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BIFURCATIONS ASSOCIATED WITH A THREE-FOLD ZERO EIGENVALUE*

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Abstract. The bifurcation and instability behavior of a nonlinear autonomous system in the vicinity of a compound critical point is studied in detail. The critical point is characterized by a triple zero of index one eigenvalue, and the system is described by three independent parameters. The analysis is carried out via a *unification technique*, leading to a simple set of differential equations for the analysis of local behavior. Incipient and secondary bifurcations as well as bifurcations into invariant tori are discussed, and the explicit asymptotic results concerning periodic solutions are presented. Moreover, the criteria leading to a sequence of bifurcations into a family of two-dimensional tori are established. An electrical network is analyzed to illustrate the analytical results.

1. Introduction. It is well known that a simple zero eigenvalue of the Jacobian of a general nonlinear autonomous system leads to static bifurcations. However, if the Jacobian matrix of a multiple-parameter system has a multiple zero eigenvalue, for example, a double zero eigenvalue, the system may exhibit both static and dynamic bifurcations in the vicinity of the critical point. These phenomena (associated with a double zero eigenvalue) have been studied by several authors [1–5]. More recently, a *unification technique* [6], which combines the *multiple-parameter perturbation method* [7] and the *intrinsic harmonic balancing* procedure [8], has been developed for the analysis of this problem which enables one to obtain analytical results in a general form. In particular, the stability conditions, the secondary Hopf bifurcations, and the asymptotic solutions of the limit cycles bifurcating from the fundamental equilibrium surface were derived and expressed explicitly in terms of the system coefficients.

If, in addition to a double zero eigenvalue, the Jacobian has another zero eigenvalue at the critical point, then it is reasonable to expect that the system is liable to exhibit even more complicated phenomena in the vicinity of such a compound critical point. The repeated zero eigenvalue is said to be of index one, index two, or index three, according to whether the number of the linearly independent eigenvectors is one, two, or three, respectively. In each case of index one, two, and three, the canonical form of the Jacobian consists of one, two, and three Jordan blocks,

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respectively [5]. This paper is concerned with the first case. Previous work concerning the compound critical point characterized by a triple zero of index one has been concentrated on static bifurcation phenomena [3, 5, 9]. Nevertheless, a number of studies have recently emerged concerning dynamic bifurcations [10, 11, 12]. The aim of this paper is to explore both static and dynamic bifurcations in general terms. The unification technique, introduced earlier [6], is applied to this system to obtain a set of simplified differential equations which govern the local bifurcation behavior of the system in the vicinity of the compound critical point. The incipient bifurcations, secondary bifurcations as well as bifurcations into a two-dimensional torus, are discussed in detail. The analytical asymptotic solutions of the bifurcating limit cycles are established for the first time. All the results are expressed explicitly in terms of the system coefficients (the derivatives of the vector field). Thus, the theory is directly applicable to special problems.

The results are illustrated in an example drawn from electrical network theory.

2. Formulation and static bifurcations. Consider an autonomous system described by

$$\frac{dz^i}{dt} = Z_i(z^j; \eta^\beta), \tag{1}$$

where the z^i are the components of the state vector z and the η^β are certain independent parameters. It is assumed that the functions Z_i are analytic, at least in the region of interest. Attention in this paper will be focussed on a critical equilibrium state where the Jacobian has a 3-fold zero eigenvalue, while all the remaining eigenvalues have negative real parts. For simplicity, therefore, it is assumed that system (1) is a 3×3 system, with "i" ranging from 1 to 3. It is also assumed that the system involves three independent parameters ($\beta = 1, 2, 3$).

Now suppose that the system has a single-valued equilibrium surface in the region of interest, which is expressed as

$$z^i = f_i(\eta^\beta), \tag{2}$$

and c is a critical point on this surface where the eigenvectors corresponding to the triple zero eigenvalue also coincide.

Next, introduce the nonsingular transformation

$$z^i = f_i(\eta^\beta) + T_{ij} w^j \tag{3}$$

such that the resulting system

$$\frac{dw^i}{dt} = W_i(w^j; \eta^\beta) \tag{4}$$

has a Jacobian matrix at c (where $\eta^\beta = \eta_c^\beta$) in the form of

$$[W_{ij}]_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

It is noted that the Jacobian (5) has a Jordan block of order three and the corresponding repeated zero eigenvalue is said to be of index one.

It is also noted that a triple zero eigenvalue may also be of index two and three, corresponding to the Jacobian matrices

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_c \tag{6}$$

and

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_c , \tag{7}$$

respectively. Systems associated with (6) and (7) are not considered in this paper.

It follows from the transformation (3) that the new system has the properties

$$W_i(0;\eta^\xi) = W_{i\beta}(0;\eta^\xi) = W_{i\beta\gamma}(0;\eta^\xi) = \cdots = 0, \tag{8}$$

where the subscripts on W_i's indicate differentiations with respect to the corresponding parameters.

Suppose the eigenvalues of the Jacobian matrix $[W_{ij}(\mu^\beta)]$ are denoted by $\lambda_1(\mu^\beta)$, $\lambda_2(\mu^\beta)$, and $\lambda_3(\mu^\beta)$; then

$$\lambda_1(0) = \lambda_2(0) = \lambda_3(0) = 0,$$

where $\mu^\beta = \eta^\beta - \eta^\beta_c$, $\mu^\beta = 0$ giving the critical point c.

First, consider the static bifurcations from the fundamental equilibrium surface. To this end, let the post-critical equilibrium solutions be expressed in the parametric form

$$w^i = w^i(\sigma^a), \qquad \eta^\beta = \eta^\beta(\sigma^a), \tag{9}$$

where the σ^a's ($a = 1, 2, 3$) are certain unidentified parameters.

Substituting the assumed solution (9) into the equilibrium equations $W_i(w^j;\eta^\beta) = 0$ results in the identities

$$W_i\left[w^j(\sigma^a);\eta^\beta(\sigma^a)\right] \equiv 0, \tag{10}$$

which are then differentiated with respect to the σ^a successively to generate a sequence of perturbation equations

$$W_{ij}w^{j,a} + W_{i\beta}\eta^{\beta,a} \equiv 0, \tag{11}$$

and

$$W_{ijk}w^{j,a}w^{k,b} + W_{ij\beta}(w^{j,a}\eta^{\beta,b} + w^{j,b}\eta^{\beta,a}) + W_{i\beta\gamma}\eta^{\beta,a}\eta^{\gamma,b} + W_{ij}w^{j,ab} + W_{i\beta}\eta^{\beta,ab} = 0, \tag{12}$$

etc., where $i, j, k = 1, 2, 3$; $\beta, \gamma = 1, 2, 3$ and $a, b = 1, 2, 3$, the subscripts on the functions W_i denote differentiations with respect to the corresponding parameters, and summation convention applies. For clarity, differentiations of the variables with respect to the σ^a are indicated by superscripts after a comma.

Now we will use the unification technique [6] to derive the equations governing incipient bifurcations. First, evaluating the first-order perturbation equation (11) at the critical point c, with the aid of (8), results in

$$w^{2,a} = w^{3,a} = 0 \quad (a = 1, 2, 3) \tag{13}$$

by virtue of (5).

Next, evaluating the second perturbation equation (12) at c with the aid of (8) yields

$$W_{ijk}w^{j,a}w^{k,b} + W_{ij\beta}(w^{j,a}\eta^{\beta,b} + w^{j,b}\eta^{\beta,a}) + W_{ij}w^{j,ab} = 0. \tag{14}$$

It is noted that (14) involves 6 equations associated with the 6 different combinations of a and b. The *unification technique* has been very helpful in dealing with such simultaneous equations [6]. Generally, in applying the perturbation procedure, one aims at determining the derivatives of various amplitudes, parameters, and the frequency which are then used to construct Taylor's expansions for these variables. Identifying some amplitudes and/or parameters as perturbation parameters in the beginning of the analysis usually simplifies the analysis considerably. While this may not be possible even in the analysis of some degenerate Hopf bifurcations, it becomes almost impossible in the case of compound critical points associated with two or more codimensions. In these cases, perturbation procedure yields simultaneous equations which are often impossible to solve explicitly for the necessary derivatives. The main idea underlying the unification technique is to produce consistent approximations for the equations themselves in terms of the basic variables rather than the derivatives, which provide the relationships governing the local bifurcation properties. Furthermore, these relationships lead to local dynamical rate equations. To this end, consider equation (14) for $(a = b = 1)$, $(a = 1, b = 2)$, $(a = 1, b = 3)$, $(a = b = 2)$, $(a = 2, b = 3)$, and $(a = b = 3)$, and multiply these equations by $(\sigma^1)^2/2$, $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $(\sigma^2)^2/2$, $\sigma^2\sigma^3$, and $(\sigma^3)^2/2$, respectively. Adding the resulting equations together yields

$$\begin{aligned}
&W_{ij}\left[\tfrac{1}{2}w^{j,11}(\sigma^1)^2 + \tfrac{1}{2}w^{j,22}(\sigma^2)^2 + \tfrac{1}{2}w^{j,33}(\sigma^3)^2 + w^{j,12}\sigma^1\sigma^2 + w^{j,13}\sigma^1\sigma^3 + w^{j,23}\sigma^2\sigma^3\right] \\
&+ W_{i1\beta}(w^{1,1}\sigma^1 + w^{1,2}\sigma^2 + w^{1,3}\sigma^3)(\eta^{\beta,1}\sigma^1 + \eta^{\beta,2}\sigma^2 + \eta^{\beta,3}\sigma^3) \\
&\qquad\qquad + \tfrac{1}{2}W_{i11}(w^{1,1}\sigma^1 + w^{1,2}\sigma^2 + w^{1,3}\sigma^3) = 0,
\end{aligned} \tag{15}$$

which takes the simple form

$$W_{ij}w^j + W_{i1\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{i11}(w^1)^2 = 0, \tag{16}$$

upon taking into account the Taylor expansions of (9).

