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QUASISTATIC PROCESSES FOR ELASTIC-VISCOPLASTIC MATERIALS*

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Introduction. An initial and boundary value problem for materials with a rate-type constitutive equation of the form[1]

$$\dot{\sigma} = \mathscr{E}\dot{\varepsilon} + F(\sigma, \varepsilon) \tag{0.1}$$

is considered. Such equations are used in order to describe the behavior of real materials like rubbers, metals, rocks, and so on. Various results and mechanical interpretations concerning constitutive equations of the form (0.1) may be found, for instance, in the papers of Freudenthal and Geringer [8], Cristescu and Suliciu [3], Gurtin, Williams, and Suliciu [9], Suliciu [16], and Podio-Guidugli and Suliciu [15].

For particular forms of F, Eq. (0.1) reduces to some classical models used in viscoelasticity and viscoplasticity. In the case when F depends only on σ, existence results for such materials may be found, for instance, in Duvaut and Lions [6], Djaoua and Suquet [5], Anzellotti [1], and Anzellotti and Giaquinta [2].

In this paper we are interested in existence results for materials obeying (0.1) for which a full coupling in stress and strain is involved in F. A relatively simple one-dimensional example of a constitutive equation of this type (see Cristescu and Suliciu [3, p. 35]) is

$$F(\varepsilon, \sigma) = \begin{cases} -k_1 F_1(\sigma - g_1(\varepsilon)) & \text{if } \sigma \geq g_1(\varepsilon), \\ 0 & \text{if } g_2(\varepsilon) < \sigma < g_1(\varepsilon), \\ k_2 F_2(g_2(\varepsilon) - \sigma) & \text{if } \sigma \leq g_2(\varepsilon), \end{cases} \tag{0.2}$$

where $k_1, k_2 > 0$ are viscosity constants and F_1, F_2 are increasing functions with $F_1(0) = F_2(0) = 0$.

The function F is supposed to be Lipschitz continuous, and no monotony properties of F are required in order to obtain an existence and uniqueness result (Theorem 3.1) and the continuous dependence of the solution upon initial and boundary data (Theorem 4.1). Since F depends both on σ and ε, the monotony arguments used in the above-mentioned papers do not work. For this reason a different technique is

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[1] Everywhere in this paper the dot represents the derivative with respect to the time variable.

used, based on the equivalence between the studied problem and an ordinary differential equation in a Hilbert space. The reduction of an evolution problem concerning elastic viscoplastic materials with internal variables to an ordinary differential equation was also used by Nečas and Kratochvil [19].[2]

In the papers of Suliciu [16] and Podio-Guidugli and Suliciu [15], it is assumed that there exists a strong monotone function G such that $F(\sigma, \varepsilon) = 0$ iff $\sigma = G(\varepsilon)$. Starting from this assumption in order to get a better insight on the model and on its connection with elasticity, we particularize F in (0.1) as

$$F(\sigma, \varepsilon) = -k(\sigma - G(\varepsilon)), \tag{0.3}$$

where $k > 0$ is a viscosity coefficient. Let us remark that if in (0.2) we put $k_1 = k_2 = k$, $F_1 = F_2 =$identity, and $g_1 = g_2 = G$ we again obtain (0.3).

In this case the asymptotic stability of every solution is obtained (Corollary 4.2), and for periodic external data, the existence of a unique periodic solution is proved (Theorem 5.1).

The study of the asymptotic behavior of the solution upon the viscosity coefficient k shows that elasticity is a proper asymptotic theory for viscoelastic materials described by (0.1) and (0.2) (Theorem 6.1). Finally it is proved that the solution of the elastic problem can characterize in some cases the large time behavior of the solution of the viscoelastic problem (Theorem 7.1).

1. Problem statement. Let Ω be a bounded domain in R^n $(n = 1, 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$, and let Γ_1 be an open subset of Γ and $\Gamma_2 = \Gamma - \overline{\Gamma}_1$. We suppose $\operatorname{mes}\Gamma_1 > 0$. Let us consider the following mixed problem:

Find the displacement function $u : R_+ \times \Omega \to R^n$ and the stress function $\sigma : R_+ \times \Omega \to \mathscr{S}$ such that

$$\operatorname{div}\sigma(t) + b(t) = 0, \tag{1.1}$$

$$\varepsilon(u(t)) = \tfrac{1}{2}(\nabla u(t) + \nabla^{\mathrm{T}} u(t)), \tag{1.2}$$

$$\dot{\sigma}(t) = \mathscr{E}\varepsilon(\dot{u}(t)) + F(\sigma(t), \varepsilon(u(t))) \quad \text{in } \Omega, \tag{1.3}$$

$$u(t)\,|_{\Gamma_1} = g(t), \tag{1.4}$$

$$\sigma(t)\upsilon\,|_{\Gamma_2} = f(t) \quad \text{for all } t > 0,. \tag{1.5}$$

$$u(0) = u_0, \tag{1.6}$$

$$\sigma(0) = \sigma_0 \quad \text{in } \Omega, \tag{1.7}$$

where $\mathscr{S}$ is the set of second-order symmetric tensors on R^n and υ is the exterior unit normal at Γ. The equations (1.1) are Cauchy's equilibrium equations in which $b : R_+ \times \Omega \to R^n$ is the given body force, and (1.2) defines the strain tensor of small deformations. (1.3) represents a rate-type viscoelastic or viscoplastic constitutive equation in which $\mathscr{E}$ is a fourth-order tensor and $F : \Omega\times\mathscr{S}\times\mathscr{S} \to \mathscr{S}$ is a constitutive function. The functions u_0 and σ_0 are the initial data and f, g are the given boundary data.

[2] It seems that this technique was also used by Laborde [20] and Suquet [21].

2. Notations and preliminaries. We denote by $\cdot$ the inner product on the spaces R^n and $\mathcal{S}$ and by $|\cdot|$ the euclidean norms on these spaces. The following notations are used:

$$\begin{aligned}\mathcal{L} &= \{\tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega), i, j = 1, \dots, n\},\\ L &= \{u = (u_i) \mid u_i \in L^2(\Omega), i = 1, \dots, n\},\\ \mathcal{H} &= \{T \in \mathcal{L} \mid \operatorname{div} T \in L\},\\ H &= \{u = (u_i) \mid u_i \in H^1(\Omega), i = 1, \dots, n\}.\end{aligned}$$

The spaces $\mathcal{L}, L, \mathcal{H}, H$ are Hilbert spaces with respect to the canonical inner products given by

$$(\sigma, \tau) = \int_\Omega \sigma \cdot \tau \, dx, \tag{2.1}$$

$$((u, v)) = \int_\Omega u \cdot v \, dx, \tag{2.2}$$

$$(\tau, \sigma)_d = (\tau, \sigma) + ((\operatorname{div} \tau, \operatorname{div} \sigma)), \tag{2.3}$$

$$(u, v)_H = ((u, v)) + (\nabla u, \nabla v). \tag{2.4}$$

The norms induced by (2.1)–(2.4) will be denoted by $\|\cdot\|$, $|||\cdot|||$, $\|\cdot\|_d$, $\|\cdot\|_H$ respectively.

