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# APPROXIMATE ANALYTICAL SOLUTION OF A STEFAN'S PROBLEM IN A FINITE DOMAIN*

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**Abstract.** A Stefan's problem in a finite domain may be given an approximate analytical solution. An example is shown with constant boundary and initial conditions. The solution is initially that of a semi-infinite domain, transits through infinitely many intermediate stage solutions, and finally becomes stationary. The solution is exact in the initial stage and also at the steady final stage, but approximate at the intermediate stages.

**Introduction.** Stefan's problem has a long history. Since 1962 a new trend appeared and the problem in a semi-infinite domain has been studied under arbitrary initial and boundary conditions by utilizing the infinite series of the elemental temperature functions, i.e., a general solution for heat conduction in a semi-infinite domain [1, 2, 3, 4]. In 1978, though unaware of [1, 2, 3], Tao [4] completed the solution method. He proved the convergence of the solution series and introduced a formula expressing the higher derivatives of a function of a series. By this formula, the conditions at the moving interface can be rewritten to a set of simultaneous linear equations of its unknown parameters. Further progress in the analysis in a semi-infinite domain has been made by Tao. The density jump that may occur at the moving interface can be introduced into the analysis [5]. Analyticity of the interfacial coordinate as a function of $\sqrt{t}$, where $t$ is time, has been proved [6].

In spite of the progress that has been made so far, difficulties yet remain in a finite domain. We need to understand the analysis in a finite domain for practical applications. The heart of the difficulties seems to be in analytically formulating the well-established practice in numerical studies (for example [7]); i.e., the solution of Stefan's problem in a finite domain must be initially the one in a semi-infinite domain but finally arrives at a stationary one. In this paper, we demonstrate how the solution in a finite domain, obtained under the simplest boundary and initial conditions, transits from the initial semi-infinite domain solution to the final stationary solution.

Three innovations are introduced. The first is the use of an integral type solution in place of the well-used serial type solution. It enables us to formulate an approximate

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temperature solution of the old phase in a two-phase Stefan's problem for a wide range of initial and boundary conditions.

The second is the inverse-Laplace-integral type expression of $i^k \operatorname{erfc}(x/\sqrt{4\kappa t})$ that is valid for any integer $k$, negative, zero, or positive. This formula is used to expand the interfacial temperature of the old phase into a series of $\sqrt{t}$, and also to sum up the series

$$\sum_{n=0}^{\infty} i^k \operatorname{erfc} \frac{2nl \pm x}{\sqrt{4\kappa t}},$$

where $l$ is the length of the finite domain and $\kappa$ the diffusivity.

The third is the concept of lead times. At appropriate times, called lead times, new terms are added successively to the initial semi-infinite domain solution. When infinitely many terms are added, the steady final temperature distribution is arrived at. We use the sequence of lead times as a parameter. The use of the parametric lead time, however, introduces factors that must be supposed, to satisfy the heat equations, to be constant in time differentiation. The solution is approximate, therefore, except at the stage where the factors are not supposedly but truly constant, i.e., at the initial stage where the interfacial coordinate is proportional to $\sqrt{t}$, $t$ being time.

The paper consists of eleven sections. As a mathematical preliminary, Section 1 presents the transformation of a serial type solution of the heat equation in a semi-infinite domain to an integral type solution. As a corollary, Duhamel's time integral for solving boundary value problems is transformed to an equivalent space integral. Section 2 states the problem to be solved as an example.

Section 3 presents the decomposition of the new phase temperature into a series of $\sqrt{t}$ on the interfacial boundary. The section contains a revised version of Faa de Bruno's formula, by use of which an analytic function of an infinite series can be developed into another infinite series. To facilitate the differentiations necessary for the application, indices of the elemental functions are extended to negative integers. A systematization of the well-used one, this procedure applies perfectly to a semi-infinite domain, but not to a finite domain without a modification because of the singularities existing in the old phase at the terminal boundary as a function of $\sqrt{t}$. The remedy is provided in Section 7.

Section 4 presents Widder's solution of heat conduction in a finite domain as a step for finding the old phase temperature, i.e., for solving the embedding problem posed in Section 2. Section 5 partially solves the embedding problem by satisfying the terminal boundary and initial conditions only. Section 6 converts the partial solution to a sequence of finite-term solutions by introducing lead times. Section 7 presents the inverse-Laplace-integral type expression of $i^n \operatorname{erfc}(x/\sqrt{4\kappa t})$. Using this formula, Section 8 describes decomposition of a finite-term old phase temperature solution at the interface into a series of $\sqrt{t}$. Satisfying the yet unused interfacial conditions, Section 9 completes the approximate solution (i.e., containing lead times) of the embedding problem. This section also shows that, like Neuman's solution [8] in the semi-infinite domain, both the new and old phase approximate solutions consist of a single term, the latter solution being a summation of binary terms.

Using again the inverse-Laplace-integral type formula, Section 10 determines the final interfacial position and the resulting steady temperature distributions. They agree with those dictated by the boundary and interfacial conditions, showing that the approximate solution approaches the exact one as the limit. Section 11 describes the numerical process for solving the transcendental equation of the approximate interfacial position. The approach to the final steady temperature profile is described. The extent of the constant coefficient zone is recognized, where the semi-infinite domain solution is applicable.

**1. Integral type solution.** The heat equation

$$\partial T/\partial t = \kappa\, \partial^2 T/\partial x^2 \tag{1.1}$$

has a serial type general solution

$$T(x, \kappa t) = \sum_{n=0}^{\infty} (\sqrt{4\kappa t})^n \left[ A_n \cdot G_n\left(\frac{x}{\sqrt{4\kappa t}}\right) + B_n \cdot i^n \operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right)\right], \tag{1.2}$$

where the elemental functions are defined by

$$i^n \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(\lambda - x)^n}{n!} e^{-\lambda^2}\, d\lambda \tag{1.3}$$

and

$$G_n(x) = \frac{1}{2}\{i^n \operatorname{erfc}(-x) + (-1)^n i^n \operatorname{erfc}(x)\}. \tag{1.4}$$

The domain of $x$ is infinite; it will be made semi-infinite later. The notation $G_n(x)$ is after Tao [4]. With two phases in Stefan's problem, we use $\kappa t$ in place of $t$, where $\kappa$ is the thermal diffusivity of a phase.

Substituting from (1.3), (1.4) transforms to

$$G_n(x) = \frac{1}{n!\sqrt{\pi}} \int_{-\infty}^{\infty} (x - \lambda)^n e^{-\lambda^2}\, d\lambda, \tag{1.5}$$

which integrates to an $n$th degree polynomial

$$G_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{(2x)^{n-2k}}{k!(n-2k)!}. \tag{1.6}$$

Solution (1.2) gives the initial condition

$$T(x, 0) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n x^n \tag{1.7}$$

and the boundary condition

$$T(0, \kappa t) = \sum_{n=0}^{\infty} A_{2n} (\sqrt{4\kappa t})^{2n} G_{2n}(0) + \sum_{n=0}^{\infty} B_n (\sqrt{4\kappa t})^{2n} i^n \operatorname{erfc}(0). \tag{1.8}$$

The serial type solution (1.2) has been used to solve Stefan's problem in a semi-infinite domain [1, 2, 3, 4]. We shall transform it to an integral type solution.

We may use the heat functions

$$u_n(x, \kappa t) = n!(\sqrt{4\kappa t})^n i^n \operatorname{erfc}(x/\sqrt{4\kappa t}) \tag{1.9}$$

and

$$v_n(x, \kappa t) = n!(\sqrt{4\kappa t})^n G_n(x\sqrt{4\kappa t}). \tag{1.10}$$

The latter is called the heat polynomial by Widder [9]. Use of (1.3) and (1.5) gives (1.9) and (1.10), the integral expressions

$$u_n(x, \kappa t) = 2\int_0^\infty y^n k(x+y, \kappa t)\, dy \tag{1.11}$$

and

$$v_n(x, \kappa t) = \int_{-\infty}^\infty y^n k(x-y, \kappa t)\, dy, \tag{1.12}$$

respectively, where

$$k(x, \kappa t) = e^{-x^2/(4\kappa t)}/\sqrt{4\pi\kappa t}. \tag{1.13}$$

We may, therefore, define integrable functions

$$f(x) = \sum_{n=0}^\infty \frac{1}{n!} A_n x^n \tag{1.14}$$

and

$$g(x) = 2\sum_{n=0}^\infty \frac{1}{n!} B_n x^n \quad \text{for } 0 < x < \infty \tag{1.15a}$$

$$= 0 \quad \text{for } -\infty < x < 0 \tag{1.15b}$$

in the infinite domain, and transform (1.2) to an integral expression

$$T(x, \kappa t) = \int_{-\infty}^\infty f(y) k(x-y, \kappa t)\, dy + \int_{-\infty}^\infty g(y) k(x+y, \kappa t)\, dy. \tag{1.16}$$

Use of the delta function [10]

$$k(x, 0) = \delta(x) \tag{1.17}$$

yields

$$T(x, 0) = f(x). \tag{1.18}$$

The second integral in (1.16) becomes 0 when $t = 0$ by virtue of (1.15b). Because $f(x)$ can be sectionally analytic, (1.18) is more general than (1.7).