Equations (16) may be expressed more explicitly as

$$\begin{aligned}
w^2 + W_{11\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{111}(w^1)^2 &= 0, \\
w^3 + W_{21\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{211}(w^1)^2 &= 0, \\
W_{31\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{311}(w^1)^2 &= 0,
\end{aligned} \tag{17}$$

which gives two equilibrium solutions. One solution is

$$w^1 = w^2 = w^3 = 0, \tag{18}$$

describing the fundamental equilibrium surface, as expected. The post-critical solution, on the other hand, is given by

$$\begin{aligned} w^1 &= -\frac{2}{W_{311}} W_{31\beta}\mu^3, \\ w^2 &= -w^1\left(W_{11\beta}\mu^\beta + \tfrac{1}{2}W_{111}w^1\right), \\ w^3 &= -w^1\left(W_{21\beta}\mu^\beta + \tfrac{1}{2}W_{211}w^1\right), \end{aligned} \tag{19}$$

which describes the static bifurcations in the vicinity of c. It is assumed here that $W_{311} \neq 0$.

3. Stability and bifurcation analysis. The structure of the second and third equations in (17) suggests that additional terms may be contributed to these equations from a third perturbation, which will be needed for a local stability analysis. Indeed, it has been observed [6] that the third perturbation equation does contribute to the local stability analysis for a system whose Jacobian has a double zero (of index one) eigenvalue at a critical point. Thus, applying the procedure described above to the third-order perturbation equations (for details see [6]) yields

$$\begin{aligned} W_{ij}w^j + W_{i1\beta}\mu^\beta w^1 + W_{i2\beta}\mu^\beta w^2 + W_{i3\beta}\mu^\beta w^3 + \tfrac{1}{2}W_{i11}(w^1)^2 + W_{i12}w^1w^2& \\ + W_{i13}w^1w^3 + \tfrac{1}{6}W_{i11}(w^1)^3 = 0.& \end{aligned} \tag{20}$$

It is noted that while the first equation in (17) remains valid to a first-order approximation, the second and third equations have to be supplemented with additional terms, taking the form

$$w^3 + W_{21\beta}\mu^\beta w^1 + W_{22\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{211}(w^1)^2 + W_{221}w^1w^2 = 0,$$

and

$$W_{31\beta}\mu^\beta w^1 + W_{32\beta}\mu^\beta w^2 + W_{33\beta}\mu^\beta w^3 + \tfrac{1}{2}W_{311}(w^1)^2 + W_{321}w^1w^2 + W_{331}w^1w^3 = 0, \tag{21}$$

respectively.

Now, in the vicinity of the critical point c, the time rate of change of the state variables may be expressed as

$$\begin{aligned} \frac{dw^1}{dt} &= w^2 + W_{11\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{111}(w^1)^2, \\ \frac{dw^2}{dt} &= w^3 + W_{21\beta}\mu^\beta w^1 + W_{22\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{211}(w^1)^2 + W_{221}w^1w^2, \\ \frac{dw^3}{dt} &= W_{31\beta}\mu^\beta w^1 + W_{32\beta}\mu^\beta w^2 + W_{33\beta}\mu^\beta w^3 + \tfrac{1}{2}W_{311}(w^1)^2 + W_{321}w^1w^2 + W_{331}w^1w^3, \end{aligned} \tag{22}$$

which can be shown to be in compliance with the original state equations to a first-order approximation (the outline of the proof is given in Appendix A).

In order to simplify the following analysis, introduce the nonlinear transformation

$$\begin{aligned} y^1 &= w^1, \\ y^2 &= w^2 + W_{11\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{111}(w^1)^2, \\ y^3 &= w^3 + W_{21\beta}\mu^\beta w^1 + (W_{11\beta} + W_{22\beta})\mu^\beta w^2 + \tfrac{1}{2}W_{211}(w^1)^2 + (W_{111} + W_{221})w^1w^2, \end{aligned} \tag{23}$$

into (22) to obtain a set of differential equations up to second-order terms as follows:

$$\begin{aligned}
\frac{dy^1}{dt} &= y^2, \\
\frac{dy^2}{dt} &= y^3, \\
\frac{dy^3}{dt} &= W_{31\beta}\mu^\beta y^1 + (W_{21\beta} + W_{32\beta})\mu^\beta y^2 + (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta y^3 \\
&\quad + \tfrac{1}{2}W_{311}(y^1)^2 + (W_{211} + W_{321})y^1 y^2 + (W_{111} + W_{221} + W_{331})y^1 y^3.
\end{aligned} \tag{24}$$

Now, based on (24), the bifurcation properties of the original system (4) in the vicinity of the critical point c can be readily investigated.

First, it is observed that the equilibrium solutions of (24) are given by

$$y^1 = y^2 = y^3 = 0 \tag{25}$$

and

$$y^1 = -\frac{2}{W_{311}} W_{31\beta}\mu^\beta, \qquad y^2 = y^3 = 0 \tag{26}$$

to a first-order approximation. These solutions may be verified, through the transformation (23), to embrace (18) and (19), respectively, by considering the terms up to second order.

The stability of the equilibrium solutions is determined by the Jacobian of (24), which is given by $J =$

$$\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\begin{array}{l}\{W_{31\beta}\mu^\beta \\ +W_{311}y^1 + (W_{211} + W_{321})y^2 \\ + (W_{111} + W_{221} + W_{331})y^3\}\end{array} & \begin{array}{l}\{(W_{21\beta} + W_{32\beta})\mu^\beta \\ \\ + (W_{211} + W_{321})y^1\}\end{array} & \begin{array}{l}\{(W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta \\ \\ + (W_{111} + W_{221} + W_{331})y^1\}\end{array}
\end{bmatrix}. \tag{27}$$

Evaluating the Jacobian on the fundamental equilibrium surface (25) leads to the characteristic polynomial

$$P(\lambda) = \lambda^3 - (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta\lambda^2 - (W_{21\beta} + W_{32\beta})\mu^\beta\lambda - W_{31\beta}\mu^\beta, \tag{28}$$

which in turn yields the stability conditions for the fundamental equilibrium solution,

$$W_{31\beta}\mu^\beta < 0, \qquad (W_{21\beta} + W_{32\beta})\mu^\beta < 0, \qquad (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta < 0,$$

and

$$(W_{11\beta} + W_{22\beta} + W_{33\beta})(W_{21\gamma} + W_{32\gamma})\mu^\beta\mu^\gamma + W_{31\beta}\mu^\beta > 0. \tag{29}$$

It follows from (28) and (29) that, in the vicinity of c, there exist two types of simple critical points and two types of compound critical points. The first critical surface, given by

$$S_1\colon W_{31\beta}\mu^\beta = 0 \quad ((W_{21\beta} + W_{32\beta})\mu^\beta < 0 \quad \text{and} \quad (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta < 0), \tag{30}$$

is a primary critical surface where *incipient static bifurcations* take place since $P(\lambda)$ has a zero eigenvalue on this critical surface. The first-order approximation of the static bifurcation solutions is expressed by (26).

The second critical surface is given by

$$S_2\colon (W_{11\beta}+W_{22\beta}+W_{33\beta})(W_{21\gamma}+W_{32\gamma})\mu^\beta\mu^\gamma+W_{31\beta}\mu^\beta=0,$$
$$((W_{21\beta}+W_{32\beta})\mu^\beta<0 \quad \text{and} \quad (W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta<0), \tag{31}$$

which describes the onset of dynamic instabilities, leading to *incipient* Hopf bifurcations ($P(\lambda)$ has a pair of pure imaginary eigenvalues on S_2) from the fundamental equilibrium surface. It is noted that (31) implies $W_{31\beta}\mu^\beta<0$. Moreover, the frequency of the periodic solutions is given by

$$\omega_{\bar{c}}=\sqrt{-(W_{21\beta}+W_{32\beta})\mu^\beta_{\bar{c}}} \quad ((W_{21\beta}+W_{32\beta})\mu^\beta_{\bar{c}}<0), \tag{32}$$

where $\bar{c}$ denotes a point on the critical surface S_2.

At this stage, it is further noted that the intersection of S_1 and S_2 gives two critical lines. One of these lines is described by

$$L_1\colon W_{31\beta}\mu^\beta=0 \quad \text{and} \quad (W_{21\beta}+W_{32\beta})\mu^\beta=0 \quad ((W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta<0), \tag{33}$$

along which the characteristic polynomial $P(\lambda)$ has *a double zero eigenvalue.* The other critical line is given by

$$L_2\colon W_{31\beta}\mu^\beta=0 \quad \text{and} \quad (W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta=0 \quad ((W_{21\beta}+W_{32\beta})\mu^\beta<0) \tag{34}$$

along which the $P(\lambda)$ has *a simple zero and a pair of pure imaginary eigenvalues.*

Next, evaluating the Jacobian (27) on the static bifurcation solution (26) results in the stability conditions

$$\begin{gathered}W_{31\beta}\mu^\beta>0, \qquad \left[(W_{21\beta}+W_{32\beta})-\frac{2}{W_{311}}(W_{211}+W_{321})W_{31\beta}\right]\mu^\beta<0,\\ \left[(W_{11\beta}+W_{22\beta}+W_{33\beta})-\frac{2}{W_{311}}(W_{111}+W_{221}+W_{331})W_{31\beta}\right]\mu^\beta<0,\end{gathered} \tag{35}$$

and

$$\begin{aligned}&\left[(W_{21\beta}+W_{32\beta})-\frac{2}{W_{311}}(W_{211}+W_{321})W_{31\beta}\right]\left[(W_{11\gamma}+W_{22\gamma}+W_{33\gamma})\right.\\&\qquad\left.-\frac{2}{W_{311}}(W_{111}+W_{221}+W_{331})W_{31\gamma}\right]\mu^\beta\mu^\gamma-W_{31\beta}\mu^\beta>0,\end{aligned}$$

which leads to the critical surface

$$\begin{aligned}S_3\colon &\left[(W_{21\beta}+W_{32\beta})-\frac{2}{W_{311}}(W_{211}+W_{321})W_{31\beta}\right]\left[(W_{11\gamma}+W_{22\gamma}+W_{33\gamma})\right.\\&\qquad\left.-\frac{2}{W_{311}}(W_{111}+W_{221}+W_{331})W_{31\gamma}\right]\mu^\beta\mu^\gamma-W_{31\beta}\mu^\beta=0,\end{aligned} \tag{36}$$

under the assumption that the remaining conditions in (35) still hold. (36) describes the onset of a secondary Hopf bifurcation from the static bifurcation solution (26). The frequency of the periodic solutions is given by

$$\omega_{\bar{c}}=\left[-(W_{21\beta}+W_{32\beta})\mu^\beta_{\bar{c}}+\frac{2}{W_{311}}(W_{211}+W_{321})W_{31\beta}\mu^\beta_{\bar{c}}\right]^{\frac{1}{2}}. \tag{37}$$

An interesting interrelationship concerning S_1, S_2, and S_3 is noted here. In fact, the intersections of S_1 and S_3, and S_2 and S_3 are again the lines L_1 and L_2 as in the case of the intersection of S_1 and S_2. In other words, in the 3-dimensional parameter space, S_1 is the common tangent plane to both S_2 and S_3 at the origin, and S_2 and S_3 are antielastic surfaces with opposite curvatures. This has been schematically illustrated in Fig. 1.

