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INITIAL-BOUNDARY VALUE PROBLEMS FOR THE EQUATION $u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$*

BY

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Abstract. Existence and uniqueness theorems are proved for global weak solutions of initial-boundary value problems corresponding to the equation

$$u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f$$

under assumptions that do not require smoothness or monotonicity of σ. The initial data are not assumed to be smooth, the boundary data are allowed to be time dependent, and f is only assumed to be in L^2.

1. Introduction. An equation used to model the longitudinal displacement in a homogeneous bar of uniform cross section and unit length is

$$u_{tt} = (\sigma(u_x))_x + \delta^2 u_{xxt}, \qquad x \in (0, 1),\ t \in [0, T]. \tag{1.1}$$

The question of the existence and uniqueness of solutions to (1.1) augmented with boundary conditions and initial conditions of various types has been dealt with in several papers. Among these are [4–11]. These papers all establish the existence and uniqueness of global, classical solutions to (1.1). In each paper, σ is assumed to be either quite smooth or monotone. The initial data are also taken to be very smooth. More recently, (1.1) has been considered under assumptions that allow σ to possess corners and fail to be monotone [2, 3, 18]. The initial data are also much rougher in these papers, which obtain weak rather than classical solutions to (1.1).

In this paper we consider a slightly more general equation,

$$u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f, \tag{1.2}$$

where $\alpha(V) \geq \delta$ for all V and α is continuous, the function f being a body force which is only assumed to be in $L^2((0, 1) \times (0, T))$. The assumptions made on σ are similar to those used in [2, 3] and the initial data are also the same.

It makes good sense physically to consider the situation where a nonzero time-dependent force is applied to one or both ends of the material. Also, this force need not be continuous. The mathematical model for this realistic physical problem would be (1.2), initial data

$$u(0, x) = u_0(x) \in W^{1\infty}(0, 1), \qquad u_t(0, x) = u_1(x) \in L^2(0, 1), \tag{1.3}$$

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along with boundary conditions of the form

$$u(t,1) = 0, \tag{1.4.1}$$

$$\sigma(u_x(t,0)) + \alpha(u_x(t,0))u_{xt}(t,0) = k_0(t), \tag{1.4.2}$$

or of the form

$$\sigma(u_x(t,0)) + \alpha(u_x(t,0))u_{xt}(t,0) = k_0(t), \tag{1.5.1}$$

$$\sigma(u_x(t,1)) + \alpha(u_x(t,1))u_{xt}(t,1) = k_1(t), \tag{1.5.2}$$

where k_i is only assumed to be bounded and measurable. The existence and uniqueness theory for (1.2), (1.3), and either (1.4) or (1.5) is currently unsolved in the available literature [20]. The solution of this problem when $\alpha(V) \geq \delta > 0$ is the main goal of this paper.

One might also consider the boundary conditions

$$u(t,1) = b_1(t), \qquad u(t,0) = b_0(t). \tag{1.6}$$

If σ is assumed to be globally Lipschitz, these boundary conditions may be obtained [12], but this assumption is overly restrictive for us in this paper. In the special case where $b_i(t) = 0$, the problem (1.2), (1.3), (1.6) may be shown to be well posed and the details are worked out in [15] without an assumption of a positive lower bound for α. Therefore, we omit the boundary conditions (1.6) and note that these type of conditions are less interesting than (1.4) and (1.5) because the thing that can actually be controlled at the ends of the material is not the displacement but the force.

Because of the discontinuities of k_i, the roughness of the initial data, and the lack of smoothness of σ and α, the traditional methods of obtaining estimates on local classical solutions are not easily applicable. Instead, we have chosen to estimate global weak solutions of an approximate problem obtained by adding the assumption that σ is globally Lipschitz. This approximate problem may be dealt with by using elliptic regularization and the Galerkin method. The solution to it has just barely enough regularity to allow us to obtain estimates on u_x. The basic ideas used are generalizations of those in [2], but because of the weakness of the solutions to this approximate problem, the implementation of these ideas requires careful attention to measurability questions.

Section 2 discusses a variational formulation for the approximate initial-boundary value problems and gives a proof of existence and uniqueness which is based on an abstract theorem whose proof is found in Sec. 5. Section 3 obtains the necessary estimates on the solutions to the approximate problem. Section 4 uses these estimates to obtain existence and uniqueness for the initial-boundary value problems of interest. For E a Banach space, $L^2(a,b;E)$ is the space of strongly measurable, square-integrable functions. The notation $\rightharpoonup$ will denote weak or weak* convergence, while $\rightarrow$ will mean strong convergence.

2. The approximate problem. In this section, we prove existence and uniqueness theorems for global weak solutions of initial-boundary value problems for the equation

$$u_{tt} = (\sigma(u_x))_x + (\alpha(u_x)u_{xt})_x + f \tag{2.1}$$

with initial conditions

$$u(0,x) = u_0(x), \qquad u_0 \in W^{1\infty}(0,1), \tag{2.2.1}$$

$$u_t(0,x) = u_1(x), \qquad u_1 \in L^2(0,1), \tag{2.2.2}$$

and boundary conditions which are sufficiently general to include

$$u(1,t) = 0, \tag{2.3.1}$$

$$\sigma(u_x(0,t)) + \alpha(u_x(0,t))u_{xt}(0,t) = k_0(t), \tag{2.3.2}$$

or

$$\sigma(u_x(0,t)) + \alpha(u_x(0,t))u_{xt}(0,t) = k_0(t), \tag{2.4.1}$$

$$\sigma(u_x(1,t)) + \alpha(u_x(1,t))u_{xt}(1,t) = k_1(t), \tag{2.4.2}$$

where k_i is only assumed to be bounded and measurable. The problem solved is called the approximate problem because

$$\text{For some } \sigma_0,\ W(V) = \int_0^V (\sigma(s) + \sigma_0)\,ds \text{ is bounded below,} \tag{2.5.1}$$