We now change the infinite domain to the semi-infinite. The boundary condition in our problem allows us to assume $f(x)$ to be odd:

$$f(-x) = -f(x). \tag{1.19}$$

Then (1.16) transforms to

$$T(x, \kappa t) = \int_0^\infty f(y)[k(x-y, \kappa t) - k(x+y, \kappa t)]\, dy + \int_0^\infty g(y) k(x+y, \kappa t)\, dy. \tag{1.20}$$

The first integral becomes zero at $x = 0$ for any $t$ in the domain $0 < t < \infty$ and the second integral becomes zero at $t = 0$ for any $x$ in the domain $0 < x < \infty$. Therefore, the first and second integrals describe the evolutions from the initial and boundary conditions, respectively, in a semi-infinite domain.

Particularly, we have an expression of the boundary value

$$T(0, \kappa t) = \int_0^\infty g(y) k(y, \kappa t)\, dy. \tag{1.21}$$

If

$$T(0, \kappa t) = \sum_{n=0}^\infty B_n (\sqrt{4\kappa t})^n i^n \operatorname{erfc}(0), \tag{1.22}$$

then the solution $g(y)$ of the integral equation (1.21) under the condition of (1.19) is (1.15a).

The solution of the boundary-value problem by use of the integral equation (1.21) is consistent with Duhamel's theorem [8]. To show this, we refer to Widder's [9] solution for heat conduction in a semi-infinite domain. In our notation, it is

$$\begin{aligned} T(x, \kappa t) = & \int_0^\infty [k(x-y, \kappa t) - k(x+y, \kappa t)] T(y, 0)\, dy \\ & + \int_0^t h(x, \kappa(t-\tau)) T(0, \kappa\tau)\, d\tau, \end{aligned} \tag{1.23}$$

where

$$h(x, \kappa t) = -2\, \partial k(x, \kappa t)/\partial x = (x/(\kappa t)) k(x, \kappa t). \tag{1.24}$$

The first integral in (1.23) is the one in (1.20). By substituting $T(0, \kappa t)$ from (1.21), the second integral, a product of Duhamel's theorem, becomes the repeated integrals

$$\int_0^\infty g(y)\, dy \int_0^t h(x, \kappa(t-\tau)) k(y, \kappa\tau)\, d\tau.$$

The convolution integral in the above simplifies to

$$\int_0^t h(x, \kappa(t-\tau)) k(y, \kappa\tau)\, d\tau = k(x+y, \kappa t), \tag{1.25}$$

because both sides yield the same Laplace transform

$$(1/\sqrt{4s}) \exp(-(x+y)/\sqrt{s}).$$

To derive this, we employ $\kappa t$ in place of $t$ in the usual Laplace transforms. Widder's solution is, therefore, the same as ours.

**2. Problem.** We consider the simplest freezing problem in a finite domain $0 \le x \le l$. The boundary temperature $T_A$ at $x = 0$ and $T_B$ at $x = l$ are constant, the latter being also the initial temperature.

At $t = 0$, a new phase emerges at $x = 0$, whose temperature we express by $T_{\mathrm{I}}(x, \kappa_{\mathrm{I}} t)$, where $\kappa_{\mathrm{I}}$ is the thermal diffusivity of the new phase. The domain of the new phase is $0 \le x \le s(t)$, where we assume $s(t)$ is given by

$$s(t) = \sum_{n=0}^\infty s_n \tau^{n+1}, \tag{2.1}$$

where

$$\tau = t^{1/2}. \tag{2.2}$$

Coefficients $s_n$ are constant in a semi-infinite domain. We shall show that $s_n$ in a finite domain is a function of time called the lead time. We express the temperature of the old phase by $T_{\mathrm{II}}(x, \kappa_{\mathrm{II}} t)$, where $\kappa_{\mathrm{II}}$ is the thermal diffusivity of the old phase. The domain of the old phase is $s(t) \leq x \leq l$. The quantities of the new and old phases are designated by the roman numerals I and II, respectively, used as a sub- or superindex.

We extend the domain of the old phase to $0 \leq x \leq l$ by introducing the embedding boundary temperature at $x = 0$. In addition to the boundary condition

$$T_{\mathrm{II}}(l, \kappa_{\mathrm{II}} t) = T_B \tag{2.3a}$$

and the initial condition

$$T_{\mathrm{II}}(x, 0) = T_B \quad \text{for } 0 < x < l, \tag{2.3b}$$

the old phase temperature with the embedding unknowns must satisfy the interfacial conditions

$$T_{\mathrm{II}}(s(t), \kappa_{\mathrm{II}} t) = T_{\mathrm{F}} \tag{2.3c}$$

and

$$K_{\mathrm{I}} \frac{\partial T_{\mathrm{I}}}{\partial x}\bigg|_s - K_{\mathrm{II}} \frac{\partial T_{\mathrm{II}}}{\partial x}\bigg|_s = L\rho \frac{ds}{dt}, \tag{2.3d}$$

where $T_{\mathrm{F}}$ is the freezing temperature, $K_{\mathrm{I}}$ and $K_{\mathrm{II}}$ are the thermal conductivities of the new and old phases, respectively, $\rho$ the assumedly equal density of the two phases, and $L$ the latent heat.

The temperature $T_{\mathrm{I}}(x, \kappa t)$ in the new phase must satisfy the boundary condition

$$T_{\mathrm{I}}(0, \kappa_{\mathrm{I}} t) = T_A, \tag{2.4a}$$

the interfacial conditions

$$T_{\mathrm{I}}(s(t), \kappa_{\mathrm{I}} t) = T_{\mathrm{F}} \tag{2.4b}$$

and (2.3d).

The embedding technique was initiated by Boley [11]. Employing elemental temperature functions, and using a space integral instead of Duhamel's time integral to represent a boundary condition, ours can be handled simpler than his.

**3. Decomposition of the new phase temperature.** The general solution of the new phase temperature is

$$T_{\mathrm{I}}(x, \kappa_{\mathrm{I}} t) = \sum_{n=0}^{\infty} (\sqrt{4\kappa_{\mathrm{I}} t})^n \left[ A_n^{\mathrm{I}} \cdot G_n\left(\frac{x}{\sqrt{4\kappa_{\mathrm{I}} t}}\right) + B_n^{\mathrm{I}} \cdot i^n \operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa_{\mathrm{I}} t}}\right)\right], \tag{3.1}$$

where $A_n^{\mathrm{I}}$ and $B_n^{\mathrm{I}}$ are to be determined. We decompose (3.1) on the interface (2.1) to a series of $\tau$,

$$T_{\mathrm{I}}(s(t), \kappa_{\mathrm{I}} t) = \sum_{p=0}^{\infty} T_p^{\mathrm{I}} \tau^p. \tag{3.2}$$

Similarly, we decompose the derivative to a series of $\tau$,

$$\tau \cdot \frac{\partial T_{\mathrm{I}}}{\partial x}(s(t), \kappa_{\mathrm{I}} t) = \sum_{p=0}^{\infty} T_p^{\mathrm{DI}} \tau^p. \tag{3.3}$$

Multiplication of $\tau$ on the left side of (3.3) makes the right side conform with that of (3.2).

To achieve the decomposition, we use a revised version of Faa de Bruno's formula [12]. Let functions $z = z(y)$ be analytic and $y = y(x)$ be an infinite series

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{3.4}$$

Then the power series expressing the composite function $z(y(x))$ is given by

$$z(y(x)) = z(y(a_0)) + \sum_{n=1}^{\infty} x^n \sum_{\nu=1}^{n} S_{n\nu}(a_*) \left( \frac{d^\nu z(y)}{dy^\nu} \right)_{x=0}, \tag{3.5}$$

where

$$S_{n\nu}(a_*) = \sum_{\lambda_i} \frac{1}{\lambda_1! \cdots \lambda_{n-\nu+1}!} a_1^{\lambda_1} \cdots a_{n-\nu+1}^{\lambda_{n-\nu+1}}. \tag{3.6}$$

The summation is over all the sets of at least one nonzero and all nonnegative integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ that simultaneously satisfy the two equations

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_{n-\nu+1} = \nu \tag{3.7a}$$

and

$$\lambda_2 + 2\lambda_3 + \cdots + (n-\nu)\lambda_{n-\nu+1} = n - \nu \tag{3.7b}$$

for every integer $\nu$ in the range

$$1 \le \nu \le n. \tag{3.7c}$$

The solutions are tabulated in [13, p. 831]. The subscript $*$ of $a_*$ in $S_{n\nu}(a_*)$ indicates that $a_*$ is a vector with components $a_1, \ldots, a_{n-\nu+1}$.

