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A VARIATIONAL FORMALISM FOR STEADY FLOW OF DUSTY FLUID PROBLEMS I*

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Abstract. A general variational formalism for the solution of steady flow of dusty fluids is given. The associated boundary conditions are enforced by suitable terms in a functional which is stationary at the solution of the given problem and consequently the expansion functions used need not satisfy any of the boundary conditions.

Introduction. Problems of mechanics of systems with more than one phase have been considered by many authors. (See Saffman [10], Rao [9], Didwania and Homsy [5] and references cited there.) Situations which occur frequently are concerned with motion of liquid or gas which contains a distribution of solid particles. Such situations occur, for example, in the movement of dusty laden air, in problems of fluidization, in the use of dust in gas-cooling systems to enhance heat transfer process, and in the process by which raindrops are formed by the coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence.

In a number of papers, variational principles have been derived for the solution of various boundary value problems: Tani [13], Sloan [11], Barrett [1], Wenger [14], and Smith [12].

More recently, variational principles have been derived for stationary advective problems where the boundary conditions are incorporated via suitable functionals and the restriction that the expansion functions need to satisfy the boundary conditions has been removed. (See Haj et al [6].)

In solving convective diffusion problems, Moult, Burley, and Rawson [8] produced a functional corresponding to a local potential approach, where the first derivative term is kept fixed during the variation.

In the present paper we propose a suitable functional for the solution of steady flow of dusty fluid problems which contain first and second derivative terms by considering a general variational formulation rather than the local potential approach of [8]. We extend the technique of [6] by enforcing the prescribed boundary conditions by the proposed functional for the solution of the two phase problems.

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1. Basic equations. The equations of motion of a dusty, steady, viscous incompressible fluid are (see Saffman [10]):

$$(\omega \cdot \nabla)\omega = -1/\rho \operatorname{grad} p + \nu \nabla^2 \omega + (KN/\rho)(\mathbf{u} - \omega), \tag{1.1}$$

$$\operatorname{div} \omega = 0, \tag{1.2}$$

$$m[(\mathbf{u} \cdot \nabla)\mathbf{u}] = K(\omega - \mathbf{u}), \tag{1.3}$$

$$\operatorname{div}(N\mathbf{u}) = 0, \tag{1.4}$$

where ω and $\mathbf{u}$ denote the local velocity vectors of the fluid and dust particles respectively, ρ the density, p the static fluid pressure, ν the kinematic viscosity, N the number density of a dust particle, K the Stokes resistance coefficient (for spherical particles of radius δ, it is $6\pi\mu\delta$, μ is the fluid viscosity) and m the mass of a dust particle.

Consider the velocity distribution of fluid and dust particles defined in the Cartesian coordinates $\mathbf{x} \equiv (x, y)$ respectively as:

$$\omega = (0, 0, \omega(\mathbf{x})), \tag{1.5}$$

$$\mathbf{u} = (U_0, U_1, u(\mathbf{x})), \tag{1.6}$$

$$N = N_0 (\text{a constant}), \tag{1.7}$$

where $(0, 0, \omega(\mathbf{x}))$ and $(U_0, U_1, u(\mathbf{x}))$ are the velocity components of the fluid and dust particles, respectively; U_0 and U_1 are constants. The equations of motion then reduce to the following coupled equations in a region R with boundary ∂R:

$$-A\nabla^2 \omega(\mathbf{x}) + D\omega(\mathbf{x}) - Du(\mathbf{x}) = G, \quad \mathbf{x} \text{ in } R, \tag{1.8}$$

$$\mathbf{C} \cdot \nabla u(\mathbf{x}) + Du(\mathbf{x}) - D\omega(\mathbf{x}) = 0, \quad \mathbf{x} \text{ in } R, \tag{1.9}$$

subject to the following boundary conditions:

$$\omega(\mathbf{x}) = g_1, \quad \mathbf{x} \text{ on } \partial R, \tag{1.10}$$

$$u(\mathbf{x}) = g_2, \quad \mathbf{x} \text{ on } \partial R, \tag{1.11}$$

where

$$A = \mu/(mN_0), \qquad D = K/m, \qquad G = -\frac{dp}{dz}/(mN_0) = \text{constant},$$
$$\text{and } \mathbf{C} \equiv (U_0, U_1). \tag{1.12}$$