$$\sigma \text{ is bounded, and } |\sigma(V_1) - \sigma(V_2)| \le K|V_1 - V_2|, \tag{2.5.2}$$

assumptions that will be weakened later in Sec. 3. We also assume that α is continuous and that

$$0 \le \delta \le \alpha(V) < M_0 < \infty. \tag{2.5.3}$$

Our approach is abstract. To motivate the definitions that we will use, let $b(t,x) = k_0(t)(1-x) + k_1(t)x$ and suppose

$$\begin{aligned} &-\int_0^T\int_0^1 u_t(t,x)\varphi_t(t,x)\,dx\,dt + \int_0^T\int_0^1 [\sigma(u_x(t,x)) - b(t,x)]\varphi_x(t,x)\,dx\,dt \\ &+\int_0^T\int_0^1 \alpha(u_x(t,x))u_{xt}(t,x)\varphi_x(t,x)\,dx\,dt = \int_0^T\int_0^1 (f(t,x) + b_x(t,x))\varphi(t,x)\,dx\,dt, \end{aligned} \tag{2.6.1}$$

$$u, u_t, u_x, u_{xt} \quad \text{are in } L^2((0,T)\times(0,1)), \tag{2.6.2}$$

for all $\varphi \in C_0^\infty(0,T;E)$ where E is a closed subspace of $H^1(0,1)$ containing $C_0^\infty(0,1)$. Thus, in particular, (2.6) holds if $\varphi \in C_0^\infty((0,T)\times(0,1))$, which implies that u is a weak solution of (2.1). Integrating (2.6.1) by parts formally yields the variational boundary conditions

$$\int_0^T (\sigma(u_x(t,1)) - k_1(t) + \alpha(u_x(t,1))u_{xt}(t,1))\varphi(t,1)\,dt = 0, \tag{2.7.1}$$

$$\int_0^T (\sigma(u_x(t,0) - k_0(t) + \alpha(u_x(t,0))u_{xt}(t,0))\varphi(t,0)\,dt = 0, \tag{2.7.2}$$

with (2.7) holding for all $\varphi \in C_0^\infty(0,T;E)$. If $E = H^1(0,1)$ these conditions yield (2.4). If $E = \{u \in H^1(0,1) : u(1) = 0\}$, the boundary conditions (2.3) are obtained. These formal manipulations may all be made precise ([19] or [16]) but we shall regard (2.6) along with the initial data as the fundamental object of our study.

Letting E be the closed subspace of $H^1(0,1)$ just described, we shall let $H = L^2(0,1)$ and identify H and H'. Therefore,

$$E \subseteq H = H' \subseteq E'. \tag{2.8}$$

For $u \in E$ and $t \in [0,T]$, let $Q(u)$, $N(t,\cdot)$, and M be operators mapping E to E' given by

$$\langle Q(u)w, v\rangle = (\alpha(u_x)w_x, v_x)_H, \tag{2.9.1}$$

$$\langle N(t,w), v\rangle = (\sigma(w_x) - b(t,\cdot), v_x)_H, \tag{2.9.2}$$

$$\langle Mw, v\rangle = (\beta(w_x), v_x)_H, \tag{2.9.3}$$

where

$$\beta(V) = \int_0^V \alpha(s)\,ds. \tag{2.10}$$

For $h \in L^1(0,T;E')$, h' may be defined as an E'-valued distribution by the rule

$$h'(\varphi) = -\int_0^T h(t)\varphi'(t)\,dt \tag{2.11}$$

for all $\varphi \in C_0^\infty(0,T)$. Also let $\mathscr{V} = L^2(0,T;E)$ and if $u \in \mathscr{V}$ consider $Q(u)$, N, and M to be operators from $\mathscr{V}$ to $\mathscr{V}' = L^2(0,T;E')$ defined by

$$(Q(u)w)(t) = Q(u(t))w(t), \tag{2.12.1}$$

$$(Nw)(t) = N(t, w(t)), \tag{2.12.2}$$

$$(Mw)(t) = M(w(t)). \tag{2.12.3}$$

With the above conventions and definitions, we can state the following theorem.

THEOREM 1. Let $u_1 \in H$, $u_0 \in E$, and $g \in \mathscr{V}'$. Then there exists a unique solution u to

$$u'' + Nu + Q(u)u' = g \quad \text{in } \mathscr{V}', \tag{2.13.1}$$

$$u(0) = u_0, \tag{2.13.2}$$

$$u'(0) = u_1, \tag{2.13.3}$$

$$u \in L^\infty(0,T;E), \qquad u' \in \mathscr{V} \cap L^\infty(0,T;H). \tag{2.13.4}$$

A theorem similar to Theorem 1 is proved in [13] but there are some differences here and so we have chosen to present a proof. This is done in Sec. 5 to avoid obscuring the main ideas.

LEMMA 1. Suppose $u, u' \in L^2(a,b;E)$ and let $\bar{u}_t(\cdot,\cdot)$ and $\bar{u}_{tx}(\cdot,\cdot)$ be Borel measurable representatives for $u'(\cdot)(\cdot)$ and $(\partial u'(\cdot)/\partial x)\,(\cdot)$, respectively. Also define

$$\bar{u}_x(t,x) = \int_a^t \bar{u}_{tx}(s,x)\,ds + u_0'(x), \tag{2.14.1}$$

$$\bar{u}(t,x) = \int_a^t \bar{u}_t(s,x)\,ds + u_0(x). \tag{2.14.2}$$

Then

(1) $\bar{u}(t,\cdot) = u(t)$ a.e., $\bar{u}_t(t,\cdot) = u'(t)$ a.e.,
(2) $\bar{u}_x(t,\cdot) = \frac{\partial u(t)}{\partial x}$ a.e., $\bar{u}_{tx}(t,\cdot) = \frac{\partial u'(t)}{\partial x}$ a.e.,
(3) $\bar{u}_x = \frac{\partial}{\partial x}(\bar{u})$ in sense of distributions,
(4) $\bar{u}_{tx} = \frac{\partial}{\partial x}(\bar{u}_t)$ in sense of distributions.