It is observed from the above analysis that the assumption $W_{311} \neq 0$ plays a significant role. Analyzing the special (degenerate) case in which $W_{311} = 0$ requires higher-order terms (to supplement (22)) and will not be considered in this paper.

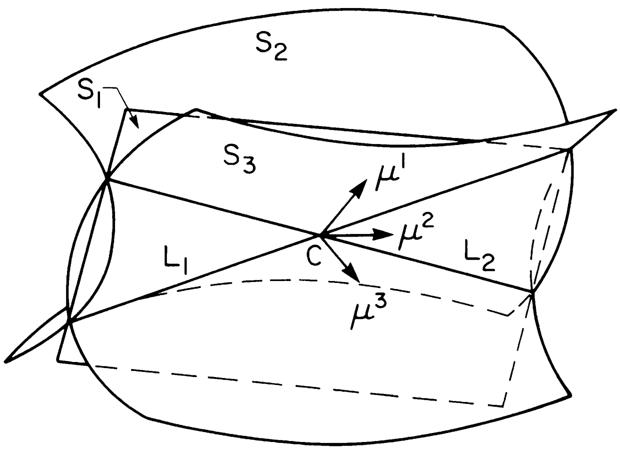

FIG. 1. Critical surfaces.

4. Hopf bifurcations and two-dimensional tori. The nontrivial equilibrium solution bifurcating from the fundamental solution along the critical surface S_1 is expressed by (26). On the other hand, dynamic bifurcations occur along the critical surface S_2. In order to obtain a more comprehensive view of the behavior characteristics of the system in the vicinity of S_2, the system will be analyzed in the neighborhood of both a regular point on S_2 as well as L_2, which represents the intersection of S_2 with S_1.

In order to study the bifurcating limit cycles, the Jacobian of the system should first be transformed into the canonical form

$$J(0; \mu_{\bar{c}}^{\beta}) = \begin{bmatrix} 0 & \omega_{\bar{c}} & 0 \\ -\omega_{\bar{c}} & 0 & 0 \\ 0 & 0 & \alpha_{o\bar{c}} \end{bmatrix}, \tag{38}$$

where $\omega_{\bar{c}}$ and $\alpha_{o\bar{c}}$ are given by

$$\omega_{\bar{c}} = \sqrt{-(W_{21\beta} + W_{32\beta})\mu_{\bar{c}}^{\beta}} \quad ((W_{21\beta} + W_{32\beta})\mu_{\bar{c}}^{\beta} < 0) \tag{39}$$

and

$$\alpha_{o\bar{c}} = (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu_{\bar{c}}^{\beta}, \tag{40}$$

respectively. Here, $\mu_{\bar{c}}^{\beta}$ represents a regular point on S_2 when $\alpha_{o\bar{c}} < 0$.

To this end, introduce the transformation

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ \omega_{\bar{c}} & 0 & \alpha_{o\bar{c}} \\ 0 & \omega_{\bar{c}}^2 & \alpha_{o\bar{c}}^2 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} \tag{41}$$

to transform system (24) into

$$\begin{aligned} \frac{dv^1}{dt} &= V_1(v^j;\mu^\beta) = -\frac{\alpha_{o\bar{c}}}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}\left[(W_{21\beta}+W_{32\beta})\mu^\beta\right]v^1+\omega_{\bar{c}}v^2 \\ &\quad + \frac{\alpha_{o\bar{c}}^3\left[\alpha_{o\bar{c}}-(W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta\right]}{\omega_{\bar{c}}(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2)}v^3-\frac{\alpha_{o\bar{c}}}{\omega_{\bar{c}}}F(v^j,\mu^\beta), \\ \frac{dv^2}{dt} &= V_2(v^j;\mu^\beta) = -\frac{\omega_{\bar{c}}}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}\left[\alpha_{o\bar{c}}^2-(W_{21\beta}+W_{32\beta})\mu^\beta\right]v^1 \\ &\quad - \frac{\alpha_{o\bar{c}}^2\left[\alpha_{o\bar{c}}-(W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta\right]}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}v^3+F(v^j,\mu^\beta), \\ \frac{dv^3}{dt} &= V_3(v^j;\mu^\beta) = -\frac{\omega_{\bar{c}}}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}\left[\omega_{\bar{c}}^2+(W_{21\beta}+W_{32\beta})\mu^\beta\right]v^1 \\ &\quad + \frac{\alpha_{o\bar{c}}\left[\alpha_{o\bar{c}}(W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta+\omega_{\bar{c}}^2\right]}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}v^3+F(v^j,\mu^\beta), \end{aligned} \tag{42}$$

where

$$\begin{aligned} F(v^j,\mu^\beta) &= \frac{1}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}\left[\omega_{\bar{c}}^2(W_{11\beta}+W_{22\beta}+W_{33\beta})\mu^\beta v^2+\alpha_{o\bar{c}}(W_{21\beta}+W_{32\beta})\mu^\beta v^3\right] \\ &+ \frac{1}{\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2}\Big\{-W_{31\beta}\mu^\beta+\omega_{\bar{c}}(W_{211}+W_{321})v^1 \\ &+ \left[\tfrac{1}{2}W_{311}-\omega_{\bar{c}}^2(W_{111}+W_{221}+W_{331})\right]v^2 \\ &- \left[\tfrac{1}{2}W_{311}+\alpha_{o\bar{c}}(W_{211}+W_{321})\right]v^3\Big\}(v^2-v^3) \end{aligned} \tag{43}$$

and the constant coefficients of the original system (22) are retained for convenience in applications. The Jacobian of (42) evaluated on the critical surface S_2 is now in the form of (38).

In order to obtain the *behavior surface* (amplitude-parameter relationship) associated with Hopf bifurcations, it is convenient to shift from (42) to the dynamic equations involving the amplitude (i.e., polar representation) of the limit cycles. This is performed by assuming a Fourier series representation for the limit cycles, given by

$$v^i(\tau,\overline{\sigma}^a) = \sum_{m=0}^{M}\left[p_{im}(\overline{\sigma}^a)\cos m\tau + r_{im}(\overline{\sigma}^a)\sin m\tau\right], \tag{44}$$

and applying the *harmonic balancing* [5, 8] and *unification techniques*. Thus, one obtains

$$\frac{d\rho}{dt} = \frac{1}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)} \Big\{ - \Big[W_{31\beta} + \alpha_{o\bar{c}}(W_{21\beta} + W_{32\beta}) - \omega_{\bar{c}}^2(W_{11\beta} + W_{22\beta} + W_{33\beta}) \Big] \varphi^\beta \rho$$
$$- W_{311}\rho v^3 + \frac{1}{8(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)(\alpha_{o\bar{c}}^2 + 4\omega_{\bar{c}}^2)} W_{311} \Big[-3(\alpha_{o\bar{c}}^2 + 2\omega_{\bar{c}}^2)(W_{211} + W_{321})$$
$$+ \frac{\alpha_{o\bar{c}}}{\omega_{\bar{c}}^2}(\alpha_{o\bar{c}}^2 + 5\omega_{\bar{c}}^2) W_{311} \Big] \rho^3 \Big\}, \tag{45}$$

$$\frac{dv^3}{dt} = \frac{1}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)} \left[\frac{\alpha_{o\bar{c}}\omega_{\bar{c}}}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)^2} (W_{211} + W_{321}) W_{31\beta}\varphi^\beta \rho + 2\alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2) v^3 + \tfrac{1}{2} W_{311}\rho^2 \right],$$

and

$$\frac{d\theta}{dt} = \omega_{\bar{c}} - \frac{1}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)} \left\{ \omega_{\bar{c}} \left[(W_{21\beta} + W_{32\beta}) - \frac{\alpha_{o\bar{c}}}{\omega_{\bar{c}}^2} W_{31\beta} \right] \varphi^\beta \right. \tag{46}$$
$$\left. + \omega_{\bar{c}} \left[(W_{211} + W_{321}) - \frac{\alpha_{o\bar{c}}}{\omega_{\bar{c}}^2} W_{311} \right] v^3 - \frac{5\alpha_{o\bar{c}}^4 + 25\alpha_{o\bar{c}}^2\omega_{\bar{c}}^2 + 2\omega_{\bar{c}}^4}{24\omega_{\bar{c}}^3(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)^2(\alpha_{o\bar{c}}^2 + 4\omega_{\bar{c}}^2)} (W_{311})^2 \rho^2 \right\},$$

where $\varphi^\beta = \mu^\beta - \mu_{\bar{c}}^\beta$, and ρ represents the amplitude of the periodic solutions. The first-order approximation of ρ is p_{11} if one sets $r_{11}(\bar{\sigma}^a) \equiv 0$, otherwise, $\rho = \sqrt{(p_{11})^2 + (r_{11})^2}$. Equation (46) gives the frequency ω for the periodic solutions. It is noted that in the equations (45) and (46) we still express the constant coefficients in terms of the coefficients of system (22).