2. A functional embodying the boundary conditions. In this section we introduce the following adjoint problem to problem (1.8)–(1.11):

$$-A\nabla^2 \psi(\mathbf{x}) + D\psi(\mathbf{x}) - Dv(\mathbf{x}) = G, \quad \mathbf{x} \text{ in } R, \tag{2.1}$$

$$-\mathbf{C} \cdot \nabla v(\mathbf{x}) + Dv(\mathbf{x}) - D\psi(\mathbf{x}) = 0, \quad \mathbf{x} \text{ in } R, \tag{2.2}$$

with the associated boundary conditions:

$$\psi(\mathbf{x}) = g_1, \quad \mathbf{x} \text{ on } \partial R, \tag{2.3}$$

$$v(\mathbf{x}) = g_2, \quad \mathbf{x} \text{ on } \partial R. \tag{2.4}$$

Next, we produce a functional which is stationary at the solution of (1.8)–(1.11) and (2.1)–(2.4) for a class of functions which do not necessarily satisfy any of the boundary conditions.

THEOREM. The solutions to the system (1.8)–(1.11) and (2.1)–(2.4) render the following functional stationary at the solution $W = \omega$, $U = u$, $\Psi = \psi$, and $V = v$:

$$
\begin{aligned}
F[W, U, \Psi, V] = \int_R \{ & A\nabla\Psi \cdot \nabla W + D\Psi W - DWV - D\Psi U - G(W + \Psi) \\
& + VC \cdot \nabla U + VDU\} d\mathbf{x} \\
& + \oint_{\partial R} [(g_1 - W)A\nabla\Psi + (g_1 - \Psi)A\nabla W] \cdot \mathbf{n}\, ds \\
& + \oint_{\partial R} (g_2 - U)[(g_2 - V) + V\mathbf{C} \cdot \mathbf{n}]\, ds
\end{aligned} \tag{2.5}
$$

where $\mathbf{n}$ is the unit outward normal, ds is the differential along ∂R with positive direction being counterclockwise.

Proof. Let ω and u be the solution to (1.8)–(1.11), ψ and v be the solution to (2.1)–(2.4); s_i, $i = 1, 2, 3, 4$ arbitrary functions.

Define

$$
\begin{aligned}
G_1(\varepsilon) &= F[\omega + \varepsilon s_1, u, \psi, v], \\
G_2(\varepsilon) &= F[\omega, u + \varepsilon s_2, \psi, v], \\
G_3(\varepsilon) &= F[\omega, u, \psi + \varepsilon s_3, v], \\
G_4(\varepsilon) &= F[\omega, u, \psi, v + \varepsilon s_4].
\end{aligned}
$$

Then F is stationary at the solution of (1.8)–(1.11) and (2.1)–(2.4) if

$$\frac{dG_k(0)}{d\varepsilon} = 0, \quad k = 1, 2, 3, 4.$$

Now,

$$
\begin{aligned}
\frac{dG_1(0)}{d\varepsilon} = \int_R & (A\nabla\psi \cdot \nabla s_1 + D\psi s_1 - Ds_1 v - Gs_1)\, d\mathbf{x} \\
& + \oint_{\partial R} [-As_1\nabla\psi + (g_1 - \psi)A\nabla s_1] \cdot \mathbf{n}\, ds.
\end{aligned} \tag{2.6}
$$

Using Green's theorem and (2.1), the first integral in (2.6) reduces to

$$\oint_{\partial R} As_1 \nabla\psi \cdot \mathbf{n}\, ds.$$

This integral cancels the last integral in (2.6) taking into consideration the boundary conditions (2.3). Hence,

$$\frac{dG_1(0)}{d\varepsilon} = 0.$$

In the same manner, it can easily be proved that

$$\frac{dG_2(0)}{d\varepsilon} = \frac{dG_3(0)}{d\varepsilon} = \frac{dG_4(0)}{d\varepsilon} = 0,$$

and hence the functional F is stationary at the solution.