LEMMA 2. Let $\bar{u}$ be the measurable representative of the solution of Theorem 1 with $g = f + b_x \in L^2(0,T;H)$ described in Lemma 1. Then (2.6) holds for each $\varphi \in C_0^\infty(0,T;E)$.

Proof. Let $\varphi \in C_0^\infty(0,T;E)$ and use (2.13.1) to write

$$\langle u'', \varphi\rangle_{\mathscr{V}'\mathscr{V}} + \langle Nu, \varphi\rangle_{\mathscr{V}'\mathscr{V}} + \langle Q(u)u', \varphi\rangle_{\mathscr{V}'\mathscr{V}} = \langle f + b_x, \varphi\rangle_{\mathscr{V}'\mathscr{V}}. \tag{2.15}$$

Consider the first term of (2.15):

$$\begin{aligned}
\langle u'', \varphi\rangle_{\mathscr{V}'\mathscr{V}} &= \int_0^T \int_0^t \langle u''(t), \varphi'(s)\rangle\, ds\, dt \\
&= \int_0^T \int_s^T \langle u''(t), \varphi'(s)\rangle\, dt\, ds \\
&= -\int_0^T \langle u'(s), \varphi'(s)\rangle\, ds + \int_0^T \langle u'(T), \varphi'(s)\rangle\, ds \\
&= -\int_0^T \langle u'(s), \varphi'(s)\rangle\, ds \\
&= -\int_0^T \int_0^1 \bar{u}_t(s,x)\varphi_t(s,x)\, dx\, ds \quad \text{from (2.13.4) and (2.8).}
\end{aligned} \tag{2.16}$$

Note that $u'(T)$ makes sense because u' and u'' are both in $\mathscr{V}' = L^2(0,T;E')$. This yields the first term of (2.6) and it is clear from (2.8), (2.9), and (2.13) that formula (2.6) is obtained.

Earlier we demonstrated that integrating (2.6) by parts and choosing E appropriately will yield boundary conditions of various sorts. It remains to discuss the initial conditions. For u the solution of Theorem 1, $u \in \mathscr{V}$, $u' \in \mathscr{V}$, and $u'' \in \mathscr{V}'$. Therefore [14] we may conclude that u equals a function in $C(0,T;E)$ a.e. and u' equals a function in $C(0,T;H)$ a.e. Therefore, the manner in which the initial conditions are satisfied may be described by

$$u(t) = v_0(t) \quad \text{and} \quad u'(t) = v_1(t) \quad \text{if } t \notin D,\ m(D) = 0, \tag{2.17}$$

$$v_0 \in C(0,T;E), \qquad v_1 \in C(0,T;H), \tag{2.18.1}$$

$$\lim_{t\to 0} |v_1(t) - u_1|_H + \|v_0(t) - u_0\|_E = 0. \tag{2.18.2}$$

If $\bar{u}$ is the measurable representative of u from Lemma 1 it follows that

$$v_0(t) = \bar{u}(t,\cdot), \qquad \bar{u}_t(t,\cdot) = v_1(t) \quad \text{a.e.} \tag{2.18.3}$$

LEMMA 3. If $\bar{u}$ is Borel measurable and satisfies (2.18), (2.6), then if $v(t) = \bar{u}(t, \cdot)$,

$$v'(t) = \bar{u}_t(t, \cdot) \quad \text{a.e.,} \tag{2.19.1}$$

$$v'' \in \mathscr{V}', \quad \text{and } v \text{ solves (2.13) with } g = f + b_x. \tag{2.19.2}$$

Proof. Let $\Psi \in C_0^\infty(0, T)$ and let $w \in E$.

$$\begin{aligned}\left\langle \int_0^T v(t)\Psi'(t)\,dt, w \right\rangle_{E'E} dt &= \int_0^T \Psi'(t)\langle v(t), w\rangle_{E'E}\,dt \\ &= \int_0^T \Psi'(t)(v(t), w)_H\,dt = \int_0^T \int_0^1 \bar{u}(t,x)w(x)\Psi'(t)\,dx\,dt \\ &= -\int_0^T \int_0^1 \bar{u}_t(t,x)w(x)\Psi(t)\,dx\,dt \\ &= \left\langle -\int_0^T \Psi(t)\bar{u}_t(t,\cdot)\,dt, w \right\rangle_{E'E}.\end{aligned}$$

Since $w \in E$ is arbitrary, (2.19.1) is proved. Now apply (2.6) to $\varphi(t,x) = \Psi(t)w(x)$,

$$\begin{aligned}\left\langle \int_0^T v'(t)\Psi'(t)\,dt, w \right\rangle &= \int_0^T \langle Nv, \varphi\rangle\,dt + \int_0^T \langle Q(v)v', \varphi\rangle\,dt - \int_0^T \langle g, \varphi\rangle\,dt \\ &= \left\langle \int_0^T (Nv(t) + Q(v)v'(t) - g(t))\Psi(t)\,dt, w \right\rangle\end{aligned}$$

where $g = f + b_x$. Since w is arbitrary

$$-v'' = Nv + Q(v)v' - g.$$

This proves (2.19.2).

COROLLARY 1. There exists a unique solution u to the problem (2.18) and (2.6).

Proof. The existence comes from Lemma 2. To obtain uniqueness, use Lemma 3 and the uniqueness part of Theorem 1.