To facilitate the differentiations in (3.5), we extend the index $n$ of the elemental functions to negative integers. We define

$$i^{-n} \operatorname{erfc}(x) = (-1)^n d^n \operatorname{erfc}(x)/dx^n = (2/\sqrt{\pi}) e^{-x^2} H_{n-1}(x) \tag{3.8}$$

and

$$G_{-n}(x) = 0 \tag{3.9}$$

for negative integers. Then the formulas

$$d^k i^n \operatorname{erfc}(x)/dx^k = (-1)^k i^{n-k} \operatorname{erfc}(x) \tag{3.10}$$

and

$$d^k G_n(x)/dx^k = G_{n-k}(x) \tag{3.11}$$

hold true for any integer $k$, larger than, equal to, or less than $n$, where $H_{n-1}(x)$ is the Hermite polynomial of $(n-1)$th degree for $n \ge 1$. The recurrence formula of $i^n \operatorname{erfc}(x)$ is valid even for negative integers $n$.

Substituting $s(t)$ from (2.1) for $x$ in $G_n(x/\sqrt{4\kappa t})$ and $i^n \operatorname{erfc}(x/\sqrt{4\kappa t})$, and applying (3.5), we find the series of $\tau$,

$$G_n\left(\frac{s(t)}{\sqrt{4\kappa t}}\right) = \sum_{k=0}^{\infty} G_k^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) \cdot \tau^k \tag{3.12a}$$

and

$$i^n \operatorname{erfc}\left(\frac{s(t)}{\sqrt{4\kappa t}}\right) = \sum_{k=0}^{\infty} I_k^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) \cdot \tau^k. \tag{3.12b}$$

The coefficients in (3.12a) and (3.12b), which we call $G$-derivatives and $I$-derivatives, respectively, are expressed for $k = 0$ by

$$G_0^{(n)}(s_*/\sqrt{4\kappa}) = G_n(s_0/\sqrt{4\kappa}) \tag{3.13a}$$

and

$$I_0^{(n)}(s_*/\sqrt{4\kappa}) = i^n \operatorname{erfc}(s_0/\sqrt{4\kappa}), \tag{3.13b}$$

and for $k \geq 1$ by

$$G_k^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) = \sum_{\nu=1}^{k} G_{n-\nu}\left(\frac{s_0}{\sqrt{4\kappa}}\right) \cdot S_{k\nu}\left(\frac{s_*}{\sqrt{4\kappa}}\right) \tag{3.14a}$$

and

$$I_k^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) = \sum_{\nu=1}^{k} (-1)^{\nu} i^{n-\nu} \operatorname{erfc}\left(\frac{s_0}{\sqrt{4\kappa}}\right) \cdot S_{k\nu}\left(\frac{s_*}{\sqrt{4\kappa}}\right). \tag{3.14b}$$

The index $n - \nu$ can be negative. Due to (3.9), therefore, the number of components of a $G$-derivative, shown on the right of (3.14a), may be less than $k$ or possibly zero. The argument $s_*/\sqrt{4\kappa}$ in the $G$-derivatives and the $I$-derivatives is a vector standing for $k - \nu + 1$ components, $s_1/\sqrt{4\kappa}, \ldots, s_{k-\nu+1}/\sqrt{4\kappa}$.

Thus the general form of $T_p^{\mathrm{I}}$ in (3.2) and $T_p^{\mathrm{DI}}$ in (3.3) are

$$T_p^{\mathrm{I}} = \sum_{n=0}^{p} (\sqrt{4\kappa})^n \left[A_n^{\mathrm{I}} \cdot G_{p-n}^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) + B_n^{\mathrm{I}} \cdot I_{p-n}^{(n)}\left(\frac{s_*}{\sqrt{4\kappa}}\right)\right] \tag{3.15}$$

and

$$T_p^{\mathrm{DI}} = \sum_{n=0}^{p} (\sqrt{4\kappa})^{n-1} \left[A_n^{\mathrm{I}} \cdot G_{p-n}^{(n-1)}\left(\frac{s_*}{\sqrt{4\kappa}}\right) - B_n^{\mathrm{I}} \cdot I_{p-n}^{(n-1)}\left(\frac{s_*}{\sqrt{4\kappa}}\right)\right]. \tag{3.16}$$

We call (3.15) and (3.16) level-$p$ coefficients of the series (3.2) and (3.3), respectively. The highest index of the parameters in the level-$p$ coefficients is $p$. We call them level-$p$ parameters. There are three level-$p$ coefficients. Two of them are $A_p^{\mathrm{I}}$ and $B_p^{\mathrm{I}}$. The third is $s_p$, hidden in $I_p^{(0)}(s_*/\sqrt{4\kappa})$ in (3.15) and in $I_p^{(-1)}(s_*/\sqrt{4\kappa})$ in (3.16), as may be found by use of the formula

$$S_{n1}(a_*) = a_n, \tag{3.17}$$

which is a particular case of (3.6). In the equations for the determination of the parameters, which we discuss subsequently, the higher-than-level-0 parameters are all linear. Only $s_0$, one of the level-0 parameters, is nonlinear.

**4. Widder's solution of heat conduction in a finite domain.** To solve the embedding problem posed in Sec. 2, we find a general expression of $T_{\mathrm{II}}(x, \kappa_{\mathrm{II}} t)$, determined under unrestricted initial and boundary conditions. We do so by transforming Widder's [9] solution in a finite domain. Using our notation, his solution is

$$T_{\rm II}(x,\kappa_{\rm II}t) = \int_0^l [\theta(x-y,\kappa_{\rm II}t) - \theta(x+y,\kappa_{\rm II}t)]T_{\rm II}(y,0)\,dy + \int_0^t \phi(x,\kappa_{\rm II}(t-\tau))T_{\rm II}(0,\kappa_{\rm II}\tau)\,d\tau + \int_0^t \phi(l-x,\kappa_{\rm II}(t-\tau))T_{\rm II}(l,\kappa_{\rm II}\tau)\,d\tau, \tag{4.1}$$

where

$$\theta(x,\kappa_{\rm II}t) = \sum_{n=-\infty}^{\infty} k(x+2nl,\kappa_{\rm II}t) \tag{4.2a}$$

and

$$\phi(x,\kappa_{\rm II}t) = \sum_{n=-\infty}^{\infty} h(x+2nl,\kappa_{\rm II}t). \tag{4.2b}$$

Function $h(x,\kappa t)$ is given in (1.24). We call the three integrals on the right side of (4.1), from the top, the parts due to initial distribution, embedding boundary, and terminal boundary, respectively.

The part due to initial distribution describes the evolution from the initial distribution $T_{\rm II}(x,0)$. This statement can be proved by applying (1.17), the reduction to the delta function, to

$$\int_{-\infty}^{\infty} T_{\rm II}(y,0)k(x-y,\kappa_{\rm II}t)\,dy, \tag{4.3}$$

which is derived by extending the range of integration of the first integral of (4.1) to the infinite domain on the assumption that $T_{\rm II}(x,0)$ is odd and periodic in $2l$.

The parts due to embedding and terminal boundaries describe the evolutions from the boundary conditions at $x=0$ and $l$, respectively. We prove this in the following by changing the time integrals to space integrals. We derive on this occasion a format for the solution we need later. We use the space-integral expressions of the boundary conditions at $x=0$ and $l$,

$$T_{\rm II}(0,\kappa_{\rm II}t) = \int_0^{\infty} g_1(y)k(y,\kappa_{\rm II}t)\,dy \tag{4.4a}$$

and

$$T_{\rm II}(l,\kappa_{\rm II}t) = \int_0^{\infty} g_2(y)k(y,\kappa_{\rm II}t)\,dy, \tag{4.4b}$$

found by applying (1.21) to two semi-infinite domains, $0 < x < \infty$ and $l > x > -\infty$, respectively, where $y$ in (4.4a) and (4.4b) stands for $x$ and $l-x$, respectively, and functions $g_1(y)$ and $g_2(y)$ are defined, like (1.15b), to be zero for negative values of $y$.

We substitute (4.4a) and (4.4b) into the second and third integrals, respectively, on the right side of (4.1), divide the range of summation on the right side of (4.2b) into the nonnegative integers and the negative integers, and use in the partial summation including the latter the oddity of the $h$-function shown by (1.24), so that (1.25), in which the arguments must be positive, may be applied. Thus we find

$$\int_0^t \phi(x, \kappa_{\rm II}(t-\tau))T_{\rm II}(0, \kappa_{\rm II}\tau)\,d\tau$$
$$= \int_0^\infty g_1(y)\sum_{n=0}^\infty \{k(2nl+x+y, \kappa_{\rm II}t) - k(2(n+1)l-x+y, \kappa_{\rm II}t)\}\,dy \quad (4.5a)$$

and

$$\int_0^t \phi(l-x, \kappa_{\rm II}(t-\tau))T_{\rm II}(l, \kappa_{\rm II}\tau)\,d\tau$$
$$= \int_0^\infty g_2(y)\sum_{n=0}^\infty \{k((2n+1)l-x+y, \kappa_{\rm II}t) - k((2n+1)l+x+y, \kappa_{\rm II}t)\,dy\}. \quad (4.5b)$$

If we let $x = 0$ or $l$ on the right side of (4.5a) or (4.5b), we find that, after the cancellation, only the $n = 0$ term remains, which is equal to $T_{\rm II}(0, \kappa_{\rm II}t)$ by (4.4a) or $T_{\rm II}(l, \kappa_{\rm II}t)$ by (4.4b), respectively. By letting $x = l$ or 0 on the right side of (4.5a) or (4.5b), we find either of them disappears. By virtue of (1.15b), which both $g_1(y)$ and $g_2(y)$ satisfy, we find that the right sides of (4.5a) and (4.5b) both disappear at $t = 0$. The two integrals therefore describe the evolutions from the respective boundary conditions.