Now, it is observed that

$$\rho = v^3 = 0 \tag{47}$$

is a solution of (45), describing the fundamental surface, as expected. The post-critical solution, on the other hand, is given by

$$\rho^2 = \frac{8\alpha_{o\bar{c}}\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)(\alpha_{o\bar{c}}^2 + 4\omega_{\bar{c}}^2)}{(W_{311})^2(\alpha_{o\bar{c}}^4 + 7\alpha_{o\bar{c}}^2\omega_{\bar{c}}^2 + 8\omega_{\bar{c}}^4)} f(\varphi^\beta),$$
$$v^3 = -\frac{W_{311}}{4\alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)}\rho^2 = -\frac{2\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + 4\omega_{\bar{c}}^2)}{W_{311}(\alpha_{o\bar{c}}^4 + 7\alpha_{o\bar{c}}^2\omega_{\bar{c}}^2 + 8\omega_{\bar{c}}^4)} f(\varphi^\beta), \tag{48}$$

which describes the *behavior surface* in the vicinity of $\bar{c}$, representing a family of bifurcating limit cycles. Here,

$$f(\varphi^\beta) = \Big[W_{31\beta} + \alpha_{o\bar{c}}(W_{21\beta} + W_{32\beta}) - \omega_{\bar{c}}^2(W_{11\beta} + W_{22\beta} + W_{33\beta}) \Big] \varphi^\beta. \tag{49}$$

For the first-order approximation of the behavior surface (48), the second-order terms in $f(\varphi^\beta)$, $(W_{11\beta} + W_{22\beta} + W_{33\beta})(W_{21\gamma} + W_{32\gamma})\varphi^\beta\varphi^\gamma$, were not needed. If this is taken into account it can be shown that

$$f(\varphi^\beta) = W_{31\beta}\mu^\beta + (W_{11\beta} + W_{22\beta} + W_{33\beta})(W_{21\gamma} + W_{32\gamma})\mu^\beta\mu^\gamma = g(\mu^\beta) \tag{50}$$

by using $\varphi^\beta = \mu^\beta - \mu_{\bar{c}}^\beta$.

The stability of the solutions (47) and (48) is determined by the Jacobian of (45), which is given by $J =$

$$\frac{1}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)} \begin{bmatrix} \begin{array}{l} -f(\varphi^\beta) - W_{311}v^3 \\ \quad - \dfrac{3W_{311}\left[3\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + 2\omega_{\bar{c}}^2)(W_{211}+W_{321}) - \alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2 + 5\omega_{\bar{c}}^2)W_{311}\right]\rho^2}{8\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2)(\alpha_{o\bar{c}}^2+4\omega_{\bar{c}}^2)} \end{array} & -W_{311}\rho \\ \dfrac{\alpha_{o\bar{c}}\omega_{\bar{c}}}{(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2)^2}(W_{211}+W_{312})W_{31\beta}\varphi^\beta + W_{311}\rho & 2\alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2) + 2W_{311}v^3 \end{bmatrix}. \tag{51}$$

Evaluating the Jacobian on the fundamental equilibrium surface ($\rho = v^3 = 0$) yields the stability conditions:

$$f(\varphi^\beta) > 0 \quad \text{and} \quad \alpha_{o\bar{c}} < 0 \quad (\text{note that } f(\varphi^\beta) = g(\mu^\beta)). \tag{52}$$

Recalling the assumption that $(W_{21\beta} + W_{32\beta})\mu_{\bar{c}}^\beta < 0$, one observes that these stability conditions are identical to the condition (29), as expected. For $g(\mu^\beta) = 0$ ($\alpha_{o\bar{c}} < 0$), Hopf bifurcations occur and the solutions are described by (46), (48), and (44).

Next, evaluate the Jacobian (51) on the behavior surface (48) to obtain $J =$

$$\frac{1}{2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)} \begin{bmatrix} -\dfrac{W_{311}\left[3\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2+2\omega_{\bar{c}}^2)(W_{211}+W_{321}) - \alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2+5\omega_{\bar{c}}^2)W_{311}\right]\rho^2}{4\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2)(\alpha_{o\bar{c}}^2+4\omega_{\bar{c}}^2)} & -W_{311}\rho \\ W_{311}\rho & 2\alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2+\omega_{\bar{c}}^2) \end{bmatrix}, \tag{53}$$

which in turn yields the stability conditions for the bifurcating limit cycles as follows:

$$\operatorname{trace} J < 0 \quad \text{and} \quad \Delta > 0, \tag{54}$$

where the trace and the determinant Δ of the Jacobian are given by

$$\begin{aligned} \operatorname{trace} J &= \alpha_{o\bar{c}} - \frac{1}{8\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)^2(\alpha_{o\bar{c}}^2 + 4\omega_{\bar{c}}^2)} W_{311} \Big[3\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + 2\omega_{\bar{c}}^2)(W_{211} + W_{321}) \\ &\qquad - \alpha_{o\bar{c}}(\alpha_{o\bar{c}}^2 + 5\omega_{\bar{c}}^2)W_{311}\Big]\rho^2, \\ \Delta &= -\frac{\rho^2}{8(\alpha_{o\bar{c}}^2 + \omega_{\bar{c}}^2)^2}\Big[3\alpha_{o\bar{c}}\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + 2\omega_{\bar{c}}^2)(W_{211} + W_{321}) \\ &\qquad - (\alpha_{o\bar{c}}^4 + 7\alpha_{o\bar{c}}^2\omega_{\bar{c}}^2 + 8\omega_{\bar{c}}^4)W_{311}\Big] W_{311}. \end{aligned} \tag{55}$$

Since $\operatorname{trace} J < 0$ ($\alpha_{o\bar{c}}$ is the main term), the stability condition $\Delta > 0$ gives

$$\Big[3\alpha_{o\bar{c}}\omega_{\bar{c}}^2(\alpha_{o\bar{c}}^2 + 2\omega_{\bar{c}}^2)(W_{211} + W_{321}) - (\alpha_{o\bar{c}}^4 + 7\alpha_{o\bar{c}}^2\omega_{\bar{c}}^2 + 8\omega_{\bar{c}}^4)W_{311}\Big] W_{311} < 0 \tag{56}$$

under which the bifurcating limit cycles (48) are stable. A lengthy analysis shows that condition (56) can be directly derived from system (42) by using the formula $\gamma_{11} - \gamma_{22}$ given in [8], and this is an independent confirmation of the results. This formula for stability of bifurcating limit cycles will be given explicitly in the sequel for a 2-dimensional system (see (71)). The asymptotic solutions can be obtained by using (41), (44), and (48), where $\tau = \omega t$, and $\omega = d\theta/dt$ is given by (46).

Next, consider the case in which $\alpha_{o\bar{c}} = 0$ in (42) and (43). This is equivalent to considering a point on L_2. In this case, instead of equations (45) and (46), one obtains

$$
\begin{aligned}
\frac{d\rho}{dt} = &-\frac{1}{2\omega_{\bar{c}}^2}\Big\{\Big[W_{31\beta} - \omega_{\bar{c}}^2(W_{11\beta} + W_{22\beta} + W_{33\beta})\Big]\varphi^\beta \rho \\
&+ \Big[W_{311} - \omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\Big]\rho v^3\Big\} \\
&- \frac{1}{32\omega_{\bar{c}}^4}(W_{211} + W_{321})\Big\{\Big[3W_{311} - 4\omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\Big]\rho^3 \\
&+ 4\Big[5W_{311} + \omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\Big]\rho(v^3)^2\Big\}, \\
\frac{dv^3}{dt} = &\frac{1}{2\omega_{\bar{c}}^2}\Big\{2W_{31\beta}\varphi^\beta v^3 + W_{311}(v^3)^2 \\
&+ \tfrac{1}{2}\Big[W_{311} - 2\omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\Big]\rho^2\Big\} \\
&+ \frac{1}{4\omega_{\bar{c}}^4}(W_{211} + W_{321})\Big\{\Big[W_{311} - \omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\Big]\rho^2 v^3 \\
&+ 2W_{311}(v^3)^3\Big\},
\end{aligned}
\tag{57}
$$

and

$$
\frac{d\theta}{dt} = \omega_{\bar{c}} + \frac{1}{2\omega_{\bar{c}}}\left[(W_{21\beta} + W_{32\beta})\varphi^\beta + (W_{211} + W_{321})v^3 - \frac{1}{48\omega_{\bar{c}}^6}(W_{311})^2\rho^2\right], \tag{58}
$$

where $\bar{c}$ represents a point on the critical line L_2 and

$$
\omega_{\bar{c}} = \sqrt{-(W_{21\beta} + W_{32\beta})\mu_{\bar{c}}^\beta} \quad ((W_{21\beta} + W_{32\beta})\mu_{\bar{c}}^\beta < 0). \tag{59}
$$

It is noted that (57) involves higher-order terms compared to (42) and (43) which will be needed for the stability analysis of the tori. Based on (57) and (58), one obtains the following first-order solutions:

$$
\text{(I)}\ \rho = v^3 = 0 \quad \text{initial equilibrium solution } (w^i = 0), \tag{60}
$$

$$
\text{(II)}\ \rho = 0, \qquad v^3 = -\frac{2}{W_{311}}W_{31\beta}\varphi^\beta \quad \text{static bifurcation solution}, \tag{61}
$$

and

$$
\text{(III)}\quad \left.\begin{aligned}
\rho^2 &= -\frac{2(2W_{31\beta}\varphi^\beta + W_{311}v^3)v^3}{W_{311} - 2\omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})}, \\
v^3 &= -\frac{[W_{31\beta} - \omega_{\bar{c}}^2(W_{11\beta} + W_{22\beta} + W_{33\beta})]\varphi^\beta}{W_{311} - \omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})},
\end{aligned}\right\} \text{Hopf bifurcation solution.} \tag{62}
$$

It is noted that the static bifurcation solution (II) is, in fact, the solution (26) as may be verified via transformation (41) (with $\alpha_{o\bar{c}} = 0$). The asymptotic periodic solutions can be obtained by using (41), (44), and (62), where $\tau = \omega t$, and $\omega = d\theta/dt$ is given by (58).