3. Estimates for the approximate problem. In this section we will estimate the solution of Theorem 1 assuming that for some constants σ_0 and $R > 0$,

$$(\sigma(V) + \sigma_0)\beta(V) \geq 0 \quad \text{if } |V| \geq R. \tag{3.1}$$

Let $|k_1(t)| + |k_0(t)| \leq L$ for all t and from now on let $g = f + b_x$. Also define

$$W(V) = \int_0^V (\sigma(s) + \sigma_0)\,ds \tag{3.2}$$

and assume

$$a \leq u_0'(x) \leq b. \tag{3.3}$$

LEMMA 4. Let q and q_t both be in $L^2((0,T) \times (0,1))$. Then there exists a measurable set D with $m(D) = 0$ such that for $x \notin D$, $t \to q(t,x)$ is equal to a continuous function a.e. t and if $q(0,x)$ is the value of this continuous function at $t = 0$,

$$q(t,x) = q(0,x) + \int_0^t q_t(s,x)\,ds \quad \text{a.e. } t. \tag{3.4}$$

LEMMA 5. For u the solution of Theorem 1.

$$|u'(t)|_H^2 + 2\int_0^1 W(\bar{u}_x(t,x))\,dx + 2\delta\int_0^t |\bar{u}_{tx}(s,\cdot)|_H^2\,ds - 2\int_0^1 W(u_0'(x))\,dx - |u_1|_H^2$$
$$- 2(L+|\sigma_0|)\int_0^t |\bar{u}_{tx}(s,\cdot)|_H\,ds \le 2||g||_{L^2(0,T;H)}\left(\int_0^t |u'(s)|_H^2\,ds\right)^{1/2}. \quad (3.5)$$

Proof. Multiply (2.13.1) by u' and integrate by parts. Let $-J \le W(V)$ for all V. Such a constant exists because of (3.1) and (3.2).

COROLLARY 2. For u the solution of Theorem 1

$$|u'(t)|_H^2 \le C, \quad (3.6.1)$$

where

$$C = \left(2J + 2\int_0^1 W(u_0'(x))\,dx + |u_1|_H^2 + ||g||^2 + T\frac{(L+|\sigma_0|)^2}{2\delta}\right)e^T. \quad (3.6.2)$$

Proof. This follows from (3.5) and Gronwall's inequality.

For the rest of the section we shall assume that

$$\{u \in H^1(0,1) : u(1) = 0\} \subseteq E. \quad (3.7)$$

LEMMA 6. Let $\bar{u}$ be the measurable representative of the solution of Theorem 1 described in Lemma 1 and let

$$q(t,x) = \int_0^x \bar{u}_t(t,z)\,dz + \int_0^t k_0(s)\,ds - \int_0^t\int_0^x f(s,z)\,dz\,ds - \beta(\bar{u}_x(t,x)) + \sigma_0 t. \quad (3.8)$$

Then

$$q_t(t,x) = \sigma_0 + \sigma(\bar{u}_x(t,x)). \quad (3.9)$$

Proof. Let $\varphi \in C_0^\infty((0,T)\times(0,1))$ and consider $-\int_0^T\int_0^1 q(t,x)\varphi_t(t,x)\,dx\,dt$. The use of Fubini's theorem and integration by parts along with (2.6.1) applied with $\Psi(t,x) = \int_x^1 \varphi(t,z)\,dz$ in place of φ will yield

$$-\int_0^T\int_0^1 q(t,x)\varphi_t(t,x)\,dx\,dt = \int_0^T\int_0^1 (\sigma(\bar{u}_x(t,x)) + \sigma_0)\varphi(t,x)\,dx\,dt$$

and this proves the lemma. It is because of (3.7) that $\Psi(t,x)$ given above may be used in (2.6.1).

Using Lemmas 4 and 6, there exists a set of measure zero, D, such that for $x \notin D$,

$$q(t,x) = \int_0^t \sigma_0 + \sigma(\bar{u}_x(s,x))\,ds + q(0,x) \quad \text{a.e. } t, \quad (3.10.1)$$

where

$$q(0,x) = \int_0^x u_1(z)\,dz - \beta(u_0'(x)). \quad (3.10.2)$$

From (3.6.1) and Lemma 1,

$$\left|\int_0^x \bar{u}_t(t,z)\,dz + \int_0^t k_0(s)\,ds - \int_0^t\int_0^x f(s,z)\,dz\,ds + \sigma_0 t\right|$$
$$\le C + LT + ||f||_{L^2(0,T;H)}\sqrt{T} + |\sigma_0|T \quad \text{a.e.} \quad (3.11)$$

Let

$$C_1 = 2(|u_1|_H + |\beta(a)| + |\beta(b)| + \beta(R) - \beta(-R) + C + LT + \|f\|\sqrt{T} + |\sigma_0|T). \quad (3.12)$$

Thus $|q(t,x) + \beta(\bar{u}_x(t,x))| < C_1/2$ a.e. t.

LEMMA 7. For all $x \notin D$, $|q(t,x)| \le C_1$ a.e. and

$$|\beta(\bar{u}_x(t,x))| \le \frac{3C_1}{2} \quad \text{for all } t. \quad (3.13)$$

Proof. Let $x \notin D$ and let $\tilde{q}(t,x) = \int_0^t \sigma_0 + \sigma(\bar{u}_x(s,x))\,ds + q(0,x)$ for all t. Thus $|\tilde{q}(0,x)| < C_1$. If $\tilde{q}(t,x) > C_1$ for some t, there exists $a \in [0,T]$ with $\tilde{q}(a,x) = C_1$ and $\tilde{q}(t,x) > C_1$ for all $t \in [a, a+\varepsilon]$. Therefore, for $t \in [a, a+\varepsilon]$,

$$\tilde{q}(t,x) = C_1 + \int_a^t \sigma_0 + \sigma(\bar{u}_x(s,x))\,ds$$

and it follows that, in a subset of $[a, a+\varepsilon]$ having positive measure, the inequality (3.11) holds and also