Let $\gamma_0 : H \to H_\Gamma$ be the trace map, where we denote by H_Γ the space $(H^{1/2}(\Omega))^n$ and its norm by $\|\cdot\|_\Gamma$. Let V_1 be the subspace of H given by $V_1 = \{u \in H \mid \gamma_0(u) = 0$ on $\Gamma_1\}$, and let E be the subspace of H_Γ defined by $E = \gamma_0(V_1) = \{\zeta \in H_\Gamma \mid \zeta = 0$ on $\Gamma_1\}$. The operator $\varepsilon : H \to \mathcal{L}$ given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^{\mathrm{T}} u) \tag{2.5}$$

is linear and continuous. Moreover, since $\operatorname{mes} \Gamma_1 > 0$, Korn's inequality holds:

$$\|\varepsilon(u)\| \geq C\|u\|_H \quad \text{for all } u \in V_1, \tag{2.6}$$

where $C > 0$ is a positive constant.

Everywhere in this paper $C, \overline{C}, C_i$, $i \in N$ will represent strictly positive generic constants that depend on $\mathcal{E}, F, \Omega, \Gamma_1, \Gamma_2$ and do not depend on time or on input data.

If $\sigma \in \mathcal{H}$ then there exists $\gamma_\upsilon \sigma \in H'_\Gamma$ (where $(H'_\Gamma, \|\cdot\|_0)$ is the strong dual of H_Γ) such that

$$\langle \gamma_\upsilon \sigma, \gamma_0 v \rangle = (\sigma, \varepsilon(v)) + ((\operatorname{div} \sigma, v)) \tag{2.7}$$

for all $v \in H$ and

$$\|\gamma_\upsilon \sigma\|_0 \leq C\|\sigma\|_d. \tag{2.8}$$

By $\sigma\upsilon \,|_{\Gamma_2}$ we shall understand the element of E' (the dual of E) that is the restriction of $\gamma_\upsilon \sigma$ on E. We denote by $\|\cdot\|_1$ the norm on E'.

Let us denote by V_2 the following subspace of $\mathcal{H}$: $V_2 = \{\sigma \in \mathcal{H} \mid \operatorname{div} \sigma = 0, \sigma\upsilon \,|_{\Gamma_2} = 0\}$; $\varepsilon(V_1)$ is the orthogonal complement of V_2 in $\mathcal{L}$. Hence,

$$(\tau, \varepsilon(v)) = 0 \quad \text{for all } v \in V_1, \tau \in V_2. \tag{2.9}$$

Let X be one of the above Hilbert spaces, and let us define the following spaces:

$$C^0(R_+, X) = \{z : R_+ \to X \mid z \text{ is continuous}\},$$
$$C^1(R_+, X) = \{z : R_+ \to X \mid \text{ there exists } \dot{z} \in C^0(R_+, X), \text{ the derivative of } z\},$$

where $R_+ = [0, +\infty)$. In a similar way the spaces $C^i(0, T, X)$, $i = 0, 1$, can be defined, and the norms on these spaces are given by

$$||z||_{T,X,0} = \max_{t\in[0,T]} ||z(t)||_X, \qquad ||z||_{T,X,1} = ||z||_{T,X,0} + ||\dot{z}||_{T,X,0}.$$

3. An existence and uniqueness result. The following hypotheses are considered: $\mathscr{E}$ is symmetric and positively defined, i.e.,

$$\begin{aligned}&\text{(a) } |\mathscr{E}_{ijkh}(x)| \le Q \text{ for all } i, j, k, h = 1, \dots, n, x \in \Omega;\\ &\text{(b) } \mathscr{E}(x)\sigma \cdot \varepsilon = \sigma \cdot \mathscr{E}(x)\varepsilon \text{ for all } \sigma, \varepsilon \in \mathscr{S}, x \in \Omega;\\ &\text{(c) there exists a strictly positive constant } d \text{ such that}\\ &\quad \text{for all } \sigma \in \mathscr{S}, x \in \Omega \text{ we have } \mathscr{E}(x)\sigma \cdot \sigma \ge d|\sigma|^2.\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\text{(a) } F \text{ is a Lipschitz function, i.e., there exists } L > 0 \text{ such that}\\ &\quad |F(x, \sigma_1, \varepsilon_1) - F(x, \sigma_2, \varepsilon_2)| \le L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|)\\ &\qquad \text{for all } \sigma_i, \varepsilon_i \in \mathscr{S}, i = 1, 2, x \in \Omega,\\ &\text{(b) } F(x, 0, 0) = 0 \text{ for all } x \in \Omega.\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\text{(a) } b \in C^1(R_+, L), f \in C^1(R_+, E');\\ &\text{(b) there exists } h \in C^1(R_+, H_\Gamma) \text{ such that } h = g \text{ on } \Gamma_1,\end{aligned} \tag{3.3}$$

$$u_0 \in H, \qquad \sigma_0 \in \mathscr{H}. \tag{3.4}$$

The initial conditions fit with boundary data, i.e.,

$$\begin{aligned}&\text{(a) } \operatorname{div} \sigma_0 + b(0) = 0;\\ &\text{(b) } \sigma_0 \upsilon |_{\Gamma_2} = f(0);\\ &\text{(c) } u_0 |_{\Gamma_1} = g(0).\end{aligned} \tag{3.5}$$

The main result of this section is given by

THEOREM 3.1. Suppose that the hypotheses (3.1)–(3.5) are fulfilled. Then there exists a unique solution $u \in C^1(R_+, H)$, $\sigma \in C^1(R_+, \mathscr{H})$ of the problem (1.1)–(1.7).

Remark 3.1. Let us observe that if the problem (1.1)–(1.7) has a solution (u, σ) such that $u \in C^1(R_+, H)$, $\sigma \in C^1(R_+, \mathscr{H})$ then the hypotheses (3.3)–(3.5) are fulfilled.

Remark 3.2. Let us consider $K \subset \mathscr{S}$ a convex closed set, $0 \in K$, $P_K : \mathscr{S} \to K$ the projector on K, and $\mu > 0$. If we put $F(\sigma, \varepsilon) = (-1/(2\mu))\mathscr{E}(\sigma - P_K\sigma)$, then (3.2) holds and problem (1.1)–(1.7) describes a quasistatic process for elastic-viscoplastic bodies studied using different methods by Duvaut and Lions [6, Chapter 5] and by Suquet [18, 17].

In order to prove Theorem 3.1 we need some preliminary results.