**5. Temperature with embedding unknowns.** We now apply our initial and boundary conditions, (2.3a) and (2.3b), respectively. To find the part TB due to the terminal boundary, we let $g_2(y) = 2T_B$ in (4.5b) and use (1.11) and (1.9) with $n = 0$. We find thus

$$\mathrm{TB} = T_B\sum_{n=0}^\infty\left[\operatorname{erfc}\frac{(2n+1)l-x}{\sqrt{4\kappa_{\rm II}t}} - \operatorname{erfc}\frac{(2n+1)l+x}{\sqrt{4\kappa_{\rm II}t}}\right]. \quad (5.1a)$$

To integrate the part ID due to the initial distribution, we divide the range of integration of (4.3) into infinitely many intervals of length $l$, let

$$T_{\rm II}(y+nl, 0) = (-1)^n T_B,$$

where $0 < y < l$, change the range of integration from 0 to $l$ to the difference of the one from 0 to $\infty$ and the one from $l$ to $\infty$, carry out the thus defined integrations by the combined use of (1.11) and (1.9) with $n = 0$, and change by use of (1.4) the terms with negative argument such as $\operatorname{erfc}(-x)$ to terms with positive argument such as $\operatorname{erfc}(x)$. Thus we find

$$\mathrm{ID} = T_B\left\{1 - \sum_{n=0}^\infty(-1)^n\operatorname{erfc}\frac{nl+x}{\sqrt{4\kappa_{\rm II}t}} + \sum_{n=1}^\infty(-1)^n\operatorname{erfc}\frac{nl-x}{\sqrt{4\kappa_{\rm II}t}}\right\}. \quad (5.1b)$$

Adding (5.1a) and (5.1b), we get

$$\mathrm{TB}+\mathrm{ID} = T_B\left\{1 - \sum_{n=0}^\infty\operatorname{erfc}\frac{2nl+x}{\sqrt{4\kappa_{\rm II}t}} + \sum_{n=1}^\infty\operatorname{erfc}\frac{2nl-x}{\sqrt{4\kappa_{\rm II}t}}\right\}. \quad (5.1c)$$

Substituting $g(x)$ in (1.15a) for $g_1(y)$ in (4.4a), rewriting $n$ and $B_n$ in the boundary condition (1.22) to $k$ and $B_k^{\mathrm{II}}/l^k$, respectively, and using (1.11) and (1.9) in combination, we integrate the part due to the embedding boundary condition.

Summing the results, we obtain

$$T_{\mathrm{II}}(x, \kappa_{\mathrm{II}}t) = T_B + \sum_{k=0}^{\infty} B_k^{\mathrm{II}} \left(\frac{\sqrt{4\kappa_{\mathrm{II}}t}}{l}\right)^k \sum_{n=0}^{\infty} \left\{ i^k \operatorname{erfc} \frac{2nl + x}{\sqrt{4\kappa_{\mathrm{II}}t}} - i^k \operatorname{erfc} \frac{(2n+2)l - x}{\sqrt{4\kappa_{\mathrm{II}}t}} \right\}. \tag{5.2}$$

The coefficients $B_0^{\mathrm{II}}$, $B_1^{\mathrm{II}}, \ldots$ are yet to be determined. Included in the summand for $k = 0$ in (5.2), the formula in the pair of braces in (5.1c) need not be listed. Differentiation yields

$$\tau \frac{\partial T_{\mathrm{II}}}{\partial x}(x, \kappa_{\mathrm{II}}t) = -\sum_{k=0}^{\infty} \frac{B_k^{\mathrm{II}}}{\sqrt{4\kappa_{\mathrm{II}}}} \left(\frac{\sqrt{4\kappa_{\mathrm{II}}t}}{l}\right)^k \sum_{n=0}^{\infty} \left\{ i^{k-1} \operatorname{erfc} \frac{2nl + x}{\sqrt{4\kappa_{\mathrm{II}}t}} + i^{k-1} \operatorname{erfc} \frac{(2n+2)l - x}{\sqrt{4\kappa_{\mathrm{II}}t}} \right\}. \tag{5.3}$$

**6. Lead times.** Reducing the second summation to a finite sum, we rewrite (5.2) to a sequence

$$T_{\mathrm{II}}(x, \kappa_{\mathrm{II}}t) = \lim_{N\to\infty} T_{\mathrm{II}}^{(N)}(x, \kappa_{\mathrm{II}}t), \tag{6.1}$$

where

$$T_{\mathrm{II}}^{(N)}(x, \kappa_{\mathrm{II}}t) = T_B + \sum_{k=0}^{\infty} B_k^{\mathrm{II}N} \left(\frac{\sqrt{4\kappa_{\mathrm{II}}t}}{l}\right)^k \sum_{n=0}^{N} \left\{ i^k \operatorname{erfc} \frac{2nl + x}{\sqrt{4\kappa_{\mathrm{II}}t}} - i^k \operatorname{erfc} \frac{(2n+2)l - x}{\sqrt{4\kappa_{\mathrm{II}}t}} \right\}, \tag{6.2}$$

which obviously satisfies conditions (2.3a) and (2.3b). Moreover, if coefficients $B_k^{\mathrm{II}N}$ are chosen to satisfy the interfacial condition (2.3c), we call (6.2) the $N$th stage solution. We express the interfacial coordinate at the $N$th stage by

$$s^{(N)}(t) = \sum_{n=0}^{N} s_n^{(N)} \tau^{(n+1)}, \tag{6.3}$$

so that it may reduce to that of the semi-infinite domain for small time.

To understand the successive emergence of infinitely many stages in (6.1), we introduce concepts of threshold and lead times. The second summand in the 0th (i.e., $n = 0$) bracket and both summands in all the subsequent ($n \geq 1$) brackets are at the initial stage less than some small number $10^{-m}$, which we call a threshold value, where $m$ is a positive number. Retaining, therefore, only the first summand in the 0th bracket, (6.2) reduces to a solution in a semi-infinite domain. At an appropriate time, called the first lead time, the second summand in the 0th bracket exceeds the threshold value. We then add it, completing the 0th bracket. At the second lead time, the first summand in the 1st (i.e., $n = 1$) bracket exceeds the threshold value. In this way the summands appear successively and the brackets are completed successively. We let mainly $m = 10$ in our numerical computation. We call $m$ the threshold power.

To describe the successive emergence of latent summands, we use a sequence of lead times as a parameter. The parametric lead time grows to infinity.

The equation for the determination of the interfacial coordinate, as we shall show, includes the lead time after the emergence of the second summand in the 0th bracket. Therefore, $s_n^{(N)}$ in (6.3) is, in general, a function of the parametric lead time. Similarly to (6.2), we rewrite (5.3) to

$$\tau\frac{\partial T_{\mathrm{II}}^{(N)}}{\partial x}(x, \kappa_{\mathrm{II}}t) = -\sum_{k=0}^{\infty}\frac{B_k^{\mathrm{II}N}}{\sqrt{4\kappa_{\mathrm{II}}}}\left(\frac{\sqrt{4\kappa_{\mathrm{II}}t}}{l}\right)^k\sum_{n=0}^{N}\left\{i^{k-1}\operatorname{erfc}\frac{2nl+x}{\sqrt{4\kappa_{\mathrm{II}}t}} + i^{k-1}\operatorname{erfc}\frac{(2n+2)l-x}{\sqrt{4\kappa_{\mathrm{II}}t}}\right\}. \tag{6.4}$$

**7. Inverse-Laplace-integral type formula.** The equations that are found by substituting (6.3) into the right sides of (6.2) and (6.4) must be developed into series of $\tau$. To do this, we apply Faa de Bruno's formula with the help of the following,

$$i^n\operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\lambda^{-n-1}\exp\left(\frac{\lambda^2}{4} - \frac{\lambda x}{\sqrt{4\kappa t}}\right)d\lambda, \tag{7.1}$$

which is valid for any integer $n$, $-\infty < n < \infty$. For a nonnegative integer $n$, we keep the real part of $c$ positive, so that the integral is convergent. (Subscript II is dropped in this section.)