$$\sigma_0 + \sigma(\bar{u}_x(t,x)) > 0, \quad (3.14.1)$$

$$\tilde{q}(t,x) = q(t,x). \quad (3.14.2)$$

Since (3.11) holds, $-\beta(\bar{u}_x(t,x)) \ge C_1/2$ and therefore

$$\beta(\bar{u}_x(t,x)) \le -C_1/2 \le \beta(-R) < 0. \quad (3.15)$$

But (3.15) implies (3.1), which requires that $\sigma(\bar{u}_x(t,x)) + \sigma_0 \le 0$, contradicting (3.14.1). Therefore, $\tilde{q}(t,x) \le C_1$ for all $t \in [0,T]$. A similar argument shows $\tilde{q}(t,x) \ge -C_1$ for all t. From (3.11) and (3.10.1), $|\beta(\bar{u}_x(t,x))| \le 3C_1/2$ a.e. From (2.14.1) of Lemma 1, $|\beta(\bar{u}_x(t,x))| \le 3C_1/2$ for all $t \in [0,T]$ and this completes the proof of Lemma 7.

4. The exact problem. In this section we use the results of the previous section to obtain existence and uniqueness theorems for (2.1), (2.2) and boundary conditions which include (2.3) and (2.4) under the assumptions

$$\alpha \text{ is continuous and } 0 < \delta \le \alpha(V) \le M_0, \quad (4.1)$$

$$\sigma \text{ is Lipschitz on every bounded interval}, \quad (4.2.1)$$

and for some constant σ_0 and $R > 0$,

$$(\sigma(V) + \sigma_0)\beta(V) \ge 0 \quad \text{if } |V| \ge R. \quad (4.2.2)$$

Also assume

$$a \le u_0'(x) \le b \quad \text{for all } x \in (0,1), \quad (4.3.1)$$

$$|k_1(t)| + |k_0(t)| \le L \quad \text{for all } t, \quad (4.3.2)$$

$$k_0 \text{ and } k_1 \text{ are measurable}, \quad (4.3.3)$$

$$u_1 \in L^2(0,1), \qquad u_0 \in E, \quad (4.3.4)$$

where $E \subseteq H^1(0,1)$ is a closed subspace satisfying

$$E \supseteq \{u \in H^1(0,1) : u(1) = 0\}. \quad (4.3.5)$$

For $m > 0$ define

$$\sigma_m(V) = \begin{cases} \sigma(V) & \text{if } |V| \le m, \\ \sigma(m) & \text{if } V > m, \\ \sigma(-m) & \text{if } V < -m. \end{cases} \tag{4.4}$$

Thus σ_m is globally Lipschitz.

In this section let u_m be the solution of Theorem 1 with σ replaced with σ_m in the definition of N and let $\bar{u}_m$ be the measurable representative of Lemma 1. Assume

$$m > R \tag{4.5}$$

and define

$$W_m(V) = \int_0^V (\sigma_m(s) + \sigma_0)\, ds. \tag{4.6}$$

Therefore, W_m is decreasing for $V < -R$ and increasing for $V > R$. Since W_m coincides with W on $[-R, R]$, it follows that W_m has a minimum which does not depend on m. Therefore

$$|u'_m(t)|_H^2 \le C, \tag{4.7}$$

where C is given in (3.6.2) and does not depend on m. From Lemma 7 there exists a set D_m with $m(D_m) = 0$ such that for all $x \notin D_m$,

$$|\beta(\bar{u}_{mx}(t, x))| < \tfrac{3}{2}C_1 \quad \text{for all } t \tag{4.8}$$

where C_1 depends only on u_1, u_0, R, $\min(W(V))$, f, k_0, and k_1. Thanks to (4.1), (4.8) implies

$$|\bar{u}_{mx}(t, x)| \le \tfrac{3}{2}C_1/\delta \quad \text{a.e.} \tag{4.9}$$

Now assume

$$m > \frac{3C_1}{2\delta}. \tag{4.10}$$

THEOREM 2. Suppose (4.1), (4.2), and (4.3) hold. Then there exist a unique solution, u, to (2.6), (2.18) and a constant N such that $|u_x(t, x)| \le N$ a.e.

Proof. Let $m > \max(R, 3C_1/(2\delta))$ and consider $\bar{u}_m$. The function $\bar{u}_m$ satisfies (2.6) and (2.18) with σ_m in place of σ. But by (4.9) $|\bar{u}_{mx}(t, x)| \le 3C_1/(2\delta) < m$ a.e. and so $\sigma_m(\bar{u}_{mx}(t, x)) = \sigma(\bar{u}_{mx}(t, x))$ a.e. Let $u = \bar{u}_m$.

COROLLARY 3. The conclusion of Theorem 2 remains true if (4.1) is replaced with the weaker assumption

$$\alpha \text{ is continuous}, \qquad \delta \le \alpha(V). \tag{4.11}$$

Proof. Let $M \ge \max\{\alpha(V)\colon V \in [-3C_1/(2\delta), 3C_1/(2\delta)]\}$ and let $\alpha_M(V) = \min(\alpha(V), M)$. If u is the solution of (2.6), (2.18) described in Theorem 2 after replacing α with α_M, then (4.9) implies $|u_x(t, x)| \le 3C_1/(2\delta)$ a.e. and so $\alpha_M(u_x(t, x)) = \alpha(u_x(t, x))$ a.e. This proves Corollary 3.

We shall now give the examples of initial-boundary value problems obtained from Theorem 2.