LEMMA 3.1. Let (3.1), (3.3) hold. Then there exists a unique couple of functions $\tilde{u} \in C^1(R_+, H)$, $\tilde{\sigma} \in C^1(R_+, \mathscr{H})$ such that

$$\operatorname{div} \tilde{\sigma}(t) + b(t) = 0, \tag{3.6}$$

$$\tilde{\sigma}(t) = \mathscr{E}\varepsilon(\tilde{u}(t)), \tag{3.7}$$

$$\tilde{u}(t)\,|_{\Gamma_1} = g(t), \tag{3.8}$$

$$\tilde{\sigma}(t)\upsilon\,|_{\Gamma_2} = f(t) \tag{3.9}$$

for all $t \in R_+$. Moreover, we have

$$\dot{\tilde{\sigma}}(t) = \mathscr{E}\varepsilon(\dot{\tilde{u}}(t)) \quad \text{for all } t \in R_+, \tag{3.10}$$

and if we denote by $\tilde{u}_i, \tilde{\sigma}_i$, $i = 1, 2$, the solution of (3.6)–(3.9) for the data b_i, f_i, g_i, the following inequalities hold:

$$\begin{aligned} &\|\tilde{u}_1(t) - \tilde{u}_2(t)\|_H + \|\tilde{\sigma}_1(t) - \tilde{\sigma}_2(t)\|_d \\ &\qquad \le C[\||b_1(t) - b_2(t)\|| + \|f_1(t) - f_2(t)\|_1 + \|h_1(t) - h_2(t)\|_\Gamma], \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\|\dot{\tilde{u}}_1(t) - \dot{\tilde{u}}_2(t)\|_H + \|\dot{\tilde{\sigma}}_1(t) - \dot{\tilde{\sigma}}_2(t)\|_d \\ &\qquad \le C[\||\dot{b}_1(t) - \dot{b}_2(t)\|| + \|\dot{f}_1(t) - \dot{f}_2(t)\|_1 + \|\dot{h}_1(t) - \dot{h}_2(t)\|_\Gamma] \end{aligned} \tag{3.12}$$

for all $t \in R_+$ (the constant C depends only on Ω, Γ_1, Q, and d).

Proof. The statement of the above lemma can be easily obtained using standard existence theorems for linear elasticity.

Denoting by $\overline{u}_0 = u_0 - \tilde{u}(0)$, $\overline{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0)$, let us homogenize the boundary conditions of (1.4)–(1.5) by considering the following problem:

Find $\overline{u} : R_+ \times \Omega \to R^n, \overline{\sigma} : R_+ \times \Omega \to \mathscr{S}$ such that

$$\operatorname{div} \overline{\sigma}(t) = 0, \tag{3.13}$$

$$\dot{\overline{\sigma}}(t) = \mathscr{E}\varepsilon(\dot{\overline{u}}(t)) + F(\overline{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\overline{u}(t)) + \varepsilon(\tilde{u}(t))), \tag{3.14}$$

$$\overline{u}(t)\,|_{\Gamma_1} = 0, \tag{3.15}$$

$$\overline{\sigma}(t)\upsilon\,|_{\Gamma_2} = 0 \quad \text{for all } t > 0, \tag{3.16}$$

$$\overline{u}(0) = \overline{u}_0, \tag{3.17}$$

$$\overline{\sigma}(0) = \overline{\sigma}_0. \tag{3.18}$$

The following lemma can be easily obtained:

LEMMA 3.2. The pair (u, σ) is a solution of (1.1)–(1.7) iff the pair $(\overline{u}, \overline{\sigma})$ defined by $\overline{u} = u - \tilde{u}$, $\overline{\sigma} = \sigma - \tilde{\sigma}$ is a solution of (3.13)–(3.18).

Let $V = V_1 \times V_2$ be the product space with the norm denoted by $\|\cdot\|_V$ which is given by the following inner product:

$$(x, y)_V = (\mathscr{E}\varepsilon(u), \varepsilon(v)) + (\mathscr{E}^{-1}\sigma, \tau) \tag{3.19}$$

for all $x, y \in V$, $x = (u, \sigma)$, $y = (v, \tau)$.

Using (3.1) and (2.6) we observe that $\|\cdot\|_V$ is equivalent with the natural norm on V. Let $A : R_+ \times V \to V$ be the operator defined as follows:

$$\begin{aligned} (A(t, x), y)_V = &- (F(\sigma + \tilde{\sigma}(t), \varepsilon(u) + \varepsilon(\tilde{u}(t))), \varepsilon(v)) \\ &+ (\mathscr{E}^{-1}F(\sigma + \tilde{\sigma}(t), \varepsilon(u) + \varepsilon(\tilde{u}(t))), \tau), \end{aligned} \tag{3.20}$$

for all $x, y \in V$, $x = (u, \sigma)$, $y = (v, \tau)$, $t \in R_+$. We let $x_0 = (\overline{u}_0, \overline{\sigma}_0)$, and using (3.5) we get $x_0 \in V$.

LEMMA 3.3. The pair $(\overline{u}, \overline{\sigma}) \in C^1(R_+, H \times \mathscr{H})$ is a solution of (3.13)–(3.18) iff $x = (\overline{u}, \overline{\sigma}) \in C^1(R_+, V)$ is a solution of the following Cauchy problem:

$$\dot{x}(t) = A(t, x(t)) \quad \text{for all } t > 0, \tag{3.21}$$

$$x(0) = x_0. \tag{3.22}$$

Proof. Let $y = (v, \tau) \in V$. Multiplying (3.14) by $\varepsilon(v)$, after integrating on Ω and using (2.9) we get

$$0 = (\mathscr{E}\varepsilon(\dot{\overline{u}}(t)), \varepsilon(v)) + (F(\overline{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\overline{u}(t)) + \varepsilon(\tilde{u}(t))), \varepsilon(v)). \tag{3.23}$$

Multiplying (3.14) by $\mathscr{E}^{-1}$ on the left and by τ on the right and integrating on Ω, from (2.9) we obtain

$$(\mathscr{E}^{-1}\dot{\overline{\sigma}}(t), \tau) = (\mathscr{E}^{-1}F(\overline{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\overline{u}(t)) + \varepsilon(\tilde{u}(t))), \tau). \tag{3.24}$$

From (3.23), (3.24), (3.20), and (3.19) we get

$$(\dot{x}(t), y)_V = (A(t, x(t)), y)_V \quad \text{for all } y \in V. \tag{3.25}$$

Conversely, let $x = (u, \sigma) \in C^1(R_+, V)$ be a solution of (3.21)–(3.22). Let $B(t) \in \mathscr{L}$ be given by

$$B(t) = \dot{\overline{\sigma}}(t) - \mathscr{E}\varepsilon(\dot{\overline{u}}(t)) - F(\overline{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\overline{u}(t)) + \varepsilon(\tilde{u}(t))). \tag{3.26}$$

Taking $y = (v, 0)$ in (3.25) and using (2.9) we get

$$(B(t), \varepsilon(v)) = 0 \quad \text{for all } v \in V_1. \tag{3.27}$$

If we put $y = (0, \tau)$ in (3.25) and we use (2.9) we obtain

$$(\mathscr{E}^{-1}B(t), \tau) = 0 \quad \text{for all } \tau \in V_2. \tag{3.28}$$

Since the orthogonal complement of $\varepsilon(V_1)$ in $\mathscr{L}$ is V_2, from (3.27) we get $B(t) \in V_2$; thus we may put $\tau = B(t)$ in (3.28) and from (3.1) we deduce $B(t) = 0$, for all $t > 0$. Hence (3.9) holds.