To prove (7.1) for such an integer $n$, we first transform $k(x, \kappa t)$ in (1.13) to

$$k(x, \kappa t) = \frac{1}{2\pi i}\frac{1}{\sqrt{4\kappa t}}\int_{c-i\infty}^{c+i\infty}\exp\left(\frac{\lambda^2}{4} - \frac{\lambda x}{\sqrt{4\kappa t}}\right)d\lambda, \tag{7.2}$$

where $c$ is a complex number whose real part is finite. Substituting $k(x+y, \kappa t)$ in (1.11) with the integral obtained from (7.2), integrating with regard to $y$, and comparing with (1.9), we find (7.1) for a positive integer. To prove (7.1) for a negative integer $n$, we begin by noting that (3.8) yields

$$i^{-n}\operatorname{erfc}(x/\sqrt{4\kappa t}) = 2(-1)^{n-1}(\sqrt{4\kappa t})^n\cdot d^{n-1}k(x, \kappa t)/dx^{n-1}. \tag{7.3}$$

Substituting $k(x, \kappa t)$ from (7.2), (7.3) becomes (7.1) for a nonnegative integer $n$. In a finite domain, we may transform (7.1) to another form,

$$\left(\frac{\sqrt{4\kappa t}}{l}\right)^n i^n\operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-n-1}\exp\left(\xi^2\frac{\kappa t}{l^2} - \xi\frac{x}{l}\right)d\xi. \tag{7.4}$$

**8. Decomposition of the $N$th stage solution.** Substituting $x$ in $i^n\operatorname{erfc}((2nl\pm x)/\sqrt{4\kappa t})$ with $s^{(N)}(t)$ and using (7.1), we obtain an integral expression

$$i^k\operatorname{erfc}\left(\frac{2nl}{\sqrt{4\kappa_{\mathrm{II}}t}} \pm \eta\right) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\lambda^{-k-1}\exp\left(\frac{\lambda^2}{4} - \lambda\frac{2nl}{\sqrt{4\kappa_{\mathrm{II}}t}} \mp \lambda\eta\right)d\lambda, \tag{8.1}$$

where

$$\eta = \sum_{p=0}^{\infty}\frac{1}{\sqrt{4\kappa_{\mathrm{II}}}}s_p^{(N)}\tau^p.$$

Use of Faa de Bruno's formula (3.5) yields the expansion

$$e^{\pm\lambda\eta} = \exp\left(\frac{\mp\lambda s_0^{(N)}}{\sqrt{4\kappa_{\mathrm{II}}}}\right)\sum_{p=0}^{\infty}\tau^p\sum_{\nu=0}^{p}(\mp\lambda)^\nu S_{p\nu}\left(\frac{s_*^{(N)}}{\sqrt{4\kappa_{\mathrm{II}}}}\right), \tag{8.2}$$

where the convention

$$S_{00}(a_*) = 1$$

and

$$S_{n0}(a_*) = 0 \quad \text{for } n \geq 1$$

is used. Substituting (8.2) into (8.1) and using (7.1), we obtain a serial development

$$i^k \operatorname{erfc}\frac{2nl \pm s^{(N)}(t)}{\sqrt{4\kappa_{\mathrm{II}}t}} = \sum_{p=0}^{\infty}\tau^p\sum_{\nu=0}^{p}(\mp 1)^\nu S_{p\nu}\left(\frac{s_*^{(N)}}{\sqrt{4\kappa_{\mathrm{II}}}}\right)\cdot i^{k-\nu}\operatorname{erfc}\frac{2nl \pm s^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\mathrm{II}}t}}. \tag{8.3}$$

Letting $x = s^{(N)}(t)$ in (6.2) and (6.4) and using (8.3), we find series of $\tau$,

$$T_{\mathrm{II}}^{(N)}(s^{(N)}(t), \kappa_{\mathrm{II}}t) = \sum_{p=0}^{\infty} T_p^{\mathrm{II}N}\tau^p \tag{8.4a}$$

and

$$\tau\cdot\frac{\partial T_{\mathrm{II}}^{(N)}}{\partial x}(s^{(N)}(t), \kappa_{\mathrm{II}}t) = \sum_{p=0}^{\infty} T_p^{\mathrm{DII}N}\tau^p. \tag{8.4b}$$

We need only the first terms

$$T_0^{\mathrm{II}N} = T_B + B_0^{\mathrm{II}N}R_N, \tag{8.5a}$$

where

$$R_N = \sum_{n=0}^{N}\left[\operatorname{erfc}\frac{2nl + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\mathrm{II}}t}} - \operatorname{erfc}\frac{(2n+2)l - s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\mathrm{II}}t}}\right], \tag{8.5b}$$

and

$$T_0^{\mathrm{DII}N} = -(B_0^{\mathrm{II}N}/\sqrt{4\kappa_{\mathrm{II}}})Q_N, \tag{8.6a}$$

where

$$Q_N = \sum_{n=0}^{N}\left[i^{-1}\operatorname{erfc}\frac{2nl + s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\mathrm{II}}t}} + i^{-1}\operatorname{erfc}\frac{(2n+2)l - s_0^{(N)}\sqrt{t}}{\sqrt{4\kappa_{\mathrm{II}}t}}\right]. \tag{8.6b}$$

Because the coefficients $B_p^{\mathrm{II}N}$ for $p \geq 1$ are, as we shall show, all equal to zero, the higher terms need not be made explicit.

**9. The $N$th stage.** Using (3.1), we write the new phase $N$th stage temperature

$$T_{\mathrm{I}}^{(N)}(x, \kappa_{\mathrm{I}}t) = \sum_{n=0}^{\infty}(\sqrt{4\kappa_{\mathrm{I}}t})^n\left[A_n^{\mathrm{I}N}G_n\left(\frac{x}{\sqrt{4\kappa_{\mathrm{I}}t}}\right) + B_n^{\mathrm{I}N}\cdot i^n\operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa_{\mathrm{I}}}}\right)\right], \tag{9.1}$$

where the coefficients $A_0^{IN}$, $B_0^{IN}$, $A_1^{IN}, \ldots$ are to be determined to satisfy conditions (2.4a), (2.4b), and (2.3d).

We write the left-hand side of condition (2.4a) to a series of $\tau$,

$$T_{\rm I}^{(N)}(0,\kappa_{\rm I}t)=\sum_{n=0}^{\infty}(\sqrt{4\kappa_{\rm I}t})^n[A_n^{{\rm I}N}G_n(0)+B_n^{{\rm I}N}\cdot i^n\,{\rm erfc}(0)]. \tag{9.2}$$

Substituting $x$ with $s^{(N)}(t)$ from (6.3), we write the left-hand side of condition (2.4b) to a series of $\tau$,

$$T_I^{(N)}(s^{(N)}(t),\kappa_{\rm I}t)=\sum_{k=0}^{\infty}T_k^{{\rm I}N}\cdot\tau^k, \tag{9.3a}$$

where

$$T_k^{{\rm I}N}=\sum_{n=0}^{k}(\sqrt{4\kappa_{\rm I}})^n A_n^{{\rm I}N}\cdot G_{k-n}^{(n)}\left(\frac{s_*}{\sqrt{4\kappa_{\rm I}}}\right)+B_n^{{\rm I}N}\cdot I_{k-n}^{(n)}\left(\frac{s_*}{\sqrt{4\kappa_{\rm I}}}\right). \tag{9.3b}$$

Similarly, we find

$$\tau\cdot\frac{\partial T_{\rm I}}{\partial x}(s^{(N)}(t),\kappa_{\rm I}t)=\sum_{k=0}^{\infty}T_k^{{\rm DI}N}\cdot\tau^k, \tag{9.4a}$$

where

$$T_k^{{\rm DI}N}=\sum_{n=0}^{k}(\sqrt{4\kappa_{\rm I}})^{n-1}A_n^{{\rm I}N}\cdot G_{k-n}^{(n-1)}\left(\frac{s_*(N)}{\sqrt{4\kappa_{\rm I}}}\right)-B_n^{{\rm I}N}\cdot I_{k-n}^{(n-1)}\left(\frac{s_*(N)}{\sqrt{4\kappa_{\rm I}}}\right). \tag{9.4b}$$

Here, to expand the elemental functions, use is made of (3.13a), (3.13b), (3.14a), and (3.14b).

We can now successively determine $A_n^{{\rm I}N}$ and $B_n^{{\rm I}N}$ as functions of $s_0^{(N)},\ldots,s_n^{(N)}$. The first pair of this sequence is

$$A_0^{{\rm I}N}=T_F-T_A\cdot{\rm erfc}(s_0^{(N)}/\sqrt{4\kappa_{\rm I}})/\,{\rm Erf}(s_0^{(N)}/\sqrt{4\kappa_{\rm I}}) \tag{9.5a}$$

and

$$B_0^{{\rm I}N}=-(T_F-T_A)/\,{\rm Erf}(s_0^{(N)}/\sqrt{4\kappa_{\rm I}}). \tag{9.5b}$$

We write Erf with a capital E lest it be confused with erfc($x$). The higher pairs of the sequence need not be described in our problem, where they are, as we shall show, all equal to zero.