Evaluating the Jacobian of (57) on the initial equilibrium solution (I) gives two critical surfaces S_1 and S_2 (see (30) and (31)). Similarly, evaluating the Jacobian on the static bifurcation solution (II) results in the critical surface S_3 (see (36)) along which a secondary Hopf bifurcation takes place. It is interesting to note that the intersection of solutions (I) and (III) results in S_2, while the intersection of (I) and (II) yields S_1, as expected. Even more remarkable is the fact that S_3 is the intersection of the solutions (II) and (III). In other words, the solution (III) which has been obtained through an analysis in the vicinity of L_2, may be viewed as a bifurcation from the static solution (II). Also, the stability conditions for the initial equilibrium solution and the static bifurcation solution given by (29) and (35), respectively, may readily be recovered here by considering the Jacobian of (57).

It is also noted that an analysis in the vicinity of the critical line L_1 (33) would not lead to new information. In fact, the solutions (26) and (48) are recovered if such an analysis is carried out. Therefore, this will not be pursued here.

Similarly, in order to consider the stability of the Hopf bifurcation solution (III), evaluate the Jacobian on the solution (III) to obtain $J =$

$$\frac{1}{2\omega_c^2}\begin{bmatrix} 0 & -\left[W_{311}-\omega_c^2(W_{111}+W_{221}+W_{331})\right]\rho \\ \left[W_{311}-2\omega_c^2(W_{111}+W_{221}+W_{331})\right]\rho & \left[W_{31\beta}-W_{311}\dfrac{W_{31\beta}-\omega_c^2(W_{11\beta}+W_{22\beta}+W_{33\beta})}{W_{311}-\omega_c^2(W_{111}+W_{211}+W_{331})}\right]\varphi^\beta \end{bmatrix}, \tag{63}$$

which in turn gives

$$\operatorname{trace} J = \frac{\left[W_{311}(W_{11\beta}+W_{22\beta}+W_{33\beta})-(W_{111}+W_{221}+W_{331})W_{31\beta}\right]\varphi^\beta}{2\left[W_{311}-\omega_c^2(W_{111}+W_{221}+W_{331})\right]},$$

and

$$\Delta = \frac{1}{4\omega_c^4}\left[W_{311}-\omega_c^2(W_{111}+W_{221}+W_{331})\right]\left[W_{311}-2\omega_c^2(W_{111}+W_{221}+W_{331})\right]\rho^2. \tag{64}$$

Since $\Delta > 0$, the bifurcating limit cycles (III) are stable if $\operatorname{trace} J < 0$.

For $\operatorname{trace} J = 0$, one has the critical surface

$$S_4\colon\ \left[W_{311}(W_{11\beta}+W_{22\beta}+W_{33\beta})-(W_{111}+W_{211}+W_{331})W_{31\beta}\right]\varphi^\beta = 0 \tag{65}$$

along which a secondary Hopf bifurcation, leading to a *two-dimensional torus*, takes place. The second frequency of the torus is given by

$$\omega_{c'} = \frac{\rho}{2\omega_c^2}\left\{\left[W_{311}-\omega_c^2(W_{111}+W_{221}+W_{331})\right]\left[W_{311}-2\omega_c^2(W_{111}+W_{221}+W_{331})\right]\right\}^{\frac{1}{2}}, \tag{66}$$

where c' denotes a point on the critical surface S_4.

Finally, we will consider the stability of the family of the two-dimensional tori bifurcating from the family of limit cycles (III) along the critical surface S_4. It is well known that the stability of a limit cycle is concerned with transient motions being attracted to or repelled from an orbit (*orbital stability* [5]). Here, however, the bifurcating torus in the state space accommodates the trajectories on its surface. Thus, one may consider the stability of the torus itself (attracting or repelling property). This stability condition can be derived from equation (45). We have tacitly assumed that the initial family of limit cycles are orbitally stable which loses stability at the critical surface S_4 where a new family of limit cycles emerges. Thus, the stability of the family of tori depends on the stability of the new family of the limit cycles.

In order to obtain the stability condition, first, introduce

$$\begin{aligned} \rho &= \rho_s + \overline{\rho}, \\ v^3 &= v_s^3 + \overline{v}^3, \end{aligned} \tag{67}$$

where ρ_s and v_s^3 are solution (III), to transform (57) into a system with $\overline{\rho} = \overline{v}^3 = 0$ as its initial equilibrium solution. Further, using an additional transformation,

$$\begin{bmatrix} \overline{\rho} \\ \overline{v}^3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2\omega_c^2 \omega_{c'}} \left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right] \rho_s & 0 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \tag{68}$$

yields

$$\begin{aligned}
\frac{dx^1}{dt} &= X_1(x^1, x^2; \varphi^\beta) \\
&= \frac{\left[(W_{111} + W_{221} + W_{331})W_{31\beta} - W_{311}(W_{11\beta} + W_{22\beta} + W_{33\beta})\right] \varphi^\beta}{W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})} x^1 + \omega_{c'} x^2 \\
&\quad + \frac{1}{2\omega_c^2} W_{311}(x^1)^2 - \frac{\omega_{c'}(W_{211} + W_{321})\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right]}{\omega_c^2\left[W_{311} - 2\omega_c^2(W_{111} + W_{221} + W_{331})\right]} x^1 x^2 \\
&\quad + \frac{1}{4\omega_c^2}\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right](x^2)^2 - \frac{1}{2\omega_c^4} W_{311}(W_{211} + W_{321})(x^1)^3 \\
&\quad - \frac{(W_{211} + W_{321})\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right]}{32\omega_c^4\left[W_{311} - 2\omega_c^2(W_{111} + W_{221} + W_{331})\right]} \\
&\qquad \times \left[3W_{311} - 4\omega_c^2(W_{111} + W_{221} + W_{331})\right] x^1(x^2)^2,
\end{aligned} \tag{69a}$$

$$\begin{aligned}
\frac{dx^2}{dt} &= X_2(x^1, x^2; \varphi^\beta) \\
&= -\omega_{c'} x^1 + \frac{\omega_{c'}(W_{211} + W_{321})\left[5W_{311} + \omega_c^2(W_{111} + W_{221} + W_{331})\right]}{4\omega_c^2\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right]}(x^1)^2 \\
&\quad - \frac{1}{2\omega_c^2}\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right] x^1 x^2 \\
&\qquad + \frac{3\omega_{c'}(W_{211} + W_{321})\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right]}{32\omega_c^4\left[W_{311} - 2\omega_c^2(W_{111} + W_{221} + W_{331})\right]} \\
&\qquad \times \left[3W_{311} - 4\omega_c^2(W_{111} + W_{221} + W_{331})\right](x^2)^2 \\
&\qquad + \frac{1}{8\omega_c^4}(W_{211} + W_{321})\left[5W_{311} + \omega_c^2(W_{111} + W_{221} + W_{331})\right](x^1)^2 x^2 \\
&\qquad + \frac{(W_{211} + W_{321})\left[W_{311} - \omega_c^2(W_{111} + W_{221} + W_{331})\right]}{32\omega_c^4\left[W_{311} - 2\omega_c^2(W_{111} + W_{221} + W_{331})\right]} \\
&\qquad \times \left[3W_{311} - 4\omega_c^2(W_{111} + W_{221} + W_{331})\right](x^2)^3
\end{aligned} \tag{69b}$$

whose Jacobian matrix evaluated on the critical surface S_4 is in the canonical form

$$J = \begin{bmatrix} 0 & \omega_{c'} \\ -\omega_{c'} & 0 \end{bmatrix}. \tag{70}$$

Now, the stability condition of the tori can be derived from system (69). This condition is related to the stability solution which is determined by $\gamma_{11} - \gamma_{22}$ given in a general form in reference [5, 8],

$$\gamma_{11} - \gamma_{22} = \frac{3}{4}\Bigg[(X_{1111} + X_{1122} + X_{2211} + X_{2222})$$
$$+ \frac{1}{\omega_{c'}}(X_{111}X_{211} + X_{211}X_{221} + X_{221}X_{222} - X_{111}X_{112} - X_{112}X_{122} - X_{122}X_{222})\Bigg]. \quad (71)$$

Thus, if $\gamma_{11} - \gamma_{22} < 0$, the bifurcation solution is stable [5, 8]. Here, $\gamma_{11} - \gamma_{22}$ takes the form

$$\gamma_{11}-\gamma_{22} = -\frac{3}{256\omega_{\bar{c}}^{8}}\left(\frac{\rho_s}{\omega_{c'}}\right)^2 (W_{211}+W_{321})\{113(W_{311})^3-\omega_{\bar{c}}^2 W_{311}(W_{111}+W_{221}+W_{331})$$
$$\times \left[278W_{311} - 129\omega_{\bar{c}}^2(W_{111} + W_{221} + W_{331})\right] - 12\omega_{\bar{c}}^6(W_{111} + W_{221} + W_{331})^3\}. \quad (72)$$

For stability $\gamma_{11} - \gamma_{22} < 0$ [5, 8]. Therefore, since $\omega_{\bar{c}}$ is small (see (59)) this condition is fulfilled when

$$W_{311}(W_{211} + W_{321}) > 0, \quad (73)$$

indicating an asymptotically stable family of tori. The simplicity of this criterion may be linked to the structure of the original Jacobian and should not cast any doubts on its validity. In fact, it can be demonstrated that retaining higher-order terms in (22) would have no effect on (73).

The bifurcation flow chart is sketched in Fig. 2.

5. An example. In this section, a nonlinear electrical network, shown in Fig. 3, is analyzed to demonstrate the applicability of the theory and formulas derived in the previous sections.

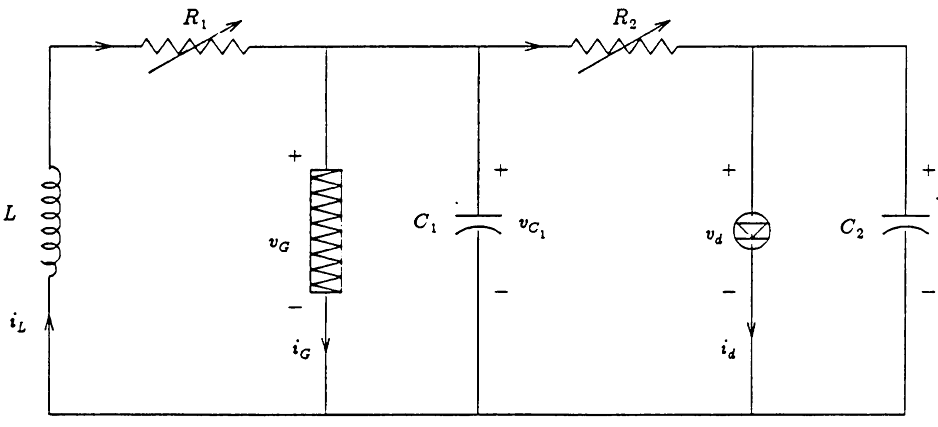

FIG. 3. A nonlinear electrical network.