Using Lemma 3.3, the problem (3.13)–(3.18) was replaced by the Cauchy problem (3.21)–(3.22) in the Hilbert space V. In order to prove the existence and the uniqueness of the solution of (3.21)–(3.22) we use the following result (see Lovelady and Martin [12] and Pavel and Ursescu [14]):

LEMMA 3.4. Let V be a Hilbert space and $A : R_+ \times V \to V$ a continuous operator such that there exists $D > 0$ with

$$(A(t, x_1) - A(t, x_2), x_1 - x_2)_V \le D\|x_1 - x_2\|_V^2, \tag{3.29}$$

for all $t > 0$ and all $x_1, x_2 \in V$. Then, for all $x_0 \in V$, there exists a unique solution $x \in C^1(R_+, V)$ of the problem (3.21)–(3.22).

LEMMA 3.5. The operator A given by (3.20) is continuous and satisfies (3.29).

Proof. Let $t_1, t_2 > 0$, $x_1 = (u_1, \sigma_1)$, $x_2 = (u_2, \sigma_2) \in V$. For all $y = (v, \tau) \in V$ we have

$$\begin{aligned}|(A(t_1, x_1) - A(t_2, x_2), y)_V| \\ &\leq |(F(\sigma_1 + \tilde{\sigma}(t_1), \varepsilon(u_1) + \varepsilon(\tilde{u}(t))), \varepsilon(v)) - (F(\sigma_2 + \tilde{\sigma}(t_2), \varepsilon(\tilde{u}(t_2))), \varepsilon(v))| \\ &+ |(\mathscr{E}^{-1}F(\sigma_1 + \tilde{\sigma}(t_1), \varepsilon(u_1) + \varepsilon(\tilde{u}(t_1))) - \mathscr{E}^{-1}F(\sigma_2 + \tilde{\sigma}(t_2), \varepsilon(u_2) + \varepsilon(\tilde{u}(t_2))), \tau)|.\end{aligned}$$

Using (3.2) in the above inequality, after some estimations we get

$$\begin{aligned}|(A(t_1, x_1) - A(t_2, x_2), y)_V| \\ &\leq C[\|x_1 - x_2\|_V + \|\tilde{u}(t_1) - \tilde{u}(t_2)\|_H + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_d]\|y\|_V.\end{aligned}$$

Therefore,

$$\begin{aligned}\|A(t_1, x_1) - A(t_2, x_2)\|_V \\ &\leq C[\|x_1 - x_2\|_V + \|\tilde{u}(t_1) - \tilde{u}(t_2)\|_H + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_d].\end{aligned} \tag{3.30}$$

Using (3.30) and Lemma 3.1, the continuity of A from $R_+ \times V$ into V follows. For $t_1 = t_2 = t > 0$, (3.30) becomes

$$\|A(t, x_1) - A(t, x_2)\|_V \leq C\|x_1 - x_2\|_V \tag{3.31}$$

for all $x_1, x_2 \in V$; hence (3.29) holds.

Proof of Theorem 3.1 follows from Lemmas 3.1–3.5.

Remark 3.3. Theorem 3.3 (as well as Corollary 4.1 for $j = 0$ below) can even be stated under weaker assumptions on the function F (see also [11]); namely, (3.2)(a) can be replaced by

(i) F is a continuous function.

(ii) There exist $L_1, L_2 > 0$ such that

$$|F(x, \sigma, \varepsilon)|^2 \leq L_1 + L_2(|\varepsilon|^2 + |\sigma|^2)$$

for all $x \in \Omega$ and $\sigma, \varepsilon \in \mathscr{S}$.

(iii) There exists $L_3 > 0$ such that

$$\begin{aligned}&-(F(x, \sigma_1, \varepsilon_1) - F(x, \sigma_2, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \\ &+ \mathscr{E}^{-1}(x)(F(x, \sigma_1, \varepsilon_1) - F(x, \sigma_2, \varepsilon_2)) \cdot (\sigma_1 - \sigma_2) \\ &\qquad \leq L_3(|\varepsilon_1 - \varepsilon_2|^2 + |\sigma_1 - \sigma_2|^2)\end{aligned}$$

for all $\sigma_1, \varepsilon_1, \sigma_2, \varepsilon_2 \in \mathscr{S}$ and $x \in \Omega$.

Indeed, (ii) is a sufficient condition in order to have $F(\sigma, \varepsilon) \in \mathscr{L}$ for all $\sigma, \varepsilon \in \mathscr{L}$, (i) assumes the continuity of A, and from (iii) we get (3.29).

4. The continuous dependence of the solution upon the input data. In this section two solutions of the problem (1.1)–(1.7) for two different input data are considered. An estimation of the difference of these solutions is given for finite time intervals that give the continuous dependence of the solution upon all input data (Theorem 4.1). In this way, the finite-time stability of the solution is obtained (Corollary 4.1).

In order to get a better insight on the model and on its connections to the elasticity, the constitutive equation (1.3) is particularized taking $F(\sigma, \varepsilon) = -k(\sigma - G(\varepsilon))$, with G a strongly monotone function and $k > 0$. In this case, the asymptotic stability of the solution is deduced (Corollary 4.2).

THEOREM 4.1. Let (3.1)–(3.2) hold and let (u_i, σ_i) be the solution of (1.1)–(1.7) for the data $b_i, f_i, g_i, u_{0i}, \sigma_{0i}$, $i = 1, 2$, such that (3.3)–(3.5) hold. For all $T > 0$ there exists $C(T) > 0$ (which generally depends on Ω, T, Q, d, L) such that

$$\|u_1 - u_2\|_{T,H,j} + \|\sigma_1 - \sigma_2\|_{T,\mathscr{H},j} \leq C(T)\left[\|b_1 - b_2\|_{T,L,j} + \|f_1 - f_2\|_{T,E',j} + \|h_1 - h_2\|_{T,H_\Gamma,j} + \|u_{01} - u_{02}\|_H + \|\sigma_{01} - \sigma_{02}\|_d\right] \quad \text{for } j = 0, 1. \tag{4.1}$$

COROLLARY 4.1. Let the hypotheses of Theorem 4.1 hold. If $b_1 = b_2$, $f_1 = f_2$, $g_1 = g_2$, then

$$\begin{aligned} &\|u_1 - u_2\|_{T,H,j} + \|\sigma_1 - \sigma_2\|_{T,\mathscr{H},j} \\ &\quad \leq C(T)[\|u_{01} - u_{02}\|_H + \|\sigma_{01} - \sigma_{02}\|_d], \qquad j = 0, 1. \end{aligned} \tag{4.2}$$

In order to avoid misunderstanding we recall some definitions of stability theory following Hahn [10, Chap. 5]. A solution (u, σ) of the problem (1.1)–(1.7) will be called:

(i) *stable* if there exists $m : R_+ \to R_+$ a continuous increasing function with $m(0) = 0$ such that

$$\|u(t) - u_1(t)\|_H + \|\sigma(t) - \sigma_1(t)\|_d < m(\|u_0 - u_{01}\|_H + \|\sigma_0 - \sigma_{01}\|_d) \tag{4.3}$$

for all $t \in R_+$ and all (u_{01}, σ_{01}) satisfying (3.5), (3.4);

(ii) *finite time stable* if (4.3) holds for a finite time interval;

(iii) *asymptotically stable* if there exists m as in (i) and $n : R_+ \to R_+$ a decreasing continuous function with $\lim_{t\to\infty} n(t) = 0$ such that

$$\|u(t) - u_1(t)\|_H + \|\sigma(t) - \sigma_1(t)\|_d \leq m(\|u_0 - u_{01}\|_H + \|\sigma_0 - \sigma_{01}\|_d) \cdot n(t)$$

for all $t \in R_+$ and all (u_{01}, σ_{01}) satisfying (3.5), (3.4), where (u_1, σ_1) is the solution of (1.1)–(1.7) for the data (u_{01}, σ_{01}).

