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ON THE LAGRANGE STABILITY OF SPATIALLY PERIODIC SOLUTIONS OF THE GINZBURG-LANDAU EQUATION*

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Abstract. We study the Lagrange stability (i.e., the pointwise boundedness) of the spatially-periodic solutions of the Ginzburg-Landau equation. There is a parameter range where the solutions blow up (instable) if their initial mean amplitudes are beyond a critical value.

1. Introduction. We study the spatially-periodic solutions of the scaled Ginzburg-Landau equation [1],

$$iA_t + (1 - ic_0)A_{xx} = (ic_0/c_1)A - (1 + ic_0/c_1)|A|^2 A, \tag{1}$$

where t is the time variable, x is the one-dimensional spatial coordinate, $A(x,t)$ represents a complex amplitude, c_0 and c_1 are real parameters, and $i = \sqrt{-1}$. This equation governs the amplitude evolution of instability waves in fluid dynamics problems such as Bénard convection [2], Taylor-Couette flow [3], and plane Poiseuille flow [4].

For a spatially-periodic solution $A(x,t)$ of equation (1) with period L:

$$A(x,t) = A(x + L, t), \tag{2}$$

set the energy functional

$$I(t) = \int_0^L |A(x,t)|^2\, dx. \tag{3}$$

Newton and Sirovich [1] proved that if $c_0, c_1 > 0$ and the spatially-periodic solution $A(x,t)$ with period L satisfies an energy growth condition $I'(t) \geq 0$ for $t \geq 0$, then $A(x,t)$ remains pointwise bounded, i.e., there is a constant $K > 0$ such that

$$\max_x |A(x,t)|^2 \leq K \quad \text{for } t \geq 0. \tag{4}$$

Hence $A(x,t)$ is Lagrange stable [6]. In the present note we show that if $c_0, c_1 > 0$, then (4) still holds under the less restricted condition

$$\inf_{t \geq 0} I'(t) > -\infty. \tag{5}$$

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Moreover, we show that in the parameter range $c_0 < 0$ and $c_1 > 0$, if $I(0) > L$, then

$$I(t) \to \infty \quad \text{as } t \to \text{ some finite } \bar{t} > 0. \tag{6}$$

So, in particular, the global spatially-periodic solution fails to exist. Our method is a combination of the formulation of Newton and Sirovich [1] and Perron's comparison principle for differential inequalities.

2. The parameter range $c_0, c_1 > 0$. In the following we state Perron's comparison principle.

LEMMA [5]. Suppose that $x(t)$ and $y(t)$ satisfy

$$dx/dt \leq G(t, x), \qquad dy/dt = G(t, y), \quad t \geq 0$$

and

$$x(0) \leq y(0).$$

Then for all $t \geq 0$, $x(t) \leq y(t)$.

Now assume that $A(x, t)$ is a spatially-periodic solution of Eq. (1) with period L and I is defined by (3). If $c_0, c_1 \neq 0$, Eqs. (1) and (2) yield that [1]

$$I'(t) = -2c_0 \int_0^L |A_x|^2 \, dx + (2c_0/c_1)I - (2c_0/c_1) \int_0^L |A|^4 \, dx. \tag{7}$$

If $c_0, c_1 > 0$, then relation (7) gives us a differential inequality

$$dI/dt \leq (2c_0/c_1)I(1 - I/L), \quad t \geq 0,$$

where we have used the inequality

$$\int_0^L |A|^4 \, dx \geq I^2/L, \tag{8}$$

which is an obvious implication of the Schwarz inequality. Consider the problem

$$\begin{cases} dJ/dt = (2c_0/c_1)J(1 - J/L), & t \geq 0, \\ J(0) = I(0). \end{cases}$$

The solution can be found explicitly:

$$J(t) = LI(0)\exp(2c_0 t/c_1)/[L + I(0)(\exp(2c_0 t/c_1) - 1)], \quad t \geq 0.$$