The network consists of an inductor L, two capacitors C_1 and C_2, two resistors R_1 and R_2, a tunnel-diode and a conductance. Suppose L, C_1, C_2, R_1, and R_2 are linear components; in addition, R_1 and R_2 may be varied, while the tunnel-diode and the conductance are nonlinear elements, and they both are voltage-controlled. The current i_L in the inductor and the voltages v_{C_1} and v_{C_2} across the capacitors C_1

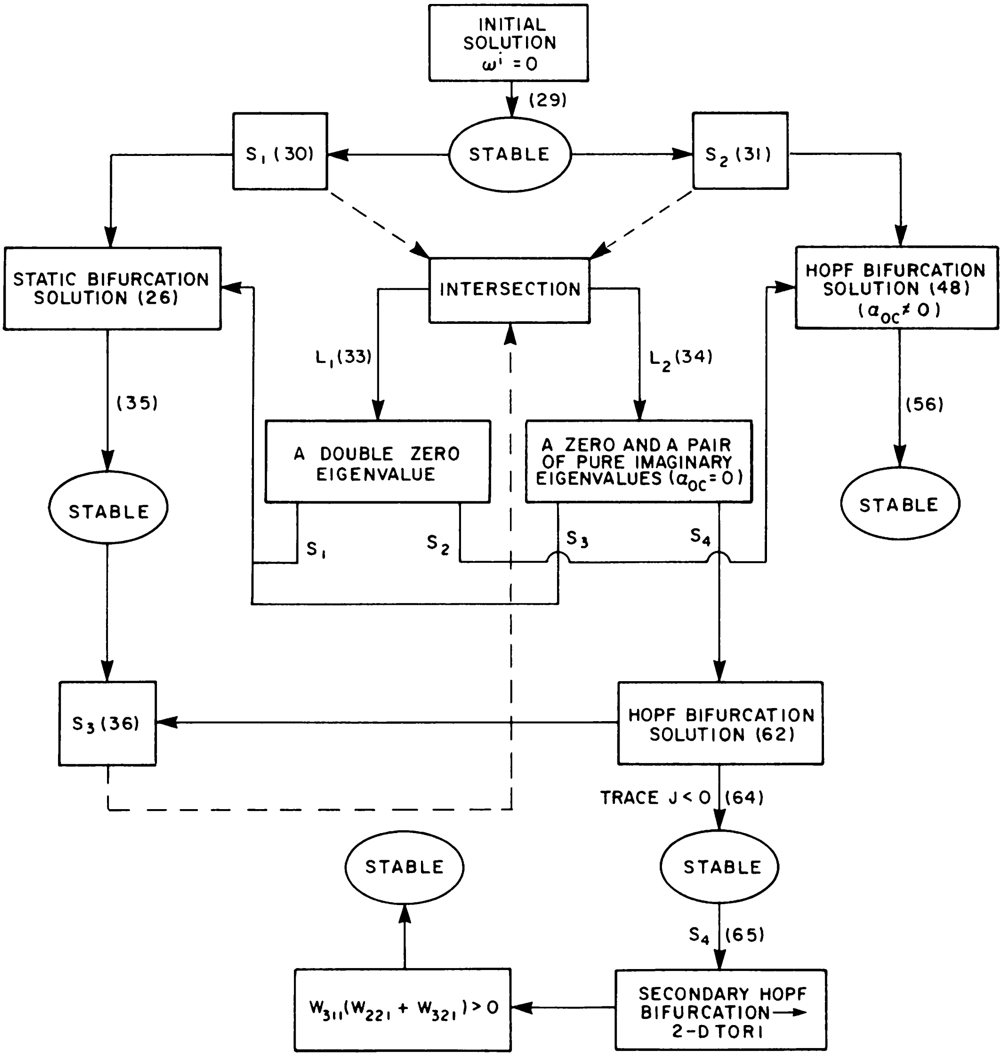

FIG 2. Bifurcation flow chart.

and C_2, respectively, are chosen as the state variables, and the state equations of the network are described by

$$\begin{aligned} L\frac{di_L}{dt} &= -R_1 i_L - v_{C_1}, \\ C_1\frac{dv_{C_1}}{dt} &= -f_1(v_{C_1}) + i_L - \frac{1}{R_2}v_{C_1} + \frac{1}{R_2}v_{C_2}, \\ C_2\frac{dv_{C_2}}{dt} &= -f_2(v_{C_2}) + \frac{1}{R_2}v_{C_1} - \frac{1}{R_2}v_{C_2}, \end{aligned} \tag{74}$$

where the functions f_1 and f_2 describe the characteristics of the tunnel-diode and the conductance, respectively, which are given by

$$f_1(v_{C_1}) = -\eta^1 v_{C_1} + (v_{C_1})^3, \tag{75}$$

and

$$\begin{aligned} f_2(v_{C_2}) &= 4.054054\left[0.01776 v_{C_2} - 0.10379(v_{C_2})^2 + 0.22962(v_{C_1})^3 - 0.22631(v_{C_2})^4 \right. \\ &\quad \left. + 0.08372(v_{C_2})^5\right] \\ &= \frac{9}{125} v_{C_2} - \frac{21}{50}(v_{C_2})^2 + \frac{14}{15}(v_{C_2})^3 - \frac{23}{25}(v_{C_2})^4 + \frac{17}{50}(v_{C_2})^5, \end{aligned} \tag{76}$$

where η^1 is a certain control parameter. The function f_1 is depicted in Fig. 4 for $\eta^1 > 0$.

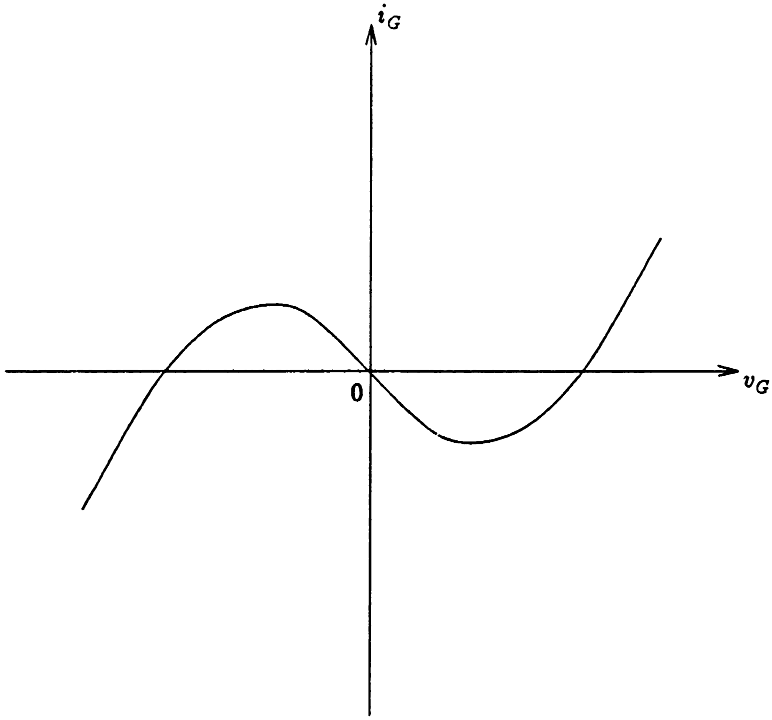

FIG 4. The characteristic of a nonlinear control element.

Denoting the state variables i_L, v_{C_1}, and v_{C_2} by z^1, z^2, and z^3, respectively, and assuming that L, C_1, and C_2 have the corresponding unit values, respectively, one obtains the following equations:

$$\begin{aligned} \frac{dz^1}{dt} &= -\eta^2 z^1 - z^2, \\ \frac{dz^2}{dt} &= z^1 + (\eta^1 - \eta^3) z^2 + \eta^3 z^3, \\ \frac{dz^3}{dt} &= \eta^3 z^2 - \left(\frac{9}{125} + \eta^3\right) z^3 + \frac{21}{50}(z^3)^2, \end{aligned} \tag{77}$$

where the third and higher-order terms have been truncated, and R_1 and R_2 are treated as additional control parameters, which are therefore replaced by η^2 and η^3.

The initial equilibrium solution is described by $z^i = 0$ (since $z^i = 0$ yields $dz^i/dt = 0$ for all values of η^β). The Jacobian matrix of (77) evaluated on this solution takes the form

$$J = \begin{bmatrix} -\eta^2 & -1 & 0 \\ 1 & \eta^1 - \eta^3 & \eta^3 \\ 0 & \eta^3 & -\left(\frac{9}{125} + \eta^3\right) \end{bmatrix}. \tag{78}$$

It can be shown that at the critical point c, defined by $\eta_c^1 = 163/125, \eta_c^2 = 4/5$, and $\eta_c^3 = 27/125$, the Jacobian (78) has a 3-fold zero eigenvalue with index one. In order to use the formulas obtained in the theory, it is required to transform system (77) to a new system such that its Jacobian will be in the canonical form (5). To this end, using the transformation of the state variables,

$$\begin{bmatrix} z^1 \\ z^2 \\ z^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{162}{125} & -\frac{12}{5} & 3 \\ \frac{648}{625} & \frac{402}{125} & 0 \\ \frac{486}{625} & -\frac{36}{125} & 1 \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix}, \tag{79}$$

and the transformation of the parameters,

$$\eta^1 = \frac{163}{125} + \mu^1, \qquad \eta^2 = \frac{4}{5} + \mu^2, \qquad \eta^3 = \frac{27}{125} + \mu^3, \tag{80}$$

yields the system

$$\begin{aligned} \frac{dw^1}{dt} &= w^2 - \frac{1}{810}(128\mu^1 + 335\mu^2 - 233\mu^3)w^1 + \frac{37989}{156250}(w^1)^2, \\ \frac{dw^2}{dt} &= w^3 + \frac{1}{75}(28\mu^1 + 10\mu^2 - 13\mu^3)w^1 + \frac{1}{405}(469\mu^1 + 100\mu^2 - 949\mu^3)w^2 \\ &\quad - \frac{30618}{390625}(w^1)^2 + \frac{4536}{78125}w^1w^2, \\ \frac{dw^3}{dt} &= \frac{9}{125}(16\mu^1 + 25\mu^2 - \mu^3)w^1 + \frac{1}{375}(268\mu^1 + 250\mu^2 - 103\mu^3)w^2 \\ &\quad - \frac{1}{18}(15\mu^2 - \mu^3)w^3 \\ &\quad + \frac{413343}{9765625}(w^1)^2 - \frac{61236}{1953125}w^1w^2 + \frac{1701}{15625}w^1w^3, \end{aligned} \tag{81}$$

whose Jacobian matrix evaluated at the critical point c is in the canonical form (5).