3. Matrix set-up for the solution. Now we seek approximate solutions to (1.8)–(1.11) and (2.1)–(2.4) of the form:

$$\begin{aligned} \omega \simeq \omega_M &= \sum_{i=1}^{M} a_i^{(1)} h_i(\mathbf{x}), & u \simeq u_M &= \sum_{i=1}^{M} a_i^{(2)} h_i(\mathbf{x}), \\ \psi \simeq \psi_M &= \sum_{i=1}^{M} b_i^{(1)} h_i(\mathbf{x}), & v \simeq v_M &= \sum_{i=1}^{M} b_i^{(2)} h_i(\mathbf{x}). \end{aligned} \tag{3.1}$$

Substituting (3.1) into the functional (2.5) and taking its stationary value with respect to $b_i^{(1)}$, $b_i^{(2)}$ leads to the defining equations for the coefficients $a_i^{(1)}$ and $a_i^{(2)}$ (equations for $b_i^{(1)}$ and $b_i^{(2)}$ are not needed):

$$\begin{pmatrix} L & H \\ H & E \end{pmatrix} \begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}^{(1)} \\ \mathbf{Q}^{(2)} \end{pmatrix} \tag{3.2}$$

where the matrices L, H, and E are $M \times M$ matrices and $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, $\mathbf{Q}^{(1)}$, and $\mathbf{Q}^{(2)}$ are M vectors. The various matrices and vectors of (3.2) are given by:

$$\begin{aligned} L_{ij} &= \int_R (\nabla h_j \cdot A \nabla h_i + h_j D h_i)\, d\mathbf{x} - \int_{\partial R} (h_i A \nabla h_j + h_j A \nabla h_i) \cdot \mathbf{n}\, ds, \\ H_{ij} &= -\int_R h_j D h_i \, d\mathbf{x}, \\ E_{ij} &= \int_R (h_j \mathbf{C} \cdot \nabla h_i + h_j D h_i)\, d\mathbf{x} + \int_{\partial R} (h_i h_j + h_i h_j \mathbf{C} \cdot \mathbf{n})\, ds, \\ Q_j^{(1)} &= \int_R G h_j \, d\mathbf{x} - \int_{\partial R} A g_1 \nabla h_j \cdot \mathbf{n}\, ds, \end{aligned}$$

and

$$Q_j^{(2)} = \int_{\partial R} g_2 (h_j - h_j \mathbf{C} \cdot \mathbf{n})\, ds.$$

It is desirable, for stability reasons (see Mikhlin [7]) to take the basis functions $h_i(\mathbf{x})$ in (3.1) as a set of orthogonal polynomials. Without loss of generality, and to keep the exposition simple, we consider the one-dimensional basis $h_i(x)$ and, for programming conveniency, we will define h_k, $k = -2, -1, i$ as

$$h_{-2}(x) = 1, \qquad h_{-1}(x) = x, \qquad h_i(x) = -(1 - x^2) T_i(x), \tag{3.3}$$

where $-1 \le x \le 1$, and $i = 0, 1, \cdots, M-3$; $T_i(x)$ is a Chebyshev polynomial of the first kind. The reason for this choice of basis is to avoid introducing artificial singularities in the matrix equation (3.2).

