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ON THE ROOTS OF $f(z) = J_0(z) - iJ_1(z)$*

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Abstract. The function $f(z) = J_0(z) - iJ_1(z)$ is examined to determine its behavior in the complex plane. It is shown that $f(z)$ has no zeroes in the upper half plane.

Introduction. The function

$$f(z) = J_0(z) - iJ_1(z) \tag{1}$$

arises frequently in problems of shallow-water waves propagating over constant depth and then encountering a sloping beach. In this context, it was first encountered by Keller and Keller **[1]** in the matching of a linear inner solution valid over the sloping beach to a linear outer solution valid over the constant depth region.

Linear theory is known to describe the evolution of long gravity waves reasonably well away from the shoreline. Carrier **[2]** and Synolakis **[3]** have used different solutions of the linear theory to specify a boundary condition for the Carrier and Greenspan transformation **[4]**, and they have obtained analytical results for the climb of relatively arbitrary long waves up sloping beaches. These results have been shown to be in good agreement with experimental data **[3]**.

The process of determining the evolution of a long wave over a sloping beach using either linear theory or a combination of linear and nonlinear theory involves calculating integrals of the form

$$\Psi(\beta, \sigma, \theta) = \oint \frac{\Phi(z)}{z} \frac{J_0(\sigma z) e^{-i\theta z}}{J_0(\beta z) - iJ_1(\beta z)} dz. \tag{2}$$

$\Phi(z)$ is the frequency distribution of the incoming wave, β is a parameter which depends on the beach angle, and σ and θ are functions of the space and time coordinates. To evaluate this integral with standard methods it is necessary to determine the behavior of $f(z) = J_0(z) - iJ_1(z)$. In this paper, I will show that $f(z)$ has no zeroes in the upper half plane.

Solution. Consider a contour $\mathscr{C}$ in the upper half plane consisting of the real axis segment $(-r, r)$ and the semicircular arc Γ defined by $z = |r|$, $0 < \arg z < \pi$. $\mathscr{D}$ is the domain bounded by $\mathscr{C}$. Let N be the number of zeroes, and let P be the number of

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poles of $f(z)$ in $\mathscr{D}$. By the principle of the argument,

$$N - P = \frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}\,dz. \tag{3}$$

Using Bessel function identities (Abramowitz and Stegun, [5]), the integral becomes

$$\frac{1}{2\pi i}\oint \frac{J_0'(z) - J_1'(z)}{J_0(z) - iJ_1(z)}\,dz = -\frac{1}{2\pi}\oint i\,dz + \frac{1}{2\pi}\oint \frac{J_1(z)/z}{J_0(z) - iJ_1(z)}\,dz. \tag{4}$$

The integral $\oint dz$ is the integral of an analytic function around a closed path inside its domain of analyticity; it is equal to zero. The second integral of the right-hand side of equation (4) cannot be evaluated in closed form. However, it is possible to obtain an upper limit for its modulus by using the appropriate asymptotic expansions.

The principal asymptotic forms for large values of the Bessel functions are given by Abramowitz and Stegun [5] as

$$J_0(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right) + e^{|\mathscr{I} z|} O\left(\frac{1}{|z|}\right), \tag{5a}$$

and

$$J_1(z) = \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4}\right) + e^{|\mathscr{I} z|} O\left(\frac{1}{|z|}\right). \tag{5b}$$

These relationships imply that

$$|J_0(z) - iJ_1(z)| = \left|\sqrt{\frac{2}{\pi z}} e^{-i(z-\frac{\pi}{4})} + e^{|\mathscr{I} z|} O\left(\frac{1}{|z|}\right)\right|. \tag{6}$$

In the upper half plane $\mathscr{I} z > 0$, and, since the magnitude of the first term of the expansion is of higher order than the second, one can write that

$$|J_0(z) - iJ_1(z)| \geq \sqrt{\frac{2}{\pi|z|}} e^{|\mathscr{I} z|} - e^{|\mathscr{I} z|}\left|O\left(\frac{1}{|z|}\right)\right|. \tag{7}$$