Furthermore, applying the transformation (23), which is now given by

$$\begin{aligned} y^1 &= w^1, \\ y^2 &= w^2 - \frac{1}{810}(128\mu^1 + 335\mu^2 - 233\mu^3)w^1 + \frac{37989}{156250}(w^1)^2, \\ y^3 &= w^3 + \frac{1}{75}(28\mu^1 + 10\mu^2 - 13\mu^3)w^1 + \frac{1}{18}(18\mu^1 - 3\mu^2 - 37\mu^3)w^2 \\ &\quad - \frac{30618}{390625}(w^1)^2 + \frac{1701}{3125}w^1w^2, \end{aligned} \tag{82}$$

(79) takes the form

$$\begin{aligned}\frac{dy^1}{dt} &= y^2,\\ \frac{dy^2}{dt} &= y^3,\\ \frac{dy^3}{dt} &= \frac{9}{125}(16\mu^1+25\mu^2-\mu^3)y^1+\frac{4}{125}(34\mu^1+25\mu^2-14\mu^3)y^2\\ &\quad+(\mu^1-\mu^2-2\mu^3)y^3+\frac{4133443}{9765625}(y^1)^2-\frac{367416}{1953125}y^1y^2+\frac{10206}{15625}y^1y^3.\end{aligned} \tag{83}$$

Now, one may apply the formulas (25)–(56) to system (81) to obtain the following results.

The stable region of the fundamental equilibrium solution,

$$y^1=y^2=y^3=0, \tag{84}$$

is defined by

$$16\mu^1+25\mu^2-\mu^3<0, \qquad 34\mu^1+25\mu^2-14\mu^3<0, \qquad \mu^1-\mu^2-2\mu^3<0,$$

and

$$20(\mu^1-\mu^2-2\mu^3)(34\mu^1+25\mu^2-14\mu^3)+9(16\mu^1+25\mu^2-\mu^3)>0, \tag{85}$$

which leads to two critical surfaces. One of these is given by

$$S_1\colon\ 16\mu^1+25\mu^2-\mu^3=0 \quad (34\mu^1+25\mu^2-14\mu^3<0 \ \text{ and } \ \mu^1-\mu^2-2\mu^3<0), \tag{86}$$

along which a static bifurcation, described by

$$y^1=-\frac{15625}{45927}(16\mu^1+25\mu^2-\mu^3), \qquad y^2=y^3=0, \tag{87}$$

occurs from the fundamental equilibrium surface.

Another critical surface is expressed by (31), which takes the form

$$\begin{aligned}S_2\colon\ &20(\mu^1-\mu^2-2\mu^3)(34\mu^1+25\mu^2-14\mu^3)+9(16\mu^1+25\mu^2-\mu^3)=0,\\ &\qquad(\mu^1-\mu^2-2\mu^3<0 \ \text{ and } \ 34\mu^1+25\mu^2-14\mu^3<0)\end{aligned} \tag{88}$$

and describes the onset of Hopf bifurcations from the fundamental equilibrium surface. The frequency of the periodic solutions is given by (32) as

$$\omega_{\bar c}=\frac{2}{25}\sqrt{-5(34\mu^1_{\bar c}+25\mu^2_{\bar c}-14\mu^3_{\bar c})} \quad (34\mu^1_{\bar c}+25\mu^2_{\bar c}-14\mu^3_{\bar c}<0). \tag{89}$$

Equations (33) give the stability conditions for the static bifurcation solution (87) as

$$16\mu^1+25\mu^2-\mu^3>0, \qquad 66\mu^1+75\mu^2-16\mu^3<0, \qquad 23\mu^1+59\mu^2+16\mu^3>0,$$

and

$$20(66\mu^1+75\mu^2-16\mu^3)(23\mu^1+59\mu^2+16\mu^3)+81(16\mu^1+25\mu^2-\mu^3)<0, \tag{90}$$

which results in the third critical surface

$$\begin{aligned}S_3\colon\ &20(66\mu^1+75\mu^2-16\mu^3)(23\mu^1+59\mu^2+16\mu^3)+81(16\mu^1+25\mu^2-\mu^3)=0,\\ &\qquad(66\mu^1+75\mu^2-16\mu^3<0 \ \text{ and } \ 23\mu^1+59\mu^2+16\mu^3>0),\end{aligned} \tag{91}$$

along which a secondary Hopf bifurcation from the bifurcating solution (87) takes place. The frequency of the periodic solutions is expressed by (35), which now takes the form

$$\omega_{\bar{c}} = \frac{2}{25}\sqrt{-(66\mu^1_{\bar{c}} + 75\mu^2_{\bar{c}} - 29\mu^3_{\bar{c}})} \quad (66\mu^1_{\bar{c}} + 75\mu^2_{\bar{c}} - 29\mu^3_{\bar{c}} < 0). \tag{92}$$

Finally, applying (41), (44), and (48) to system (83), one obtains the asymptotic solutions of the limit cycles bifurcating from the fundamental equilibrium surface along the critical surface S_2 as follows:

$$\begin{aligned} y^1 &= -\rho\sin\omega t - \frac{78125}{45927}\frac{\omega^2_{\bar{c}}(\alpha^2_{o\bar{c}} + 4\omega^2_{\bar{c}})}{\alpha^4_{o\bar{c}} + 7\alpha^2_{o\bar{c}}\omega^2_{\bar{c}} + 8\omega^2_{\bar{c}}} f(\varphi^\beta), \\ y^2 &= \omega_{\bar{c}}\rho\cos\omega t - \frac{78125}{45927}\frac{\alpha_{o\bar{c}}\omega^2_{\bar{c}}(\alpha^2_{o\bar{c}} + 4\omega^2_{\bar{c}})}{\alpha^4_{o\bar{c}} + 7\alpha^2_{o\bar{c}}\omega^2_{\bar{c}} + 8\omega^2_{\bar{c}}} f(\varphi^\beta), \\ y^3 &= -\omega^2_{\bar{c}}\rho\sin\omega t - \frac{78125}{45927}\frac{\alpha^2_{o\bar{c}}\omega^2_{\bar{c}}(\alpha^2_{o\bar{c}} + 4\omega^2_{\bar{c}})}{\alpha^4_{o\bar{c}} + 7\alpha^2_{o\bar{c}}\omega^2_{\bar{c}} + 8\omega^2_{\bar{c}}} f(\varphi^\beta), \end{aligned} \tag{93}$$

where

$$\alpha_{o\bar{c}} = \mu^1_{\bar{c}} - \mu^2_{\bar{c}} - 2\mu^3_{\bar{c}} < 0, \qquad \omega^2_{\bar{c}} = -\frac{4}{125}(34\mu^1_{\bar{c}} + 25\mu^2_{\bar{c}} - 14\mu^3_{\bar{c}}), \tag{94}$$

$$f(\varphi^\beta) = \left(16 + \frac{136}{125}\alpha_{o\bar{c}} - \omega^2_{\bar{c}}\right)\varphi^1 + \left(25 + \frac{4}{5}\alpha_{o\bar{c}} + \omega^2_{\bar{c}}\right)\varphi^2 - \left(1 + \frac{56}{125}\alpha_{o\bar{c}} - 2\omega^2_{\bar{c}}\right)\varphi^3, \tag{95}$$

and the $\bar{c}$ indicates a point on the critical surface (76).

The frequency ω in (93) is determined by $\omega = d\theta/dt$ from (46) and the amplitude ρ is given by

$$\rho^2 = \frac{2}{5}\left(\frac{1953125}{137781}\right)^2 \frac{\alpha_{o\bar{c}}\omega^2_{\bar{c}}(\alpha^2_{o\bar{c}} + \omega^2_{\bar{c}})(\alpha^2_{o\bar{c}} + 4\omega^2_{\bar{c}})}{\alpha^4_{o\bar{c}} + 7\alpha^2_{o\bar{c}}\omega^2_{\bar{c}} + 8\omega^4_{\bar{c}}} f(\varphi^\beta). \tag{96}$$

The stability conditions associated with the Hopf bifurcation solution (93) can be obtained from (56) as follows:

$$\alpha_{o\bar{c}}\omega^2_{\bar{c}}(\alpha^2_{o\bar{c}} + 2\omega^2_{\bar{c}}) + 1.5(\alpha^4_{o\bar{c}} + 7\alpha^2_{o\bar{c}}\omega^2_{\bar{c}} + 8\omega^4_{\bar{c}}) > 0. \tag{97}$$

Next, (65) gives the critical surface

$$S_4\colon 459283\varphi^1 - 9874318\varphi^2 - 3903808\varphi^3 = 0, \tag{98}$$

along which a secondary Hopf bifurcation occurs from the first Hopf bifurcation solution (81), leading to a family of two-dimensional tori. Here,

$$\omega^2_{\bar{c}} = -\frac{4}{125}(34\mu^1_{\bar{c}} + 25\mu^2_{\bar{c}} - 14\mu^3_{\bar{c}}) \tag{99}$$

and $\bar{c}$ denotes a point on the critical line L_2 determined by

$$L_2\colon 16\mu^1 + 25\mu^2 - \mu^3 = 0 \quad \text{and} \quad \mu^1 - \mu^2 - 2\mu^3 = 0 \quad (34\mu^1 + 25\mu^2 - 14\mu^3 < 0). \tag{100}$$

The second frequencies of this family of tori are given by (66) as

$$\omega_{c'} = \frac{\rho}{9765625\omega^2_{\bar{c}}}\left[(4133443 - 3189375\omega^2_{\bar{c}})(4133443 - 6378750\omega^2_{\bar{c}})\right]^{1/2}, \tag{101}$$

where ρ can be obtained from (62) as

$$\rho^2 = \frac{1}{10}\left(\frac{1953125}{4133443}\right)^2 \left[81(16\varphi^1 + 25\varphi^2 - \varphi^3)^2 - 15625(\varphi^1 - \varphi^2 - 2\varphi^3)^2\right]. \quad (102)$$

The stability condition (73) gives

$$W_{311}(W_{211} + W_{321}) = -\frac{4133443}{9765625}\frac{734832}{1953125} < 0; \quad (103)$$

therefore, the bifurcating two-dimensional tori are unstable.