In calculating the elements of the various matrices and vectors in (3.2) we need the following:

$$d_n^{(1)} = (2/\pi) \int_{-1}^{1} [(1 - x^2) T_n(x) / (1 - x^2)^{1/2}]\, dx, \quad n \ge 0, \tag{3.4}$$

where

$$d_n^{(1)} = \begin{cases} -\frac{1}{2} & \text{if } n = 2, \\ 1 & \text{if } n = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.5}$$

using the results,

$$1 - x^2 = \tfrac{1}{2}[T_0(x) - T_2(x)]$$

and (3.6)

$$T_n T_r = \tfrac{1}{2}[T_{n+r} + T_{n-r}]$$

and the orthogonality property of Chebyshev polynomials. We also need

$$d_n^{(2)} = (2/\pi)\int_{-1}^{1} [(1-x^2)^2 T_n(x)/(1-x^2)^{1/2}]\,dx \tag{3.7}$$

which can be related to $d_n^{(1)}$ by the following:

$$d_n^{(2)} = \tfrac{1}{4}[-d_{n+2}^{(1)} + 2d_n^{(1)} - d_{|n-2|}^{(1)}], \tag{3.8}$$

where we have used (3.7), (3.6), and the property $T_{-n} = T_n$. Similarly, it can be shown that

$$d_n^{(r+\frac{1}{2})} = (2/\pi)\int_{-1}^{1} [(1-x^2)^{r+\frac{1}{2}} T_n(x)/(1-x^2)^{\frac{1}{2}}]\,dx \tag{3.9}$$

can be related to $d_n^{(r)}$ by the following:

$$d_n^{(r+\frac{1}{2})} = (2/\pi)\left[d_n^{(r)} - \sum_{m\geq 1} \frac{1}{4m^2-1}\left(d_{2m+n}^{(r)} + d_{|2m-n|}^{(r)}\right)\right] \tag{3.10}$$

which follows from (3.6) and the well-known Chebyshev relation:

$$\frac{\pi}{2}(1-x^2)^{\frac{1}{2}} = 1 - \sum_{m\geq 1}[2/(4m^2-1)]T_{2m}(x). \tag{3.11}$$

At this stage we are able to relate the elements of the matrices and vectors in (3.2) to $d_n^{(r)}$ and $d_n^{(r+\frac{1}{2})}$, $r = 1, 2$. For example, the elements of the matrix H:

$$\begin{aligned} H_{ij} &= -D\int_{-1}^{1} h_j(x)h_i(x)\,dx \\ &= -D\int_{-1}^{1} [(1-x^2)^{5/2} T_i T_j/(1-x^2)^{\frac{1}{2}}]\,dx \\ &= -\frac{\pi D}{4}[d_{i+j}^{(5/2)} + d_{|i-j|}^{(5/2)}], \quad i, j \geq 0, \end{aligned} \tag{3.12}$$

and

$$H_{-2,-2} = -D\int_{-1}^{1} h_{-2}h_{-2}\,dx = -2D, \tag{3.13}$$

$$H_{-1,-1} = -D\int_{-1}^{1} h_{-1}h_{-1}\,dx = (-2/3)D, \tag{3.14}$$

$$H_{-1,-2} = -D\int_{-1}^{1} h_{-1}h_{-2}\,dx = 0, \tag{3.15}$$

$$H_{ij} = H_{ji}. \tag{3.16}$$

In the same manner, we can relate the elements of the form

$$\widehat{L}_{ij} = \int_{-1}^{1} h'_j \cdot A h'_i \, dx \quad \text{and} \quad \widehat{E}_{ij} = \int_{-1}^{1} h_j \widehat{C} h'_i \, dx \quad (\text{where } h' \equiv dh/dx)$$

to $d_n^{(r)}$ using the identity

$$\frac{d}{dx}[(1-x^2)T_i(x)] = \tfrac{1}{2}[(i-2)T_{i-1}(x) - (i+2)T_{i+1}(x)]. \tag{3.17}$$