Example 1. Let $E = H^1(0,1)$. Then if u is the solution described in Theorem 2, u is a weak solution of

$$u_{tt} = \sigma(u_x)_x + (\alpha(u_x)u_{xt})_x + f, \tag{4.12.1}$$
$$u(0,x) = u_0(x), \tag{4.12.2}$$
$$u_t(0,x) = u_1(x), \tag{4.12.3}$$

$$\sigma(u_x(t,0)) + \alpha(u_x(t,0))u_{xt}(t,0) = k_0(t), \tag{4.12.4}$$
$$\sigma(u_x(t,1)) + \alpha(u_x(t,1))u_{xt}(t,1) = k_1(t). \tag{4.12.5}$$

Example 2. Let $E = \{u \in H^1(0,1) \colon u(1) = 0\}$. Then u solves (4.12.1)–(4.12.4) and

$$u(t,1) = 0. \tag{4.13}$$

5. Proof of Theorem 1. Let $N(t,\cdot)$, M, and Q be described in Sec. 2.

LEMMA 8. For $v, u \in E$,

$$\|N(t,u) - N(t,v)\|_{E'} \le K\|u - v\|_E, \tag{5.1.1}$$
$$\|Mu - Mv\|_{E'} \le M_0\|u - v\|_E, \tag{5.1.2}$$
$$\delta\|u - v\|_E^2 \le \langle Mu - Mv, u - v\rangle \le M_0\|u - v\|_E^2, \tag{5.1.3}$$

and if $u, u' \in L^2(a, a + T_1; E)$ where $T_1 > 0$ and u' is understood in the sense of E'-valued distributions ((2.11), (2.8)), then

$$M(u(t)) = M(u(a)) + \int_a^t Q(u(s))u'(s)\,ds. \tag{5.1.4}$$

(Note that $u(a)$ has meaning because u, u' are in $L^2(a, a + T_1; E)$.)

Proof. Parts (5.1.1)–(5.1.3) are obvious. Let $\varphi \in C_0^\infty(a, a + T_1)$ and let $v \in E$. Also let $\bar{u}$ and $\bar{u}_x$ be given in Lemma 1, and let $b = a + T_1$.

$$\begin{aligned}
\left\langle \int_a^b M(u(t))\varphi'(t)\,dt, v \right\rangle &= \int_a^b \langle M(u(t)), v\rangle \varphi'(t)\,dt \\
&= \int_a^b \left(\beta\left(\frac{\partial u(t)}{\partial x}\right), v_x \right)_H \varphi'(t)\,dt \\
&= \int_a^b \int_0^1 \beta(\bar{u}_x(t,x))v_x(x)\varphi'(t)\,dx\,dt \\
&= \int_0^1 \int_a^b \beta(\bar{u}_x(t,x))\varphi'(t)v_x(x)\,dt\,dx \\
&= -\int_a^b \int_0^1 \alpha(\bar{u}_x(t,x))\bar{u}_{tx}(t,x)\varphi(t)v_x(x)\,dx\,dt \\
&= -\int_a^b \langle Q(\bar{u}(t,\cdot))\bar{u}_t(t,\cdot), v\rangle \varphi(t)\,dt \\
&= \left\langle -\int_a^b Q(u(t))u'(t)\varphi(t)\,dt, v \right\rangle.
\end{aligned} \tag{5.2}$$

Since $v \in E$ was arbitrary, it follows that

$$(Mu)' = Q(u)u' \tag{5.3}$$

and therefore

$$M(u(t)) = M(u(a)) + \int_0^t Q(u(s))u'(s)\,ds \quad \text{a.e.} \tag{5.4}$$

We need a few more definitions. Let T_1 satisfy

$$0 < T_1 < 1, \qquad \delta e^{-\delta T_1} - KT_1 \geq \delta/2, \tag{5.5}$$

and let $\{E_n\}$ be a sequence of finite-dimensional subspaces of E satisfying

$$E = \overline{\bigcup_{n<\infty} E_n}, \qquad E_{n+1} \supseteq E_n \quad \text{for all } n. \tag{5.6}$$

Let $u_0 \in E$ and $u_1 \in H$ be given and let $\{u_{0n}\}$ and $\{u_{1n}\}$ be sequences satisfying

$$u_{0n}, u_{1n} \in E_n, \tag{5.7.1}$$

$$\lim_{n\to\infty} \|u_{0n} - u_0\|_E + |u_{1n} - u_1|_H = 0. \tag{5.7.2}$$

Also, for g given in $L^2(a, a+T_1; E')$, let

$$f_n(t) = \int_a^t g(s)\,ds + M(u_{0n}) + u_{1n}, \tag{5.8.1}$$

$$f(t) = \int_a^t g(s)\,ds + M(u_0) + u_1. \tag{5.8.2}$$

Thus $\lim_{n\to\infty} \|f_n - f\|_{L^2(a,a+T_1;E')} = 0$. Let $\{w_1, \ldots, w_n\}$ be a basis for E_n satisfying $(w_k, w_r)_H = \delta_{kr}$.

LEMMA 9. There exists a unique function u_n satisfying

$$u_n, u_n' \in L^2(a, a+T_1; E_n), \tag{5.9.1}$$

and for all $w \in E_n$,

$$(u_n'(t), w)_H + \left\langle \int_a^t N(s, u_n(s))\,ds, w \right\rangle + \langle Mu_n(t), w\rangle = \langle f_n(t), w\rangle \quad \text{a.e.,} \tag{5.9.2}$$

$$u_n(a) = u_{0n}. \tag{5.9.3}$$

Proof. Let $u_n(t) = \sum_{k=1}^n y_k(t)w_k$ and let $\mathscr{A} : L^2(a, a+T_1; R^n) \to L^2(a, a+T_1; R^n)$ be defined by

$$(\mathscr{A}\mathbf{y})_r(t) = \int_a^t \left\langle N\left(s, \sum_{k=1}^n y_k(s)w_k\right), w_r \right\rangle ds + \left\langle M\left(\sum_{k=1}^n y_k(t)w_k\right), w_r \right\rangle. \tag{5.10}$$