Remark 4.1. From (4.2) we deduce the finite-time stability of every solution of (1.1)–(1.7). Generally, stability does not hold (see Remark 4.2).

In order to prove Theorem 4.1, the following lemma will be useful.

LEMMA 4.1. Let $p : [0, T] \to R_+$ be a positive continuous function and $\alpha \in R$. If $\vartheta : [0, T] \to R_+$ is an absolutely continuous positive function such that $\dot{\vartheta}(t) \leq 2\alpha\vartheta(t) + 2\sqrt{\vartheta(t)}p(t)$ a.e. $t \in [0, T]$, then

$$\sqrt{\vartheta(t)} \leq \exp(\alpha t)\sqrt{\vartheta(0)} + \int_0^t p(s)\exp(\alpha(t - s))\, ds \tag{4.4}$$

for all $t \in [0, T]$.

Proof of Theorem 4.1. Let $(\tilde{u}_i, \tilde{\sigma}_i)$, $i = 1, 2$, be the functions given by Lemma 3.1 for the data f_i, b_i, g_i, $i = 1, 2$. We let $\overline{u}_i = u_i - \tilde{u}_i$, $\overline{\sigma}_i = \sigma_i - \tilde{\sigma}_i$, and $x_i = (\overline{u}_i, \overline{\sigma}_i) \in V$, $i = 1, 2$. Let A_i be given by (3.20) (replacing $\tilde{u}, \tilde{\sigma}$ by $\tilde{u}_i, \tilde{\sigma}_i$). For all $y_i \in V$, $i = 1, 2$, and all $t \in R_+$, from (3.11), (3.2) it follows that

$$\begin{aligned} &\|A_1(t, y_1) - A_2(t, y_2)\|_V \\ &\quad \leq C(\|y_1 - y_2\|_V + \||b_1(t) - b_2(t)\|| + \|f_1(t) - f_2(t)\|_1 + \|h_1(t) - h_2(t)\|_\Gamma). \end{aligned} \tag{4.5}$$

Let $x_{0i} = (u_{0i} - \tilde{u}_i(0), \sigma_{0i} - \tilde{\sigma}_i(0)) \in V$. From Lemma 3.3, (4.5), and Lemma 4.1 for $\vartheta(t) = ||x_1(t) - x_2(t)||_V^2$ we obtain

$$||x_1(t) - x_2(t)||_V \leq \exp(Ct)[||x_{01} - x_{02}||_V + ||b_1 - b_2||_{T,L,0} + ||f_1 - f_2||_{T,E',0} + ||h_1 - h_2||_{T,H_\Gamma,0}] \quad \text{for all } t \in [0,T]. \tag{4.6}$$

Using (4.6) we deduce (4.1) for $j = 0$. From (4.6), (2.6), and (3.12) we get (4.1) for $j = 1$.

Further on we particularize the function F from (1.3) as

$$F(\sigma, \varepsilon) = -k(\sigma - G(\varepsilon)) \tag{4.7}$$

for all $\sigma, \varepsilon \in \mathscr{S}$, where $k > 0$ is a viscosity coefficient and $G: \Omega \times \mathscr{S} \to \mathscr{S}$. In order to satisfy (3.2) we suppose that

$$\begin{aligned} |G(x,\varepsilon_1) - G(x,\varepsilon_2)| &\leq L|\varepsilon_1 - \varepsilon_2| && \text{for all } x \in \Omega \text{ and } \varepsilon_1, \varepsilon_2 \in \mathscr{S}, \\ G(x,0) &= 0, && \text{for all } x \in \Omega. \end{aligned} \tag{4.8}$$

Moreover, we suppose that G is a strongly monotone function, i.e., there exists a constant $a > 0$ such that

$$(G(x,\varepsilon_1) - G(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq a|\varepsilon_1 - \varepsilon_2|^2 \quad \text{for all } x \in \Omega \text{ and } \varepsilon_1, \varepsilon_2 \in \mathscr{S}. \tag{4.9}$$

THEOREM 4.2. Let (3.1), (4.7)–(4.9) hold, and let (u_i, σ_i) be two solutions of (1.1)–(1.7) for the data $b_i, f_i, g_i, u_{i0}, \sigma_{0i}$, $i = 1, 2$, for which (3.3)–(3.5) hold. Then there exist two constants $C, \overline{C} > 0$ (depending only on $\Omega, \Gamma_1, Q, d, L$, and a) such that for all $T > 0$ we have

$$\begin{aligned} ||u_1(t) - u_2(t)||_H &+ ||\sigma_1(t) - \sigma_2(t)||_d \\ &\leq \overline{C}[(||u_{01} - u_{02}||_H + ||\sigma_{01} - \sigma_{02}||_d)\exp(-Ckt) \\ &\quad + ||b_1 - b_2||_{T,L,0} + ||f_1 - f_2||_{T,E',0} + ||h_1 - h_2||_{T,H_\Gamma,0}], \end{aligned} \tag{4.10}$$

for all $t \in [0, T]$.

COROLLARY 4.2. Let the hypotheses of Theorem 4.2 hold. If $b_1 = b_2$, $f_1 = f_2$, $g_1 = g_2$, then

$$\begin{aligned} ||u_1(t) - u_2(t)||_H &+ ||\sigma_1(t) - \sigma_2(t)||_d \\ &\leq \overline{C}(||u_{01} - u_{02}||_H + ||\sigma_{01} - \sigma_{02}||_d)\exp(-Ckt), \end{aligned} \tag{4.11}$$

for all $t \in R_+$.

Remark 4.2. From (4.11) we deduce the asymptotic stability of every solution of the problem (1.1)–(1.7), (4.7). One-dimensional examples can be given in order to prove that if G is not a monotone function, stability generally does not hold.

In order to prove Theorem 4.2, the following lemma is useful.