In two dimensions, we can write $h_i(x, y)$ as

$$h_i(x, y) = h_p(x)h_q(y) \quad p = 1, \cdots, M_p; \; q = 1, \cdots, M_q \tag{3.18}$$

where $i \to (p, q)$ is a bijection and $M = M_p M_q$. In this case, we can read the elements of the matrices (3.2) directly from the one-dimensional version, see Delves and Phillips [4], where a similar treatment is considered there. Note that the assumption that $x \in [-1, 1]$ causes no problem because it can be extended to a general closed interval $[a, b]$ by using the change of variables $X = \frac{1}{2}[(b-a)x + (a+b)]$ to transform the number $x \in [-1, 1]$ into the corresponding number $X \in [a, b]$.

While we do not attempt an error analysis here, the properties of our formalism are well understood; see for example [4], where a similar treatment is given for integral equation problems. According to this reference, our formalism yields a rapid convergence provided that the integrands involved in (3.4) are smooth; this is so with the procedure given in this paper.

4. A special case of zero Stokes resistance coefficient. Here, we consider spherical particles of radius δ, when $\delta \to 0$, i.e., $K \to 0$, $D \to 0$, and $\mathbf{C} \cdot \nabla u = 0$. In this case the functional (2.5) reduces to

$$F(W) = \int_R [A(\nabla W)^2 - 2GW] \, d\mathbf{x} + 2\int_{\partial R} (g_1 - W) A \nabla W \cdot \mathbf{n} \, ds. \tag{4.1}$$

The approximate solution of the system is

$$\omega \simeq \omega_M = \sum_{i=1}^{M} a_i h_i(\mathbf{x}), \tag{4.2}$$

where the coefficients a_i are defined by

$$L\mathbf{a} = \mathbf{Q}, \tag{4.3}$$

where the $M \times M$ matrix L and the M vector $\mathbf{Q}$ are given by

$$L_{ij} = \int_R \nabla h_j \cdot A \nabla h_i \, d\mathbf{x} - \int_{\partial R} (h_i A \nabla h_j + h_j A \nabla h_i) \cdot \mathbf{n} \, ds,$$

$$\mathbf{Q} = \int_R G h_j \, d\mathbf{x} - \int_{\partial R} A g_1 \nabla h_j \cdot \mathbf{n} \, ds,$$

where $A = +\mu/(mN_0)$, and $G = -(dp/dz)/(mN_0)$ as in Eq. (1.12). The functional (4.1) yields the solution (4.2) and (4.3) of the problem for steady, viscous, incompressible fluid subject to the conditions (1.5) and (1.10). We note here that this stationary functional is very similar to that given by Delves and Hall [3], whose numerical technique and solution is found in Delves and Phillips [4]. For completeness, however, we give in the next section a numerical solution for the simple zero Stokes resistance coefficients in one dimension followed by a two-dimensional problem to test the validity of the numerical technique introduced in this paper.

5. A numerical example. To apply the above technique, we now consider the following simple problem for the case of zero Stokes resistance coefficients:

$$d^2\omega/dx^2 = 2, \tag{5.1}$$

together with the associated boundary conditions:

$$\omega(0) = 0; \qquad \omega(1) = 1 \quad 0 \le x \le 1. \tag{5.2}$$

The theoretical solution is

$$\omega = x^2.$$

With the choice of basis functions (see 3.3):

$$h_{-2}(x) = 1, \qquad h_{-1}(x) = \xi; \qquad h_i(x) = (1-\xi^2)T_i(\xi), \quad i = 0, 1, \cdots,$$

where ξ is a linear map of x onto $[-1, 1]$, calculations were carried out for different numbers of expansion functions M (see Eq. (4.2)) from which we list the results for $M = 5$ in Table I. In Table II, we list the numerical results for $M = 5$ obtained by applying our technique for the following two-dimensional problem:

$$\begin{aligned} \nabla^2\omega &= 0 && \text{in } R \\ \frac{\partial\omega}{\partial n} &= 0 && \text{on } OA, OB \\ \omega &= r^4\cos 4\theta && \text{on } AB \end{aligned}$$