Thus, u_n is a solution to (5.9) if and only if $\mathbf{y}(t)$ solves

$$\mathbf{y}' + \mathscr{A}\mathbf{y} = \mathbf{F} \quad \text{in } L^2(a, a+T_1; R^n), \tag{5.11.1}$$

$$\mathbf{y}(a) = \mathbf{y}_0, \tag{5.11.2}$$

$$\mathbf{y}, \mathbf{y}' \in L^2(a, a+T_1; R^n), \tag{5.11.3}$$

where

$$y_{0r} = (u_{0n}, w_r)_H, \qquad F_r(t) = \langle f_n(t), w_r \rangle. \tag{5.12}$$

Letting $\mathbf{y}(t) = e^{\lambda(t-a)}\mathbf{z}(t)$, it follows that $\mathbf{y}$ solves (5.11) if and only if z solves

$$\mathbf{z}' + \lambda\mathbf{z} + \mathscr{A}_\lambda \mathbf{z} = e^{-\lambda(\cdot - a)}\mathbf{F} \tag{5.13.1}$$

$$\mathbf{z}(a) = \mathbf{y}_0 \tag{5.13.2}$$

$$\mathbf{z}, \mathbf{z}' \in L^2(a, a + T_1; R^n) \tag{5.13.3}$$

where

$$\mathscr{A}_\lambda(\mathbf{z})(t) = e^{-\lambda(t-a)}\mathscr{A}(e^{\lambda(\cdot - a)}\mathbf{z})(t). \tag{5.14}$$

Using the equivalence of all norms on R^n and (5.5), a little analysis yields for all $\mathbf{y}, \mathbf{z} \in L^2(a, a + T_1; R^n)$,

$$\langle (\lambda I + \mathscr{A}_\lambda)\mathbf{y} - (\lambda I + \mathscr{A}_\lambda)\mathbf{z}, \mathbf{y} - \mathbf{z} \rangle_{L^2(a,a+T_1;R^n)} \geq \eta \|\mathbf{y} - \mathbf{z}\|^2 \tag{5.15}$$

for some $\eta > 0$, the norm in (5.15) being the norm of $L^2(a, a + T_1; R^n)$.

It follows from [14, 17] that there exists a unique solution $\mathbf{z}$ to (5.13). Therefore there exists a unique solution to (5.11) and consequently there exists a unique solution to (5.9). This proves Lemma 9.

Since y, y' are in $L^2(a, a + T_1; R^n)$, y is absolutely continuous and $\mathbf{y}(t) = \mathbf{y}_0 + \int_a^t \mathbf{y}'(s)\, ds$. It follows that for $w \in E_n$

$$\langle Mu_n(t), w \rangle = \int_a^t \langle Qu_n(s)u_n'(s), w \rangle\, ds + \langle M(u_{0n}), w \rangle. \tag{5.16}$$

Therefore, from (5.8.1), (5.12), (5.10), (5.11.1) yields

$$y_r' = \int_a^{(\cdot)} I_r(s)\, ds + (u_{1n}, w_r)_H \tag{5.17}$$

where $I_r(\cdot) \in L^2(a, a + T_1)$ is given by

$$\langle g(s), w_r \rangle - \langle Q(u_n(s))u_n'(s), w_r \rangle - \langle N(s, u_n(s)), w_r \rangle. \tag{5.18}$$

It follows that $\mathbf{y}''$ exists and is in $L^2(a, a+T_1; R^n)$. Thus $y_r'(a)$ makes sense and equals $(u_{1n}, w_r)_H$. Differentiating (5.17), we have proved the following lemma.

LEMMA 10. The function, u_n, of Lemma 9 satisfies for all $w \in E_n$:

$$(u_n''(t), w)_H + \langle N(t, u_n(t)), w \rangle + \langle Q(u_n(t))u_n'(t), w \rangle = \langle g(t), w \rangle \text{ a.e.,} \tag{5.19.1}$$

$$u_n'(a) = u_{1n}, \tag{5.19.2}$$

$$u_n(a) = u_{0n}, \tag{5.19.3}$$

$$u_n'' \in L^2(a, a + T_1; H) \subseteq L^2(a, a + T_1; E'), \tag{5.19.4}$$

$$u_n, u_n' \in L^2(a, a + T_1; E). \tag{5.19.5}$$

For $t \in [a, a + T_1]$ and $w(\cdot)$ a simple function, (5.19.1) implies

$$\int_a^t (u_n''(s), w(s))_H\, ds + \int_a^t \langle N(s, u_n(s)), w(s) \rangle\, ds + \int_a^t \langle Q(u_n(s))u_n'(s), w(s) \rangle\, ds = \int_a^t \langle g(s), w(s) \rangle\, ds. \tag{5.20}$$

Therefore (5.20) holds for all $w \in L^2(a, a+T_1; E_n)$. Replacing w by u_n' and integrating by parts,

$$|u_n'(t)|_H^2 + 2\int_0^1 W(\bar{u}_{nx}(t,x))\,dx + 2\delta\int_a^t \left|\frac{\partial u_n'(s)}{\partial x}\right|_H^2 ds - |u_{1n}|^2$$
$$- 2\int_0^1 W(u_{0n}'(x))\,dx - 2(L+|\sigma_0|)\int_a^t \left|\frac{\partial u_n'(s)}{\partial x}\right|_H ds$$
$$\leq 2\|g\|_{L^2(a,a+T_1;E')}\left(\|u_n'\|_{L^2(a,t;H)} + \left\|\frac{\partial u_n'}{\partial x}\right\|_{L^2(a,t;H)}\right). \quad (5.21)$$

Let $-J$ be the lower bound of (2.5.1). The boundedness of σ implies $|\int_0^1 W(u_{0n}'(x))\,dx| \leq C_1\|u_{0n}\|_E$, a quantity which is bounded independent of n. Therefore, there exists a constant C_2 which does not depend on $t \in [a, a+T_1]$ or n such that

$$|u_n'(t)|_H^2 + 2\delta\left\|\frac{\partial u_n'}{\partial x}\right\|_{L^2(a,t;H)}^2 - 2(L+|\sigma_0|+\|g\|)\left\|\frac{\partial u_n'}{\partial x}\right\|_{L^2(a,t;H)}$$
$$\leq C_2 + \int_a^t |u_n'(s)|_H^2\,ds. \quad (5.22)$$

Here $C_2 \geq 2J + \|g\|_{L^2(a,a+T_1;E')}^2 + |\int_0^1 W(u_{0n}'(x))\,dx| + |u_1|_H^2$.