Applying the lemma to $I(t)$ and $J(t)$, we can conclude that $I(t) \leq J(t)$, $t \geq 0$. It is easy to see that $J(t)$ is increasing if $I(0) \leq L$ and decreasing if $I(0) > L$, and, moreover,

$$\lim_{t \to \infty} J(t) = L.$$

Therefore, we always have the following estimate:

$$0 \leq I(t) \leq J(t) \leq C = \max(I(0), L). \tag{9}$$

Now, following the line of Newton and Sirovich [1], for any fixed $t \geq 0$, we choose $x_0 \in [0, L]$ such that

$$|A(x_0, t)| = \min_{0 \leq x \leq L} |A(x, t)|.$$

Then we can write

$$\begin{aligned}|A(x,t)|^2 &= |A(x_0,t)|^2 + \int_{x_0}^{x} (AA_x^* + A_xA^*)\,dx \\ &\le \frac{1}{L}\int_0^L |A(x_0,t)|^2\,dx + 2\int_0^L |AA_x|\,dx \\ &\le \frac{I}{L} + 2\int_0^L |AA_x|\,dx.\end{aligned} \tag{10}$$

Applying the Schwarz inequality and the inequality $2ab \le a^2 + b^2$ to (10) we reach

$$|A(x,t)|^2 \le I/L + I^2 + \int_0^L |A_x|^2\,dx. \tag{11}$$

But relation (7) gives us

$$\begin{aligned}\int_0^L |A_x|^2\,dx &= -I'(t)/(2c_0) + I/c_1 - \frac{1}{c_1}\int_0^L |A|^4\,dx \\ &\le -I'(t)/(2c_0) + I/c_1.\end{aligned} \tag{12}$$

Consequently, the desired estimate (4) follows from combining (11), (12), (5), and (9). Thus in the range $c_0, c_1 > 0$, any spatially-periodic solution satisfying (5) is Lagrange stable, and estimate (9) is unconditionally true for all spatially-periodic solutions.

3. The parameter range $c_0 < 0$, $c_1 > 0$. Put $c_2 = -c_0$. Now relation (7) becomes

$$I'(t) = 2c_2\int_0^L |A_x|^2\,dx + (2c_2/c_1)\int_0^L |A|^4\,dx - (2c_2/c_1)I. \tag{13}$$

Relation (13) gives us another differential inequality:

$$dI/dt \ge (2c_2/c_1)I(I/L - 1),$$

where we have used the inequality (8) in (13). As in the last section, consider the initial value problem:

$$\begin{cases} dJ/dt = (2c_2/c_1)J(J/L-1), & t \ge 0, \\ J(0) = I(0). \end{cases} \tag{14}$$

Assume that $I(0) > L$. The solution $J(t)$ of problem (14) can be found explicitly:

$$J(t) = LI(0)/[I(0) - (I(0)-L)\exp(2c_2t/c_1)].$$

Therefore, we can conclude that $J(t)$ is increasing for $t \ge 0$ and

$$\lim_{t\to t_0} J(t) = \infty, \tag{15}$$

where

$$t_0 = (c_1/(2c_2))\log(I(0)/(I(0)-L)). \tag{16}$$

Now applying the lemma in the last section but with reversed inequality we get

$$I(t) \ge J(t), \quad t \ge 0. \tag{17}$$

Finally, from (15), (16), and (17) we conclude that there exists $\bar{t}: 0 < \bar{t} \le t_0$ such that the solution $A(x,t)$ blows up when t approaches $\bar{t}$ from the left. In particular, we have shown that the global existence of spatially-periodic solutions to Eq. (1) falls in the parameter range $c_0 < 0$, $c_1 > 0$. This is a somewhat astonishing situation.

REFERENCES

[1] P. K. Newton and L. Sirovich, *Instabilities of the Ginzburg-Landau equation: Periodic solutions*, Quart. Appl. Math. **44**, 49–58 (1986)

[2] A. C. Newell and J. A. Whitehead, *Finite bandwidth, finite amplitude convection*, J. Fluid Mech. **38**, 279–303 (1969)

[3] S. Kogelman and R. C. Diprima, *Stability of spatially periodic supercritical flows in hydrodynamics*, Phys. Fluids **13**, 1–11 (1970)

[4] K. Stewartson and J. T. Stuart, *A nonlinear instability theory for a wave system in plane Poiseuille flow*, J. Fluid Mech. **48**, 529–545 (1971)

[5] D. H. Sattinger, *Topics in Stability and Bifurcation Theory*, Lect. Notes in Math. **309**, Springer-Verlag, New York, p. 27

[6] V. V. Nemytiskii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton Univ. Press, 1960