LEMMA 4.2. Let $T > 0$, and let $\vartheta \in C^1(0, T, R)$ be a positive function and $\beta, \gamma \geq 0$, $\alpha > 0$ constants. If

$$\dot{\vartheta}(t) \leq -2\alpha\vartheta(t) + 2\beta\sqrt{\vartheta(t)} + 2\gamma \quad \text{for all } t \in [0, T], \tag{4.12}$$

then

$$\sqrt{\vartheta(t)} \leq \sqrt{\vartheta(0)}\exp(-\alpha t) + (\beta + \sqrt{\beta^2 + 4\alpha\gamma})(2\alpha)^{-1} \quad \text{for all } t \in [0, T]. \tag{4.13}$$

Proof. Suppose $\gamma \neq 0$ (if $\gamma = 0$ Lemma 4.1 can be used); let $\delta > 0$, and let M_δ be the set defined by

$$M_\delta = \{t \in [0,T] \mid \sqrt{\vartheta(t)} \geq \delta + \sqrt{\vartheta(0)}\exp(-\alpha t) + (\beta + \sqrt{\beta^2 + 4\gamma\alpha})(2\alpha)^{-1}\}.$$

If M_δ is not empty, since $0 \notin M_\delta$ and M_δ is a closed set we get $0 < t_\delta = \inf M_\delta$ and $\sqrt{\vartheta(t_\delta)} = \delta + \sqrt{\vartheta(0)}\exp(-\alpha t_\delta) + (\beta + \sqrt{\beta^2 + 4\gamma\alpha})(2\alpha)^{-1}$. Using (4.12) and the above equality we get

$$d/dt(\sqrt{\vartheta(t)} - \sqrt{\vartheta(0)}\exp(-\alpha t))\,\Big|_{t=t_\delta} \leq (-\alpha\delta^2)(\vartheta(t_\delta))^{-1/2}.$$

Hence $M_\delta = \varnothing$ for all $\delta > 0$.

Proof of Theorem 4.2. The same notations as in the proof of Theorem 4.1 are used. From (3.14) and (4.7) we get

$$\dot{\overline{\sigma}}_i(t) = \mathscr{E}\varepsilon(\dot{\overline{u}}_i(t)) - k[\overline{\sigma}_i(t) + \tilde{\sigma}_i(t) - G(\varepsilon(\overline{u}_i(t)) + \varepsilon(\tilde{u}_i(t)))] \tag{4.14}$$

for all $t \in R_+$, $i = 1, 2$.

If we take the difference in (4.14) for $i = 1$ and $i = 2$ and we put $\overline{u} = \overline{u}_1 - \overline{u}_2$, $\overline{\sigma} = \overline{\sigma}_1 - \overline{\sigma}_2$, $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, $\tilde{\sigma} = \tilde{\sigma}_1 - \tilde{\sigma}_2$, after multiplying by $\varepsilon(\overline{u}(t))$ and integrating the result on Ω, from (2.9) we get

$$(\mathscr{E}\varepsilon(\dot{\overline{u}}(t)), \varepsilon(\overline{u}(t)) = k(\tilde{\sigma}(t), \varepsilon(\overline{u}(t))) - k(G(\varepsilon(u_1(t))) - G(\varepsilon(u_2(t))), \varepsilon(\overline{u}(t))). \tag{4.15}$$

We let $\vartheta(t) = (\mathscr{E}\varepsilon(\overline{u}(t)), \varepsilon(\overline{u}(t)))$ and $\beta(T) = ||b_1 - b_2||_{T,L,0} + ||f_1 - f_2||_{T,E',0} + ||h_1 - h_2||_{T,H_\Gamma,0}$. Then from (4.15), (4.8), (4.9), and (3.11), after some algebra we obtain

$$\dot{\vartheta}(t) \leq k(-2C_1\vartheta(t) + 2C_2\beta(T)\sqrt{\vartheta(t)} + 2C_2\beta^2(T)) \tag{4.16}$$

for all $t \in [0,T]$ with $C_1, C_2 > 0$, $C_1 < 1$.

From (4.16) and Lemma 4.2 it follows

$$\sqrt{\vartheta(t)} \leq \sqrt{\vartheta(0)}\exp(-kC_1 t) + \beta(T)(C_2 + \sqrt{C_1^2 + 4C_1C_2})(2C_1)^{-1} \tag{4.17}$$

for all $t \in [0,T]$. Using (4.17), (3.1), and (2.6) in (4.17) we get

$$||u_1(t) - u_2(t)||_H \leq C_3(||u_{01} - u_{02}||_H \exp(-kC_1 t) + \beta(T)). \tag{4.18}$$

In a similar way, if we take the difference in (4.14) for $i = 1$ and $i = 2$ and we multiply by $\mathscr{E}^{-1}$ on the left and by $\overline{\sigma}(t)$ on the right and integrate the result on Ω, using (2.9) we obtain

$$\begin{aligned}(\mathscr{E}^{-1}\dot{\overline{\sigma}}(t), \overline{\sigma}(t)) &= -k(\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t)) - k(\mathscr{E}^{-1}\tilde{\sigma}(t), \overline{\sigma}(t)) \\ &\quad + k(\mathscr{E}^{-1}G(\varepsilon(u_1(t))) - \mathscr{E}^{-1}G(\varepsilon(u_2(t))), \overline{\sigma}(t)) \quad \text{for all } t \in R_+.\end{aligned} \tag{4.19}$$

Let $\vartheta(t) = (\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t))$. From (4.19), (3.11), and (4.8) we obtain

$$\dot{\vartheta}(t) \leq -2k\vartheta(t) + 2kC_4(\beta(T) + ||u_1(t) - u_2(t)||_H)\sqrt{\vartheta(t)}$$

for all $t \in [0,T]$. Using (4.18) in the above inequality we have

$$\dot{\vartheta}(t) \leq -2k\vartheta(t) + 2kC_5(\beta(T) + ||u_{01} - u_{02}||_H \exp(-kC_1 t))\sqrt{\vartheta(t)}$$

for all $t \in [0,T]$, and from Lemma 4.1 we get

$$\sqrt{\vartheta(t)} \leq \sqrt{\vartheta(0)}\exp(-kt) + C_5\beta(T) + C_5(1 - C_1)^{-1}\exp(-kC_1 t)||u_{01} - u_{02}||_H. \tag{4.20}$$

Using (3.1), (4.20), and (3.11) we deduce

$$||\sigma_1(t) - \sigma_2(t)||_d \leq C_6(||u_{01} - u_{02}||_H + ||\sigma_{01} - \sigma_{02}||_d) \exp(-kC_1 t) + C_6\beta(T) \quad \text{for all } t \in [0, T]. \tag{4.21}$$

From (4.18) and (4.21) we obtain (4.10).

Periodic solutions. In this section we are interested in periodic solutions of the problem (1.1)–(1.7), (4.9). The main result of this section is

THEOREM 5.1. Let (3.1), (4.7)–(4.9) hold and let the data b, f, g satisfying (3.3) be periodic functions with the same period ω. Then there exists a unique initial data (u_0, σ_0) satisfying (3.4)–(3.5) such that the solution of (1.1)–(1.7), (4.7) is a periodic function with the same period ω.

Remark 5.1. In the hypotheses of Theorem 5.1, using Corollary 4.2 we deduce that for all initial data the solution of the problem (1.1)–(1.7), (4.9) approaches a unique periodic function when $t \to +\infty$. In other words, if the external data are oscillating, then the body will "begin" after a while to oscillate too.

Theorem 5.1 is a direct consequence of the asymptotic stability result (Corollary 4.2). For details of the proof see [11].

6. Approach to elasticity. The purpose of this section is to prove the convergence when $k \to +\infty$ of the solution $(u_k(t), \sigma_k(t))$ of (1.1)–(1.7), (4.7) for all $t > 0$ to the solution of the following boundary value problem for an elastic body.