By substitution of $T_0^{{\rm II}N}$ from (8.5a), the coefficient of $\tau^0$ in the condition (2.3c) yields the evaluation of $B_0^{{\rm II}N}$,

$$B_0^{{\rm II}N}=-(T_B-T_F)/R_N. \tag{9.6}$$

By substitution of $T_0^{{\rm DI}N}$ (that is found from (9.4b) by letting $k=0$) and $T_0^{{\rm DII}N}$ (that is given in (8.6a)) into the boundary condition (2.3d), the coefficient of $\tau^0$ in (2.3d) yields the equation for the determination of $s_0^{(N)}$,

$$\frac{K_{\rm I}(T_F-T_A)}{\sqrt{4\kappa_{\rm I}}}\frac{i^{-1}\,{\rm erfc}(s_0^{(N)}/\sqrt{4\kappa_{\rm I}})}{{\rm Erf}(s_0^{(N)}/\sqrt{4\kappa_{\rm I}})}-\frac{K_{\rm II}(T_B-T_F)}{\sqrt{4\kappa_{\rm II}}}\frac{Q_N}{R_N}=\frac{1}{2}L\rho s_0^{(N)}, \tag{9.7}$$

where $Q_N$ and $R_N$ are given by (8.6b) and (8.5b), respectively. Because $Q_N$ or $R_N$ contains the lead time after the second summand in the 0th bracket has become effective, $s_0^{(N)}$ also does so. Then $A_0^{\mathrm{I}N}$, $B_0^{\mathrm{I}N}$, and $B_0^{\mathrm{II}N}$ are also functions of the $N$th lead time.

We have thus evaluated the level-0 parameters. We call the equations for determining the level-$n$ parameters level-$n$ equations. For $n \geq 1$, the level-$n$ parameters are linear in the level-$n$ equations. The level-1 equations are homogeneous with respect to the level-1 parameters, which are therefore all equal to zero. Proceeding this way, we find that the higher-than-level-0 parameters are all equal to zero.

Thus we find

$$T_{\mathrm{I}}^{(N)}(x, \kappa_{\mathrm{I}} t) = T_A + (T_F - T_A)\operatorname{Erf}\frac{x}{\sqrt{4\kappa_{\mathrm{I}} t}} \Big/ \operatorname{Erf}\frac{s_0^{(N)}}{\sqrt{4\kappa_{\mathrm{I}}}} \tag{9.8a}$$

and

$$\tau \cdot \frac{\partial T_{\mathrm{I}}^N}{\partial x}(x, \kappa_{\mathrm{I}} t) = -\frac{T_F - T_A}{\sqrt{4\kappa_{\mathrm{I}}}}\left[i^{-1}\operatorname{erfc}\frac{x}{\sqrt{4\kappa_{\mathrm{I}} t}}\right] \Big/ \operatorname{Erf}\frac{s_0^{(N)}}{\sqrt{4\kappa_{\mathrm{I}}}} \tag{9.8b}$$

for the new phase, and

$$T_{\mathrm{II}}^{(N)}(x, \kappa_{\mathrm{II}} t) = T_B - \frac{T_B - T_F}{R_N}\sum_{n=0}^{N}\left[\operatorname{erfc}\frac{2nl + x}{\sqrt{4\kappa_{\mathrm{II}} t}} - \operatorname{erfc}\frac{(2n+2)l - x}{\sqrt{4\kappa_{\mathrm{II}} t}}\right] \tag{9.9a}$$

and

$$\tau \cdot \frac{\partial T_{\mathrm{II}}^N}{\partial x}(x, \kappa_{\mathrm{II}} t) = -\frac{1}{\sqrt{4\kappa_{\mathrm{II}}}}\frac{T_B - T_F}{R_N}\sum_{n=0}^{N}\left[i^{-1}\operatorname{erfc}\frac{2nl + x}{\sqrt{4\kappa_{\mathrm{II}} t}} + i^{-1}\operatorname{erfc}\frac{(2n+2)l - x}{\sqrt{4\kappa_{\mathrm{II}} t}}\right] \tag{9.9b}$$

for the old phase. The solutions (9.8a) and (9.9a) do not satisfy the heat equation (1.1) except for the case where $N = 0$ and the 0th bracket includes only the first but not the second summand, because in other cases the parametric lead time is contained explicitly in $R_N$ and implicitly in $s_0^{(N)}$. If we may adopt a hypothetical procedure that the parametric lead time may be regarded constant in the time differentiation, the equation (1.1) is always satisfied.

**10. The final stage.** To formulate the final stage, we transform the infinite series,

$$L_k = \left(\frac{\sqrt{4\kappa t}}{l}\right)^k \sum_{n=0}^{\infty}\left[i^k\operatorname{erfc}\frac{2nl + x}{\sqrt{4\kappa t}} - i^k\operatorname{erfc}\frac{(2n+2)l - x}{\sqrt{4\kappa t}}\right] \tag{10.1}$$

and

$$M_k = \left(\frac{\sqrt{4\kappa t}}{l}\right)^k \sum_{n=0}^{\infty}\left[i^k\operatorname{erfc}\frac{2nl + x}{\sqrt{4\kappa t}} + i^k\operatorname{erfc}\frac{(2n+2)l - x}{\sqrt{4\kappa t}}\right], \tag{10.2}$$

contained in (6.2) and (6.4), respectively. (We drop the subscript II in this computation.) Although we need only $L_0$ and $M_{-1}$ in our problem, we transform the general cases $L_k$ and $M_k$ at this stage of the exposition.

We begin with the transformation of the infinite series

$$\left(\frac{\sqrt{4\kappa t}}{l}\right)^k \sum_{n=0}^{\infty} i^k\operatorname{erfc}\frac{2nl + x}{\sqrt{4\kappa t}}.$$

Using (7.4), this becomes

$$\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2\frac{\kappa t}{l^2} - \xi\frac{x}{l}\right)\sum_{n=0}^{\infty} e^{-2n\xi}\, d\xi$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2\frac{\kappa t}{l^2} + \xi\frac{l-x}{l}\right) \operatorname{Cosech}\xi\, d\xi.$$

We write $\operatorname{Cosech} x$ with a capital $C$, lest it be confused with $\operatorname{cosec} x$. Similarly,

$$\left(\frac{\sqrt{4\kappa t}}{l}\right)^k \sum_{n=0}^{\infty} i^k \operatorname{erfc}\frac{(2n+2)l - x}{\sqrt{4\kappa t}}$$

transforms to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2\frac{\kappa t}{l^2} - \xi\frac{l-x}{l}\right) \operatorname{Cosech}\xi\, d\xi.$$

Thus we find

$$L_k = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2\frac{\kappa t}{l^2}\right) \operatorname{Sinh}\left(\xi\frac{l-x}{l}\right) \operatorname{Cosech}\xi\, d\xi. \tag{10.3}$$

Similarly, we find

$$M_k = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \xi^{-k-1} \exp\left(\xi^2\frac{\kappa t}{l^2}\right) \operatorname{Cosh}\left(\xi\frac{l-x}{l}\right) \operatorname{Cosech}\xi\, d\xi. \tag{10.4}$$

In our problem, the temperatures at the final stage are stationary and linear in space. To show this, we begin by letting

$$\phi(\xi) = \exp\left(\xi^2\frac{\kappa t}{l^2}\right) \operatorname{Sinh}\left(\xi\frac{l-x}{l}\right) / \operatorname{Sinh}\xi \tag{10.5a}$$

in (10.3) and

$$\phi(\xi) = \exp\left(\xi^2\frac{\kappa t}{l^2}\right) \operatorname{Cosh}\left(\xi\frac{l-x}{l}\right) (\xi/\operatorname{Sinh}\xi) \tag{10.5b}$$

in (10.4). Noting that the functions $\phi(\xi)$ are even, we find that $L_0$ and $M_{-1}$ are given by the respective residues at $\xi = 0$. Thus we find

$$L_0 = 1 - x/l \tag{10.6a}$$

and

$$M_{-1} = 1. \tag{10.6b}$$

To formulate the final temperature, we note that the final interfacial coordinate, $s(\infty)$, is given by

$$s(\infty) = \lim_{N\to\infty} s_0^{(N)}\sqrt{t}. \tag{10.7}$$

Note that as $N \to \infty$, also $t \to \infty$. Noting that

$$\operatorname{Erf}(x) \doteqdot \frac{2}{\sqrt{\pi}}x \tag{10.8}$$

for small $x$, and taking the limit in (9.8a), we find that the final temperature in the new phase is given by

$$T_{\mathrm{I}}^{(\infty)}(x, \kappa_{\mathrm{I}} t) = T_A + (T_F - T_A)x/s(\infty). \tag{10.9}$$

Using $M_{-1}$ in (10.6b) and $L_0$ in (10.6a), we find

$$\lim_{N\to\infty} Q_N = \sqrt{4\kappa_{\mathrm{II}} t}/l \tag{10.10a}$$

and

$$\lim_{N\to\infty} R_N = 1 - s(\infty)/l, \tag{10.10b}$$

respectively. Dividing all the terms in (9.7) by $t^{1/2}$, and letting $t \to \infty$, we find that the limit of (9.7) is stationary, given by

$$\frac{K_{\mathrm{I}}(T_F - T_A)}{s(\infty)} - \frac{K_{\mathrm{II}}(T_B - T_F)}{l - s(\infty)} = 0. \tag{10.11}$$

The phase change therefore finally stops. Using (10.10b) in (9.6), we find that the limit of (9.9a) is given by

$$T_{\mathrm{II}}^{(\infty)}(x, \kappa_{\mathrm{II}} t) = T_B - (T_B - T_F)(l - x)/(l - s(\infty)). \tag{10.12}$$

Solutions (10.9), (10.11), and (10.12) are exact, because they may be found directly by applying the linearity of the final temperature profile to the boundary and interfacial conditions in Sec. 2.