Replacing the second term of the right-hand side of this inequality with its upper limit $\mathscr{L}_{\mathscr{D}} e^{|\mathscr{I} z|}/|z|$ strengthens the inequality. Thus

$$|J_0(z) - iJ_1(z)| \geq \sqrt{\frac{2}{\pi|z|}} e^{|\mathscr{I} z|} - \frac{\mathscr{L}_{\mathscr{D}}}{|z|} e^{|\mathscr{I} z|}. \tag{8}$$

The asymptotic form (5b) also implies that

$$|J_1(z)| \leq \left|\sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4}\right)\right| + e^{|\mathscr{I} z|}\left|O\left(\frac{1}{|z|}\right)\right|. \tag{9}$$

Again, replacing the residue by its upper limit $\mathscr{L}_{\mathscr{N}}/|z|$ strengthens the inequality.

All terms of the inequalities (8) and (9) are positive definite. Dividing by terms, one obtains that

$$\left|\frac{J_1(z)/z}{J_0(z) - iJ_1(z)}\right| \leq \frac{1}{|z|}\frac{\sqrt{\frac{1}{2\pi|z|}}\left(e^{|\mathscr{I} z|} + e^{-|\mathscr{I} z|}\right) + \frac{\mathscr{L}_{\mathscr{N}}}{|z|} e^{|\mathscr{I} z|}}{\left(1\sqrt{|z|}\right)\left[\sqrt{\frac{2}{\pi}} - \frac{\mathscr{L}_{\mathscr{D}}}{\sqrt{|z|}}\right] e^{|\mathscr{I} z|}}, \tag{10}$$

and, simplifying and taking the limit when $|z| = r \to \infty$, it follows that

$$\left| \frac{J_1(z)/z}{J_0(z) - iJ_1(z)} \right| \le \frac{1}{2r}, \quad \text{when } 0 < \arg z < \pi. \tag{11}$$

Therefore, the modulus of the integral over the semicircular arc Γ is given by

$$\left| \int_\Gamma \frac{J_1(z)/z}{J_0(z) - iJ_1(z)} \, dz \right| \le \frac{\pi}{2}. \tag{12a}$$

This result is consistent with the asymptotic evaluation of the integral for large $|z|$ and $0 < \arg z < \pi$,

$$\int_\Gamma \frac{J_1(z)/z}{J_0(z) - iJ_1(z)} \, dz \sim -\frac{\pi}{2}. \tag{12b}$$

When $\mathscr{I} z = 0$, one can easily show that

$$\frac{|J_1(\tilde{x})/\tilde{x}|}{|J_0(\tilde{x}) - iJ_1(\tilde{x})|} \le \frac{1}{|\tilde{x}|}, \quad \forall \tilde{x} \neq 0, \tag{13}$$

where $\mathscr{R} z = \tilde{x}$. However, this result cannot be used directly to derive an upper limit of the integral along a real axis. Numerical computations suggest that

$$\int_{-\infty}^{\infty} \frac{J_1(\tilde{x})/\tilde{x}}{J_0(\tilde{x}) - iJ_1(\tilde{x})} \, d\tilde{x} = 1.57069, \tag{14}$$

and one may conjecture that the integral is less than or equal to $\pi/2$.

These upper bounds, equations (12a) and (14), imply that

$$\frac{1}{2\pi} \left| \oint \frac{J_1(z)/z}{J_0(z) - iJ_1(z)} \, dz \right| < \frac{\frac{\pi}{2} + \frac{\pi}{2}}{2\pi} < 1. \tag{15}$$

Since $f(z)$ does not have any poles in $\mathscr{D}$, and, since $|N - P| < 1$, then $f(z)$ has no zeroes in the upper half plane.

Discussion. The fact that $f(z)$ has no zeroes in the upper half plane allows direct evaluation of the integral in equation (2) and other similar forms. In [**3**] I presented one example using $\Phi(z) = z \operatorname{cosech} z$ and $\sigma = 0$. The contour integration result was found in excellent agreement with numerical calculations of the same integral.

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