Find the displacement function $\hat{u} : R_+ \times \Omega \to R^n$ and the stress function $\hat{\sigma} : R_+ \times \Omega \to \mathscr{S}$ such that

$$\operatorname{div} \hat{\sigma}(t) + b(t) = 0, \tag{6.1}$$

$$\varepsilon(\hat{u}(t)) = \tfrac{1}{2}(\nabla \hat{u}(t) + \nabla^{\mathrm{T}} \hat{u}(t)), \tag{6.2}$$

$$\hat{\sigma}(t) = G(\varepsilon(\hat{u}(t))) \quad \text{in } \Omega, \tag{6.3}$$

$$\hat{u}(t)\,|_{\Gamma_1} = g(t), \tag{6.4}$$

$$\hat{\sigma}(t)\nu\,|_{\Gamma_2} = f(t) \quad \text{for all } t \in R_+. \tag{6.5}$$

LEMMA 6.1. Let us suppose that (3.3), (4.10), (4.11) hold. Then the problem (6.1)–(6.5) has a unique solution $\hat{u} \in C^0(R_+, H)$, $\hat{\sigma} \in C^0(R_+, \mathscr{H})$. Moreover, for all $T > 0$, $\hat{u}, \hat{\sigma}$ are absolutely continuous functions on $[0, T]$, and there exists $C > 0$ (which depends only on Ω, Γ_1, L, and a) such that

$$||\dot{\hat{u}}(t)||_H + ||\dot{\hat{\sigma}}(t)||_d \leq C(||\dot{f}(t)||_1 + |||\dot{b}(t)||| + ||\dot{h}(t)||_\Gamma) \quad \text{a.e. in } R_+. \tag{6.6}$$

Remark 6.1. The elastic problem (6.1)–(6.5) was considered by many authors with different assumptions on the function G (see, for instance, Fichera [7], Duvaut and Lions [6], Dincă [4], Mazilu and Sburlan [13], and others). However, a sketch of the proof of this lemma can be found in [11].

The following lemma (which will also be useful in Sec. 7) evaluates the difference between the solutions of (1.1)–(1.7), (4.7) and those of (6.1)–(6.5) for the same data b, f, g.

Lemma 6.2. Let (3.1), (3.3), (4.9) hold, and let (u, σ) be the solution of (1.1)–(1.7), (4.7) and $(\hat{u}, \hat{\sigma})$ the solution of (6.1)–(6.5). For all $t \in R_+$ we have

$$\|u(t) - \hat{u}(t)\|_H \le C[\|u_0 - u(0)\|_H \exp(-Ckt)] + \int_0^t (|||\dot{b}(s)||| + \|\dot{f}(s)\|_1 + \|\dot{h}(s)\|_\Gamma) \exp(-Ck(t-s))\, ds, \quad (6.7)$$

$$\|\sigma(t) - \hat{\sigma}(t)\|_d \le \overline{C}[\|\sigma_0 - \sigma(0)\|_d \exp(-kt)] + \int_0^t [k\|u(s) - \hat{u}(s)\|_H + |||\dot{b}(s)||| + \|\dot{f}(s)\|_1 + \|\dot{h}(s)\|_\Gamma] \exp(k(s-t))\, ds, \quad (6.8)$$

where the strictly positive constants $C, \overline{C} > 0$, $C > 1$ depend only on $\Omega, \Gamma_1, Q, d, L$, and a.

Proof. We let $\overline{u} = u - \hat{u}$, $\overline{\sigma} = \sigma - \hat{\sigma}$, and from (1.3), (4.7), and Lemma 6.1 we get

$$\dot{\hat{\sigma}}(t) + \dot{\overline{\sigma}}(t) = \mathscr{E}\varepsilon(\dot{\hat{u}}(t)) + \mathscr{E}\varepsilon(\dot{\overline{u}}(t)) - k[\overline{\sigma}(t) + G(\varepsilon(\hat{u}(t))) - G(\varepsilon(\overline{u}(t)) + \varepsilon(\hat{u}(t)))] \quad \text{a.e. } t \in R_+. \quad (6.9)$$

Multiplying (6.9) by $\varepsilon(\overline{u}(t))$, after integration on Ω and use of (2.9) we get

$$(\dot{\hat{\sigma}}(t) - \mathscr{E}\varepsilon(\dot{\hat{u}}(t)), \varepsilon(\overline{u}(t))) = (\mathscr{E}\varepsilon(\dot{\hat{u}}(t)), \varepsilon(\overline{u}(t))) + k[(G(\varepsilon(\overline{u}(t)) + \varepsilon(\hat{u}(t))) - G(\varepsilon(\hat{u}(t))), \varepsilon(\overline{u}(t)))] \quad \text{a.e. } t \in R_+.$$

Hence, from (6.6), (3.1), and (4.9) we obtain

$$(\mathscr{E}\varepsilon(\dot{\overline{u}}(t)), \varepsilon(\overline{u}(t))) \le -C_1 k (\mathscr{E}\varepsilon(\overline{u}(t)), \varepsilon(\overline{u}(t))) + C_2(|||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\|_\Gamma)(\mathscr{E}\varepsilon(\overline{u}(t)), \varepsilon(\overline{u}(t)))^{1/2} \quad \text{a.e. } t \in R_+.$$

We let $\vartheta(t) = (\mathscr{E}\varepsilon(\overline{u}(t)), \varepsilon(\overline{u}(t)))$ and use Lemma 4.1, (2.6), and (3.1) to get (6.7).

Applying $\mathscr{E}^{-1}$ to the left of (6.9) and taking the scalar product of it with $\overline{\sigma}(t)$, integrating the result on Ω and using (2.9) we get

$$(\mathscr{E}^{-1}\dot{\overline{\sigma}}(t), \overline{\sigma}(t)) = -k(\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t)) - (\mathscr{E}^{-1}\dot{\hat{\sigma}}(t) - \varepsilon(\dot{\hat{u}}(t)), \overline{\sigma}(t)) + k(\mathscr{E}^{-1}G(\varepsilon(\overline{u}(t)) + \varepsilon(\hat{u}(t))) - \mathscr{E}^{-1}G(\varepsilon(\hat{u}(t))), \overline{\sigma}(t)) \quad \text{a.e. } t \in R_+.$$

Using Lemma 6.1, (4.8), and (2.6) we have

$$(\mathscr{E}^{-1}\dot{\overline{\sigma}}(t), \overline{\sigma}(t)) \le -k(\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t)) + C_3(\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t))^{1/2}[k\|\overline{u}(t)\|_H + |||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\|_\Gamma],$$

a.e. $t \in R_+$. Using Lemma 4.1 here again for $\vartheta(t) = (\mathscr{E}^{-1}\overline{\sigma}(t), \overline{\sigma}(t))$, we obtain (6.8).

Theorem 6.1. Suppose (3.1)–(3.5), (4.7)–(4.9) hold. Let (u_k, σ_k) be the solution of the problem (1.1)–(1.7) for any $k > 0$, and let $(\hat{u}, \hat{\sigma})$ be the solution of (6.1)–(6.5). Then, for all $t > 0$ we have $\|u_k(t) - \hat{u}(t)\|_H \to 0$, $\|\sigma_k(t) - \hat{\sigma}(t)\|_d \to 0$ when $k \to +\infty$.