**11. Interfacial coordinates.** Introducing the nondimensional interfacial coordinates $\xi$ and $\eta$ by

$$\xi = s^{(N)}(t)/l \tag{11.1}$$

and

$$\eta = (4\kappa_{\mathrm{II}} t)^{1/2}/l, \tag{11.2}$$

and defining

$$U_k(\xi, \eta) = \sum_{n=0}^{N} \left[ i^{-k} \operatorname{erfc} \frac{2n + \xi}{\eta} + (-1)^{k-1} \cdot i^{-k} \operatorname{erfc} \frac{2n + 2 - \xi}{\eta} \right] \tag{11.3}$$

for integers $k \geq 0$, we rewrite the transcendental equation (9.7) to

$$W(\xi, \eta) = \operatorname{Erf}(\beta\xi/\eta) - (2a/\sqrt{\pi}) \exp(-(\beta\xi/\eta)^2) \cdot U_0(\xi, \eta)/\{(b\xi/\eta)U_0(\xi, \eta) + U_1(\xi, \eta)\} = 0, \tag{11.4}$$

where

$$\beta = (\kappa_{\mathrm{II}}/\kappa_{\mathrm{I}})^{1/2}, \tag{11.5}$$

$$a = \beta(K_{\mathrm{I}}/K_{\mathrm{II}})(T_F - T_A)/(T_B - T_F), \tag{11.6}$$

and

$$b = 2L\rho\kappa_{\mathrm{II}}/(K_{\mathrm{II}}(T_B - T_F)). \tag{11.7}$$

The domains of $\xi$ and $\eta$ in $W(\xi, \eta)$ are

$$0 \leq \xi < \xi_\infty \tag{11.8}$$

and

$$0 \leq \eta < \infty, \tag{11.9}$$

where

$$\xi_\infty = s(\infty)/l, \tag{11.10}$$

at which $\eta$ becomes infinite.

The derivatives of $U_k(\xi, \eta)$ are given by

$$\eta \cdot \partial U_k / \partial \eta = -U_{k+1}, \tag{11.11}$$

$$\eta \cdot \partial U_k / \partial \eta = kU_k + \frac{1}{2} U_{k+2}. \tag{11.12}$$

The derivatives $W_\xi$ and $W_\eta$ are given by

$$\frac{\sqrt{\pi}}{2} \frac{\eta}{\beta} e^{(\beta\xi/\eta)^2} \cdot W_\xi(\xi, \eta)$$

$$= 1 + \frac{2a\xi}{\eta} U_0 \Big/ \left(\frac{b\xi}{\eta} U_0 + U_1\right) + \frac{a}{\beta}(bU_0^2 + 2U_1^2 - U_0U_2) \Big/ \left(\frac{b\xi}{\eta} U_0 + U_1\right)^2 \tag{11.13}$$

and

$$\frac{\sqrt{\pi}}{2} \eta e^{(\beta\xi/\eta)^2} \cdot W_\eta(\xi, \eta) = -\frac{\beta\xi}{\eta} - a\left\{U_0^2 \cdot \frac{b\xi}{\eta}\left(1 + \frac{2\beta^2\xi^2}{\eta^2}\right)\right.$$

$$\left. - U_0U_1 \cdot \left(1 - \frac{2\beta^2\xi^2}{\eta^2}\right) - \frac{1}{2} U_0U_3 + \frac{1}{2} U_1U_2 \right\} \Big/ \left(\frac{b\xi}{\eta} + U_1\right)^2. \tag{11.14}$$

The first summand in the $N$th bracket of $U_0(\xi, \eta)$ becomes effective when the inequality

$$\mathrm{erfc}((2N + \xi)/\eta) \geq 10^{-m} \tag{11.15}$$

is satisfied, or, when

$$(2N + \xi)/\eta \leq z, \tag{11.16}$$

where $\mathrm{erfc}(z) = 10^{-m}$. We call $z$ the threshold root, whose values are shown in Table 1 for several values of threshold powers $m$. The second summand in the $N$th bracket of $U_0(\xi, \eta)$ becomes effective when

$$(2N + 2 - \xi)/\eta \leq z. \tag{11.17}$$

Given $\xi$ and $\eta$ that satisfy $\eta z - \xi \geq 0$, we find the maximum $N$ among the nonnegative integers that satisfy (11.16), and sum up (11.3) to obtain $U_0(\xi, \eta), \ldots, U_3(\xi, \eta)$. The second summand in the $N$th bracket in (11.3) is simply added if it does not underflow.

The graph of $W(\xi, \text{const.})$ runs as shown in Figure 1. It cuts the $\xi$-axis from below. Numerical computation shows that the graph is steadily increasing in the domain $0 \leq \xi \leq 1$ in the shape convex upward. If the tangent at a point on the curve in this domain cuts the $\xi$-axis to the right of the origin, we may apply the Newton iteration to find the root $\xi$.

The graph of $W$ (const., $\eta$) runs as shown in Figure 2. As $\eta$ tends to $\infty$, the graph approaches to the $W$-axis from below. To find the root $\eta$ we first discover, as shown in the figure, such points $P$ and $Q$ on the $\eta$-axis that satisfy $W(P) \cdot W(Q) < 0$, where $Q$ needs to be located to the left of the minimum $B$. Let $S$ be the *regula falsi* [14,

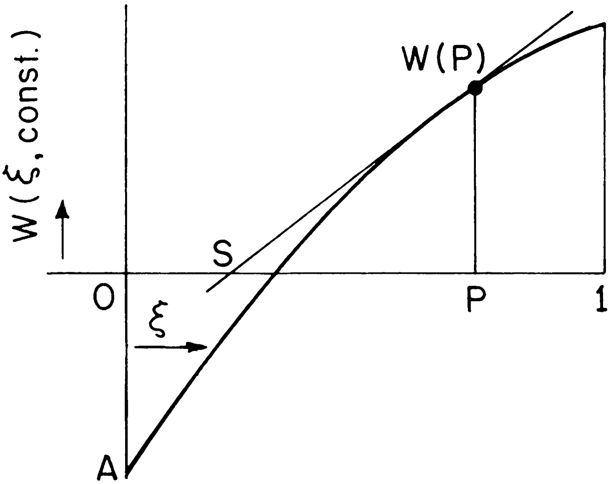


FIG. 1. $W(\xi, \text{const.})$ as function of $\xi$.

15], i.e., the intersection of the $\eta$-axis with the straight line connecting points $W(P)$ and $W(Q)$. Interval *PS* is narrower in this figure than interval *PQ*; therefore, we use the former to locate the root $R$ in this case. If the tangent drawn at point $W(S)$ falls in the interval *PS*, we apply Newton iteration at $S$. Otherwise we subdivide the range *PS* by a new *regula falsi*, and repeat the procedure.

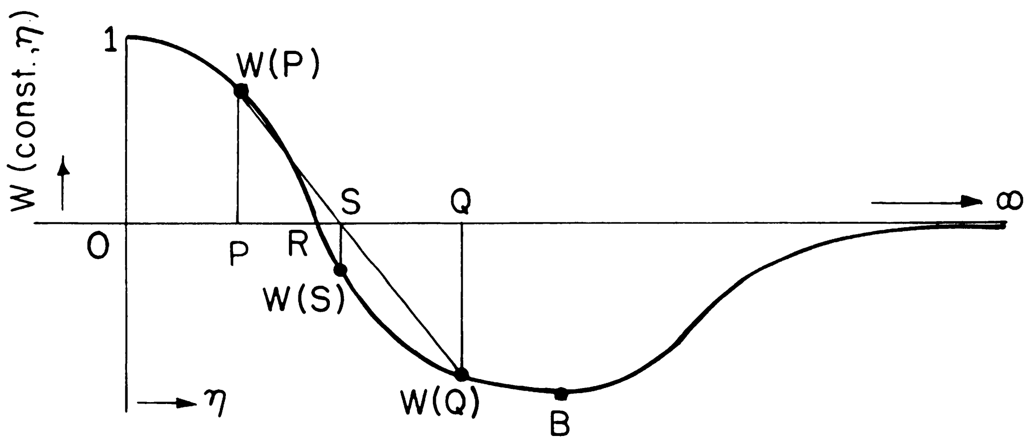


FIG. 2. $W(\text{const.}, \eta)$ as function of $\eta$.