Appendix. In this appendix, it will be demonstrated that the differential equation (22) is equivalent to the original system (4) up to first order. Before tackling this problem, however, consider a simpler case associated with a double zero eigenvalue problem (see [6]).

Suppose the system

$$\frac{dw^i}{dt} = W_i(w^j; \eta^\beta) \quad (j = 1, 2; \beta = 1, 2) \quad (A1)$$

has a Jacobian (evaluated at a critical point c) in the canonical form

$$J = \left[W_{ij}\right]_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (A2)$$

and the initial equilibrium solution is given by

$$w^i = 0. \quad (A3)$$

The simplified differential equations in this case [6] are described by

$$\begin{aligned} \frac{dw^1}{dt} &= w^2 + W_{11\beta}\mu^\beta w^1 + \tfrac{1}{2}W_{111}(w^1)^2, \\ \frac{dw^2}{dt} &= W_{21\beta}\mu^\beta w^1 + W_{22\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{211}(w^1) + W_{212}w^1w^2. \end{aligned} \quad (A4)$$

Introducing the transformation

$$\begin{aligned} w^1 &= y^1, \\ w^2 &= y^2 - W_{11\beta}\mu^\beta y^1 - \tfrac{1}{2}W_{111}(y^1)^2 \end{aligned} \quad (A5)$$

into (A4), one obtains

$$\begin{aligned} \frac{dy^1}{dt} &= y^2, \\ \frac{dy^2}{dt} &= W_{21\beta}\mu^\beta y^1 + (W_{11\beta} + W_{22\beta})\mu^\beta y^2 + \tfrac{1}{2}W_{211}(y^1)^2 + (W_{111} + W_{212})y^1y^2, \end{aligned} \quad (A6)$$

which is equivalent to (A4) up to second-order terms.

In order to demonstrate that the local dynamics and bifurcation behavior of the original system (A1) is embraced by the differential equation (A6), let (A1) be expanded into a Taylor series around the critical point $(w^i; \eta^\beta) = (0; \eta_c^\beta)$, which takes the form

$$\begin{aligned}\frac{dw^1}{dt} &= w^2 + W_{11\beta}\mu^\beta w^1 + W_{12\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{111}(w^1)^2 + W_{112}w^1w^2 \\ &\quad + \tfrac{1}{2}W_{122}(w^2)^2 + O\left(\left|(w^i;\mu^\beta)\right|^3\right), \\ \frac{dw^2}{dt} &= W_{21\beta}\mu^\beta w^1 + W_{22\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{211}(w^1)^2 + W_{212}w^1w^2 \\ &\quad + \tfrac{1}{2}W_{222}(w^2)^2 + O\left(\left|(w^i;\mu^\beta)\right|^3\right)\end{aligned} \tag{A7}$$

upon using (A2).

Next, introducing the transformation

$$\begin{aligned} w^1 &= x^1 + \tfrac{1}{2}(W_{112} + \tfrac{1}{2}W_{222})(x^1)^2 + \tfrac{1}{2}W_{122}x^1x^2 + W_{12\beta}\mu^\beta x^1, \\ w^2 &= x^2 - \tfrac{1}{2}W_{111}(x^1)^2 + \tfrac{1}{2}W_{222}x^1x^2 - W_{11\beta}\mu^\beta x^1\end{aligned} \tag{A8}$$

into (A7), and keeping terms up to second order in the resulting equations, one obtains

$$\begin{aligned}\frac{dx^1}{dt} &= x^2, \\ \frac{dx^2}{dt} &= W_{21\beta}\mu^\beta x^1 + (W_{11\beta} + W_{22\beta})\mu^\beta x^2 + \tfrac{1}{2}W_{211}(x^1)^2 + (W_{111} + W_{212})x^1x^2,\end{aligned} \tag{A9}$$

which is identical to (A6).

It is observed, by comparing the transformation (A5) with the transformation (A8), that the first equation of (A5) approximates the first equation of (A8) up to first order, and the second equation of (A5) approximates the second equation of (A8) up to second order.

Now, consider the original system (4) and the differential equation (24) which is equivalent to (22) up to second-order terms via (23).

The Jacobian of (4) evaluated at the critical point c is in the canonical form

$$J = [W_{ij}]_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A10}$$

and the initial equilibrium solution is given by $w^i = 0$.

Expanding (A1) into a Taylor series around the critical point c (where $\eta^\beta = \eta_c^\beta$) with the aid of (A10) gives

$$\begin{aligned}\frac{dw^1}{dt} &= w^2 + W_{11\beta}\mu^\beta w^1 + W_{12\beta}\mu^\beta w^2 + W_{13\beta}\mu^\beta w^3 + \tfrac{1}{2}W_{111}(w^1)^2 + \tfrac{1}{2}W_{122}(w^2)^2 \\ &\quad + \tfrac{1}{2}W_{133}(w^3)^2 + W_{112}w^1w^2 + W_{113}w^1w^3 + W_{123}w^2w^3 + O\left(\left|(w^i;\mu^\beta)\right|^3\right), \\ \frac{dw^2}{dt} &= w^3 + W_{21\beta}\mu^\beta w^1 + W_{22\beta}\mu^\beta w^2 + W_{23\beta}\mu^\beta w^3 + \tfrac{1}{2}W_{211}(w^1)^2 + \tfrac{1}{2}W_{222}(w^2)^2 \\ &\quad + \tfrac{1}{2}W_{233}(w^3)^2 + W_{212}w^1w^2 + W_{213}w^1w^3 + W_{223}w^2w^3 + O\left(\left|(w^i;\mu^\beta)\right|^3\right),\end{aligned}$$

$$\frac{dw^3}{dt} = W_{31\beta}\mu^\beta w^1 + W_{32\beta}\mu^\beta w^2 + W_{33\beta}\mu^\beta w^2 + \tfrac{1}{2}W_{311}(w^1)^2 + \tfrac{1}{2}W_{322}(w^2)^2 + \tfrac{1}{2}W_{333}(w^3)^2 + W_{312}w^1w^2 + W_{313}w^1w^3 + W_{323}w^2w^3 + O\left(\left|(w^i;\mu^\beta)\right|^3\right). \quad \text{(A11)}$$

Similarly, introducing a transformation

$$\begin{aligned} w^1 &= x^1 + W_{12\beta}\mu^\beta x^1 + W_{13\beta}\mu^\beta x^2 + \tfrac{1}{2}W_{112}(x^1)^2 + \tfrac{1}{2}W_{123}(x^2)^2 \\ &\quad + \tfrac{1}{3}(W_{122} + W_{113} + \tfrac{1}{2}W_{333})x^1x^2 + \tfrac{1}{2}W_{133}x^2x^3, \\ w^2 &= x^2 - W_{11\beta}\mu^\beta x^1 - \tfrac{1}{2}W_{111}(x^1)^2 + \tfrac{1}{3}(W_{113} + W_{223} + \tfrac{1}{2}W_{333} - \tfrac{1}{2}W_{122})(x^2)^2 \\ &\quad + \tfrac{1}{3}(W_{222} + W_{213} + W_{323})x^1x^2 + \tfrac{1}{3}(W_{122} + W_{223} + \tfrac{1}{2}W_{333} - 2W_{113})x^1x^3, \\ w^3 &= x^3 - W_{21\beta}\mu^\beta x^1 - (W_{11\beta} + W_{22\beta})\mu^\beta x^2 - W_{23\beta}\mu^\beta x^3 - \tfrac{1}{2}W_{211}(x^1)^2 - \tfrac{1}{2}W_{233}(x^3)^2 \\ &\quad - \tfrac{1}{2}W_{333}x^2x^3 + \tfrac{1}{3}(W_{213} + W_{323} - \tfrac{1}{2}W_{222})(x^2)^2 - (W_{111} + W_{212})x^1x^2 \\ &\quad + \tfrac{1}{3}(W_{222} + W_{323} - 2W_{213})x^1x^3 \end{aligned} \quad \text{(A12)}$$

into (A11), one obtains a transformed system

$$\begin{aligned} \frac{dx^1}{dt} &= x^2, \\ \frac{dx^2}{dt} &= x^3, \\ \frac{dx^3}{dt} &= W_{31\beta}\mu^\beta x^1 + (W_{21\beta} + W_{32\beta})\mu^\beta x^2 + (W_{11\beta} + W_{22\beta} + W_{33\beta})\mu^\beta x^3 + \tfrac{1}{2}W_{311}(x^1)^2 \\ &\quad + (W_{211} + W_{312})x^1x^2 + (W_{111} + W_{212} + W_{313})x^1x^3 \\ &\quad + (W_{111} + W_{212} + \tfrac{1}{2}W_{322})(x^2)^2. \end{aligned} \quad \text{(A13)}$$

It is not difficult to verify that dropping the last term in the third equation, $(W_{111} + W_{212} + \frac{1}{2}W_{322})(x^2)^2$, does not have an effect on the analysis presented in this paper. Thus, (A13) becomes identical to (24). Also, note that (23) is a first-order approximation for the nonlinear transformation (A12).

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