Remark 6.1. The behavior of (u_k, σ_k) when $k \to +\infty$ was also studied (in the dynamical case) in the papers of Suliciu [16] and Podio-Guidugli and Suliciu [15], where $F(\sigma, \varepsilon) = -\mathscr{K}(\sigma, \varepsilon)(\sigma - G(\varepsilon))$ with $\mathscr{K}(\sigma, \varepsilon)\tau \cdot \tau \geq k|\tau|^2$ for all $\tau \in \mathscr{S}$, $k =$ constant > 0.

For isolated bodies, assuming the existence and the smoothness of the solution and using the energy function, in [15], [16] it is proved that $||\sigma_k(t) - G(\varepsilon_k(t))|| \to 0$ when $k \to +\infty$, for all $t > 0$. In [16] ε_k represents the small strain tensor ($\varepsilon_k = \varepsilon(u_k)$), and in [15] ε_k represents the finite strain tensor.

Proof of Theorem 6.1. From (6.7), for $t > 0$ we get

$$||u_k(t) - \hat{u}(t)||_H \leq C[||u_0 - \hat{u}(0)||_H \exp(-Ckt) + (Ck)^{-1}(||\dot{b}||_{t,L,0} + ||\dot{f}||_{t,E',0} + ||\dot{h}||_{t,H_\Gamma,0})]. \quad (6.10)$$

From (6.8) and (6.10) we obtain

$$\begin{aligned} ||\sigma_k(t) - \hat{\sigma}(t)||_d &\leq \overline{C}[||\sigma_0 - \hat{\sigma}(0)||_d \exp(-kt) + (C/(1-C)) \exp(-Ckt)||u_0 - \hat{u}(0)||_H \\ &\quad + (C+1)(Ck)^{-1}(||\dot{b}||_{t,L,0} + ||\dot{f}||_{t,E',0} + ||\dot{h}||_{t,H_\Gamma,0})] \quad (6.11) \end{aligned}$$

and from (6.10), (6.11) the theorem follows.

7. Large-time behavior of the solution. In this section we consider the problem (1.1)–(1.7), (4.7) for a fixed $k > 0$, and we study the behavior of the solution when $t \to \infty$. The main result is the following.

THEOREM 7.1. Let (3.1), (3.3)–(3.5), (4.7)–(4.9) hold; we denote by (u, σ) the solution of (1.1)–(1.7) and by $(\hat{u}, \hat{\sigma})$ the solution of (6.1)–(6.5). If

$$\lim_{t \to +\infty} (|||\dot{b}(t)||| + ||\dot{h}(t)||_\Gamma + ||\dot{f}(t)||_1) = 0,$$

then

$$\lim_{t \to +\infty} (||u(t) - \hat{u}(t)||_H + ||\sigma(t) - \hat{\sigma}(t)||_d) = 0. \quad (7.1)$$

Remark 7.1. Let us observe that for all $t \geq 0$ the functions $\hat{u}(t), \hat{\sigma}(t)$ are uniquely determined by the data $b(t), f(t)$, and $g(t)$. From Theorem 6.1 we get that if $|||\dot{b}(t)||| + |||\dot{f}(t)|||_1 + ||\dot{h}(t)||_\Gamma \to \dot{0}$ when $t \to +\infty$, then after a large enough time the solution (u, σ) will be "determined" only by the present values of b, f, g. Hence, in this case the initial data and the history of external data have "no influence" upon the large-time behavior of the solution.

Remark 7.2. If $\lim_{t \to +\infty}(|||\dot{b}(t)||| + ||\dot{f}(t)||_1 + ||\dot{h}(t)||_\Gamma) \neq 0$, the statement of Theorem 6.1 cannot generally hold. For example, let b, f, g be periodic functions with the same period. Then $\hat{u}, \hat{\sigma}$ are periodic functions, and from Theorem 5.1 we get that there exists an initial data (u_0, σ_0) for which the solution (u, σ) of (1.1)–(1.7), (4.7) is periodic. If we suppose that $||u(t) - \hat{u}(t)||_H \to 0$ and $||\sigma(t) - \hat{\sigma}(t)||_d \to 0$ when $t \to +\infty$, we get $u = \hat{u}$, $\sigma = \hat{\sigma}$, and from (1.3) we obtain $\dot{\hat{\sigma}}(t) = \mathscr{E}(\varepsilon(\dot{\hat{u}}(t)))$ for all $t \in R_+$; this equality is generally false if $\dot{\hat{\sigma}} \neq 0$, $\dot{\hat{u}} \neq 0$, and $G(\varepsilon) \neq \mathscr{E}\varepsilon$. Hence, if the external data are periodic, then there exists a phase shift between the periodic solutions (u, σ) and $(\hat{u}, \hat{\sigma})$.

COROLLARY 7.1. Let the hypotheses of Theorem 7.1 hold. If in addition we suppose that there exists $\overline{b} \in L$, $\overline{f} \in E'$, and $\overline{h} \in H$ such that

$$\lim_{t \to +\infty} (|||b(t) - \overline{b}||| + ||f(t) - \overline{f}||_1 + ||h(t) - \overline{h}||_\Gamma) = 0 \tag{7.2}$$

and if we denote by $(\overset{\Delta}{u}, \overset{\Delta}{\sigma})$ the solution of (6.1)–(6.5) for the data $\overline{b}, \overline{f}$, and $\overline{h}$, then

$$\lim_{t \to +\infty} (||u(t) - \overset{\Delta}{u}||_H + ||\sigma(t) - \overset{\Delta}{\sigma}||_d) = 0. \tag{7.3}$$

Proof. From the continuous dependence of the solution of (6.1)–(6.5) upon the data f, b, h and (7.2), we get

$$\lim_{t \to +\infty} (||\hat{u}(t) - \overset{\Delta}{u}||_H + ||\hat{\sigma}(t) - \overset{\Delta}{\sigma}||_d) = 0,$$

and from (7.1) we deduce (7.3).

Remark 7.3. Let the data b, f, h be constant in time, and let (u, σ) be the solution of (1.1)–(1.7), (4.7) and $(\hat{u}, \hat{\sigma})$ the solution of (6.1)–(6.5). In this case the differential equation (3.21), (3.22) is an autonomous one, and $(\hat{u}, \hat{\sigma})$ is a stationary point of A. From Theorem 4.2 we can obtain

$$||u(t) - \hat{u}||_H + ||\sigma(t) - \hat{\sigma}||_d \leq \overline{C} \exp(-Ckt)(||u_0 - \hat{u}||_H + ||\sigma_0 - \sigma||_d), \tag{7.4}$$

where C and $\overline{C}$ are defined in (6.7).

In order to give the proof of Theorem 7.1, the following lemma is useful.

LEMMA 7.1. Let $r : R_+ \to R_+$ be a continuous function such that $\lim_{t \to +\infty} r(t) = 0$ and $p : R_+ \to R_+$ given by $p(t) = \int_0^t r(s) \exp(-C(t-s))\, ds$ with $C > 0$. Then $\lim_{t \to +\infty} p(t) = 0$.

Proof of Theorem 7.1 easily follows from (6.7)–(6.8) and Lemma 7.1.

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