It follows from (5.22) and Gronwall's inequality that $|u_n'(t)|_H$ is bounded independent of n and t. Therefore (5.22) implies that $\|\partial u_n'/\partial x\|_{L^2(a,a+T_1;H)}$ is bounded independent of n. Summarizing this, there exists a constant C_3 which does not depend on n or t such that

$$|u_n'(t)|_H + \left\|\frac{\partial u_n'}{\partial x}\right\|_{L^2(a,a+T_1;H)} \leq C_3. \quad (5.23)$$

This implies

$$\|u_n'\|_{L^2(a,a+T_1;E)} + \|u_n\|_{L^\infty(a,a+T_1;E)} + \|u_n'\|_{L^\infty(a,a+T_1;H)} \leq C_4 \quad (5.24)$$

where C_4 does not depend on n.

Let $A : L^2(a, a+T_1; E) \to L^2(a, a+T_1; E')$ be given by

$$Au(t) = \int_a^t N(s, u(s))\,ds + Mu(t). \quad (5.25)$$

Thanks to the size of T_1, A is monotone. It is also clear that A is hemicontinuous ($\lim_{t\to 0}\langle A(u+tw), v\rangle = \langle Au, v\rangle$) and bounded. For convenience, define

$$\mathscr{U} = L^2(a, a+T_1; E), \qquad \mathscr{U}_n = L^2(a, a+T_1; E_n), \qquad \mathscr{H} = L^2(a, a+T_1; H). \quad (5.26)$$

By (5.9), the following holds for all $w \in \mathscr{U}_n$:

$$(u_n', w)_{\mathscr{H}} + \langle Au_n, w\rangle_{\mathscr{U}} = \langle f_n, w\rangle_{\mathscr{U}}, \quad (5.27.1)$$
$$u_n(a) = u_{0n}. \quad (5.27.2)$$

Because of (5.24) there is a subsequence still denoted by u_n satisfying

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } L^\infty(a, a+T_1; E), \\ u_n' &\rightharpoonup u' \quad \text{in } \mathscr{U}, \\ u_n' &\rightharpoonup u' \quad \text{in } L^\infty(a, a+T_1; H), \\ Au_n &\rightharpoonup \xi \quad \text{in } \mathscr{U}', \\ u_n(a) &\rightharpoonup u(a) \quad \text{in } H. \end{aligned} \tag{5.28}$$

Now $u_n(a) = u_{0n}$ and $u_{0n} \to u_0$ in E. Since $u_n(a) \rightharpoonup u(a)$ in H, it follows that $u_0 = u(a)$ and $\|u_n(a) - u(a)\|_E \to 0$.

By density of $\bigcup_{n<\infty} \mathscr{U}_n$ in $\mathscr{U}$, (5.27.1), (5.8), and (5.28) imply $\xi = f - u'$. Also

$$\begin{aligned} \langle Au_n, u_n\rangle_{\mathscr{U}} \le \langle f_n, u_n\rangle_{\mathscr{U}} + \tfrac{1}{2}|u_n(a) - u(a)|_H^2 \\ + (u', u)_{\mathscr{H}} - (u', u_n)_{\mathscr{H}} - (u_n', u)_{\mathscr{H}}. \end{aligned} \tag{5.29}$$

Therefore $\overline{\lim}_{n\to\infty}\langle Au_n, u_n\rangle_{\mathscr{U}} \le \langle f, u\rangle_{\mathscr{U}} - (u', u)_{\mathscr{H}} = \langle \xi, u\rangle_{\mathscr{U}}$. Since A is type M [17], it follows that $Au = \xi = f - u'$.

LEMMA 11. Let $u_0 \in E$, $u_1 \in H$, and let $g \in \mathscr{U}' = L^2(a, a+T_1; E')$. Then there exists a unique solution to

$$u'' + Nu + Q(u)u' = g \quad \text{in } \mathscr{U}', \tag{5.30.1}$$

$$u(a) = u_0, \tag{5.30.2}$$

$$u'(a) = u_1, \tag{5.30.3}$$

$$u, u' \in \mathscr{U}, \qquad u \in L^\infty(a, a+T_1; E), \qquad u' \in L^\infty(a, a+T_1; H). \tag{5.30.4}$$

Proof. The discussion preceding Lemma 11 has shown the existence of a solution u satisfying (5.30.4) and

$$u' + Au = f, \qquad u(0) = u_0. \tag{5.31}$$

Since A is strictly monotone, the solution to (5.31) is unique [14]. (This is even true in a larger class of functions than those of (5.30.4).) However, since $u' \in \mathscr{U}$, Lemma 8 implies

$$Au(t) = \int_a^t N(s, u(s))\,ds + \int_a^t Q(u(s))u'(s)\,ds + M(u_0), \tag{5.32}$$

and it follows that $u'' \in \mathscr{U}'$ and (5.30) holds. This proves Lemma 11.

Proof of Theorem 1. Choose T_1 small enough that $\delta e^{-\delta T_1} - KT_1 \ge \delta/2$ and use Lemma 11 to solve (5.30.1)–(5.30.3) on $[0, T_1]$, obtaining a solution which satisfies (5.30.4) with $a = 0$. Next use $u(T_1) \in E$ and $u'(T_1) \in H$ as new initial data and solve (5.30.1)–(5.30.3) with these new initial data on $[T_1, 2T_1]$. Continuing in this way proves Theorem 1.

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