as shown in Fig. 1,

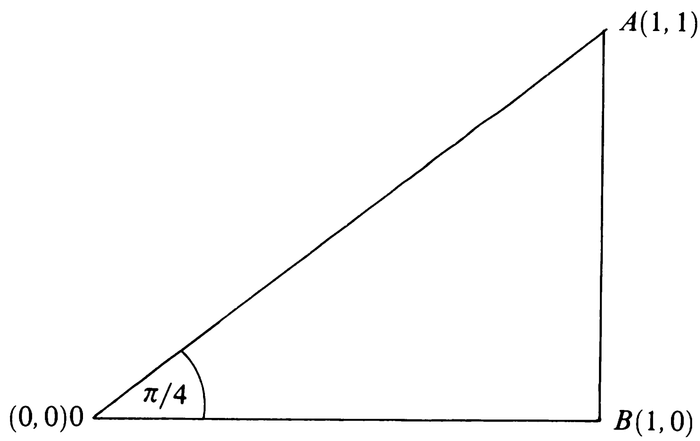

FIG. 1.

and the exact solution is

$$\omega = r^4\cos 4\theta \quad \text{everywhere in } R.$$

These results show that our technique retains the high convergence rate associated with the conventional global variational methods.

TABLE I. Computed solutions for $M = 5$.

x	Approx. Sol.	Exact Sol.	Error
0.000	− 7.766703, −13	0.000000	+ 7.766703, −13
0.125	+ 1.562499, −2	+ 1.256250, −2	+ 1.591615, −12
0.250	+ 6.249999, −2	+ 6.250000, −2	+ 4.547473, −12
0.375	+ 1.406249, −1	+ 1.406250, −1	+ 7.275957, −12
0.500	+ 2.499999, −1	+ 2.500000, −1	+ 9.094947, −12
0.625	+ 3.906249, −1	+ 3.906250, −1	+ 7.275957, −12
0.750	+ 5.625000, −1	+ 5.625000, −1	+ 7.275957, −12
0.875	+ 7.656250, −1	+ 7.656250, −1	+ 7.275957, −12
1.000	+ 1.000000, 0	+ 1.000000, 0	+ 0.000000, 0

TABLE II. Computed Solutions for $M = 5$,
$X = r\cos\theta,\ Y = r\sin\theta$.

X	Y	Approx. Sol.	True Value	Error
1.77, −6	1.77, −6	− 1.6606, − 9	− 3.906, − 23	1.6606, − 9
8.84, −2	8.84, −2	− 2.2395, − 4	− 2.4414, − 4	− 2.01907, − 5
1.77, −1	1.77, −1	− 3.9050, − 3	− 3.9062, − 3	− 1.2965, − 6
2.65, −1	2.65, −1	− 1.9826, − 2	− 1.9775, − 2	+ 5.1585, − 5
3.54, −1	3.54, −1	− 6.2136, − 2	− 6.250, − 2	− 3.645, − 4
2.08, −6	1.39, −6	1.56, − 9	− 2.762, − 23	− 1.560, − 9
1.04, −1	6.94, −2	− 1.7434, − 4	− 1.7263, − 4	1.709, − 6
4.16, −1	2.78, −1	− 4.4048, − 2	− 4.4194, − 2	− 1.4662, − 4
1.15, −1	4.78, −2	+ 3.2168, − 6	+ 0.000, + 0	− 3.2168, − 6
4.62, −1	1.91, −1	− 8.3342, − 5	+ 7.1432, − 13	+ 8.3342, − 5
2.50, −6	0.00, +0	+ 1.7867, − 9	+ 3.9063, − 23	− 1.7867, − 9
5.00, −1	0.00, +0	+ 6.2368, − 2	+ 6.250, − 2	+ 1.3175, − 4

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