{P})$ under the integral sign. We then use (4.10) and obtain, in place of (4.11),

$$\phi(\mathbf{V}) = \int g(X_1 - T + V_1, X_2 + V_2, X_3 + V_3) d_3\mathbf{X}/X_1, \tag{10.2}$$

the integral again taken over the $\mathbf{X}$-domain common to the disc D and the lens L. This remains unchanged, but the function g goes to zero smoothly as we pass to the boundary of the disc. If this transition is very rapid, it does not essentially alter the conclusions we have come to about focussing.

11. Conclusion. I have described a source which gives a concentrated pulse of scalar radiation, using some ideas associated with Minkowskian geometry. I thank my colleague Professor J. T. Lewis for useful discussions.

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