In the 0th bracket, $\xi/\eta$, or, equivalently, $s_0^{(0)}$, is constant prior to the entrance of the second summand. The constant zone satisfies the condition

$$(2-\xi)/\eta \leq z, \tag{11.18}$$

which defines a domain contiguous to the one found by letting $N = 0$ in (11.17). The end of the constant coefficient zone may therefore be defined by the solution of the simultaneous equations $W(\xi, \eta) = 0$ and $\xi + \eta z - 2 = 0$. Table 1 shows the end of the constant coefficient zone for various threshold powers in the case of the freezing of water, $T_A = -5°C$, $T_F = 0°C$, and $T_B = 5°C$. The material constants used are: $K_{\rm I} = 2.2180$ J/(m s C), $K_{\rm II} = 0.5688$ J/(m s C), $\kappa_{\rm I} = 1.15 \times 10^{-6}$ m$^2$/s , $\kappa_{\rm II} = 1.44 \times 10^{-7}$ m$^2$/s, $L = 3.35176 \times 10^5$ J/kg. These values yield $\xi_\infty = 0.7958985952$, where use is made of (10.11) and (11.10).

| Threshold power | Threshold root | $\xi_{\rm end}$ | $\eta_{\rm end}$ | $\xi_{\rm end}/\eta_{\rm end}$ | $T$ at $x = l$ in a semi-infinite domain |
|---|---|---|---|---|---|
| 3 | 2.326753766 | 0.2386364804 | 0.7570046928 | 0.3152377821 | 4.529240866 |
| 4 | 2.751063906 | 0.2056840653 | 0.6522261918 | 0.3153569542 | 4.770161762 |
| 6 | 3.458910737 | 0.1671161673 | 0.5299020333 | 0.3153718175 | 4.941947352 |
| 8 | 4.052237244 | 0.1444140153 | 0.4579164232 | 0.3153719935 | 4.984650553 |
| 10 | 4.572824967 | 0.1290340786 | 0.4091488161 | 0.3153719955 | 4.995826039 |
| 12 | 5.042029746 | 0.1177331889 | 0.3733152929 | 0.3153719956 | 4.998842966 |
| 15 | 5.675846347 | 0.1052780845 | 0.3338219183 | 0.3153719955 | 4.999826798 |

Table 1. End of the Constant Coefficient Zone, $\xi_{\rm end}$ and $\eta_{\rm end}$, for Various Threshold Powers.

$s_0^{(0)} = 0.3153719956$, which is identical to $\xi/\eta$ in the constant coefficient zone.

On a renewed assumption that the domain is semi-infinite whose temperature at $x = \infty$ is 5°C, we have computed, on the basis of the threshold powers shown in Table 1, temperatures at $x = l$ at the times when the interface reaches the end of the constant coefficient zone, which are included in the table. We accept that the temperature in a semi-infinite domain is applicable prior to a time at which one of the computed temperatures is deemed close enough to 5°C.

Table 2 shows the nondimensional interfacial coordinates for $m = 10$. The table also shows the differences of the temperature gradients at the terminals of both phases, a quantification for demonstrating the approach to the final steady temperature distribution.

The minimum absolute value of $W(\xi, \eta)$ chosen in this computation for defining the root of the equation (11.4) is $10^{-10}$. If the power is higher than 10, the solution process described above does not necessarily converge because of the error bound in the subroutine for evaluating $E_0(x)$, which we have defined through the formula

$$\mathrm{erfc}(x) = e^{-x^2} \cdot E_0(x). \tag{11.19}$$

The subroutine $E_0(x)$ produces 12 digits for any nonnegative $x$. The program uses Erf($x$) continued fraction [16] from $x = 0$ to 0.5, ERFC 5707 RATIONAL APPROXIMATION [17] from $x = 0.5$ to 8.0, and erfc($x$) continued fraction [16] from $x = 8.0$ to $\infty$.

| $\xi/\xi_\infty$ | $\eta$ | $\xi/\eta$ | $N$ | $\frac{\partial T_I}{\partial x_l}\Big\vert_A - \frac{\partial T_I}{\partial x_l}\Big\vert_S$ | $\frac{\partial T_{II}}{\partial x_l}\Big\vert_S - \frac{\partial T_{II}}{\partial x_l}\Big\vert_B$ |
|---|---|---|---|---|---|
| .00200000000 | .005047363787 | 0.3153719956 | 0 | − 19.51917859 | − 1543.583865 |
| .02000000000 | .05047363787 | 0.3153719957 | 0 | − 1.951917859 | − 154.3583865 |
| 0.1621237672* | 0.4091488161 | 0.3153719955 | 0 | − 0.2407935481 | − 18.93498242 |
| 0.2000000000 | 0.5047364533 | 0.3153719491 | 1 | − 0.1951917284 | − 14.76285098 |
| 0.3000000000 | 0.7574358477 | 0.3152340615 | 1 | − 0.1300142903 | − 6.371580060 |
| 0.4000000000 | 1.016674337 | 0.3131380683 | 2 | − 0.9622097573$E$ − 01 | − 1.878077954 |
| 0.5000000000 | 1.297904755 | 0.3066090143 | 2 | − 0.7380648687$E$ − 01 | − 0.3476011954 |
| 0.6000000000 | 1.618361976 | 0.2950756160 | 3 | − 0.5697351501$E$ − 01 | − 0.3204738620$E$ − 01 |
| 0.7000000000 | 2.006635332 | 0.2776433803 | 4 | − 0.4324388911$E$ − 01 | − 0.8985652141$E$ − 03 |
| 0.8000000000 | 2.543809268 | 0.2503013430 | 5 | − 0.3076205766$E$ − 01 | − 0.1869328571$E$ − 05 |
| 0.9000000000 | 3.576474772 | 0.2002834582 | 7 | − 0.1751584018$E$ − 01 | 0.4986304702$E$ − 09# |
| 0.9900000000 | 10.75615472 | .07325476716 | 24 | − 0.2131743132$E$ − 02 | Less than 1.$E$ − 10 |
| 0.9990000000 | 33.73851528 | .02356661785 | 76 | − 0.2186607922$E$ − 03 | Less than 1.$E$ − 10 |
| 0.9999000000 | 106.6011158 | .007465390953 | 243 | − 0.2192274630$E$ − 04 | Less than 1.$E$ − 10 |
| 0.9999900000 | 337.1043118 | .002360962492 | 770 | − 0.2192448449$E$ − 05 | Less than 1.$E$ − 10 |

Table 2. An Example of the Interfacial Coordinates and the Approach to the Steady State.

1. * shows the $\xi/\xi_\infty$ at the end of the constant coefficient zone for the threshold power 10.
2. $N + 1$ is the number of brackets used to compute $U_0(\xi, \eta)$.
3. The positive value marked by # shows that this is in the error bound.
4. Suffixes $A$, $S$, and $B$ of the temperature gradients mean the cold side, interface, and warm side, respectively.
5. $x_l$ is the nondimensional space coordinate $x/l$.

**Conclusion.** We have found an approximate analytical solution of Stefan's problem in a finite domain. An example is shown with constant boundary and initial conditions. The solution is expressed as a sequence of infinitely many stages. The new and old phase $N$th stage temperatures are given by (9.8a) and (9.9a), respectively.

The $N$th stage is defined when the $N$th, but not the $(N+1)$th, bracket of $U_0(\xi, \eta)$ is numerically significant. $U_0(\xi, \eta)$ is found by letting $k = 0$ in (11.3). The $N$th stage begins when the inequality (11.16) is satisfied, where $\xi$ and $\eta$ are the nondimensional coordinates defined by (11.1) and (11.2), respectively. In this paper the threshold value $10^{-10}$ is mainly used to let a new term enter into $U_0(\xi, \eta)$.

The interfacial coordinate $s_0^{(N)}(t)$ of the approximate $N$th stage solution is found by solving (9.7), or the nondimensional equation (11.4). As long as the inequality $2 - \xi \geq \eta z \geq \xi$ is satisfied, i.e., in the stage where the first summand in the 0th bracket is effective but not the second summand, $s_0^{(0)}(t)/\sqrt{t}$ is constant, showing that the initial stage solution is that of a semi-infinite domain and is exact. The end of the constant coefficient zone is recognized. Its numerical values are computed for several threshold powers and shown in Table 1. Beyond the end of the constant coefficient zone, the solution is approximate. The final interfacial position $s(\infty)$ is given by solving (10.11). The final new and old phase temperatures are linear as shown by (10.9) and (10.12), respectively. Approach to the final temperature profile is numerically described in Table 2. The final temperature profile solutions are exact, because (10.9), (10.11), and (10.12) can be found by applying the linearity of the final temperature profile on the assigned boundary and interfacial conditions.

Because the interfacial coordinate $s(t)$ reduces initially to that of the semi-infinite domain, it must be assumed even in a finite domain in the form of (2.1). Because of the singularities that are introduced at the terminal boundaries, however, the temperature in the old phase cannot be developed into a Taylor series of $\sqrt{t}$ in the neighborhood of $t = 0$. In this situation the only choice we can make for obtaining a series of $\sqrt{t}$ is that the nondevelopable time variable be included in the coefficients $s_n^{(N)}$. This is the way we have arrived at the use of the parametric lead time, inclusion of which makes, however, our temperature solution, (9.8a) and (9.9a), approximate. The approximate temperature solution satisfies the boundary, initial, interfacial, and final temperature conditions, but not always the differential equation (1.1). If the parametric lead time were handled as if constant in the time differentiation, our solution satisfies all the conditions.

The computer programs are written in Fortran 77 and run on the PRIME 9750 by use of the double precision that retains fourteen significant digits.

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