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DYNAMIC PHASE TRANSITIONS IN A VAN DER WAALS GAS*

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1. Introduction. The system of equations

$$u_t - v_x = 0, \qquad v_t - \sigma(u)_x = 0 \tag{1.1}$$

is of mixed hyperbolic and elliptic type when σ is a monotonically increasing function except in an interval (α, β) (see Fig. 1). System (1.1) has been used to describe dynamic changes of phase in a van der Waals gas [10] and to model elastic deformations in a rod under tension [2].

Mathematically, changes of phase for Eq. (1.1) are associated with jump discontinuities (shocks) in weak solutions (u, v) of system (1.1), in which u jumps across the interval (α, β). There is a continuing problem of how to distinguish the phase jumps that are physically relevant. Mathematically, one would like to impose an entropy admissibility condition on all jump discontinuities that selects the physically relevant shocks, including the correct phase jumps, while giving well-posedness of the Cauchy problem. For systems of mixed type, it is not known in general what the appropriate admissibility condition should be, even if the initial data are restricted to lie entirely in the hyperbolic regions. This is largely due to the presence of noncompressive shocks, which fail to satisfy the classical entropy conditions of the theory of conservation laws [3, 4] . In the context of (1.1), noncompressive shocks are phase jumps that are typically nearly stationary (i.e., with nearly zero shock speed). When the shock speed is exactly zero, the phase jump is referred to as the Maxwell line. A requirement of an admissibility condition is that the Maxwell line should be admissible.

This paper is a continuation of the study of the viscosity-capillarity criterion for shocks, introduced by Slemrod [11]. In particular, I discuss solutions of the Riemann initial value problem for (1.1). This is the Cauchy problem with piecewise constant initial data having a single jump. The main result is that for initial data near the Maxwell line, the Riemann problem has a solution consisting of two weak shock or rarefaction waves, separated by a slowly moving phase jump. All the jump discontinuities are required to satisfy the viscosity-capillarity criterion. Alternative admissibility criteria for shocks in solutions of the Riemann problem for (1.1) are discussed in [1, 5, 9].

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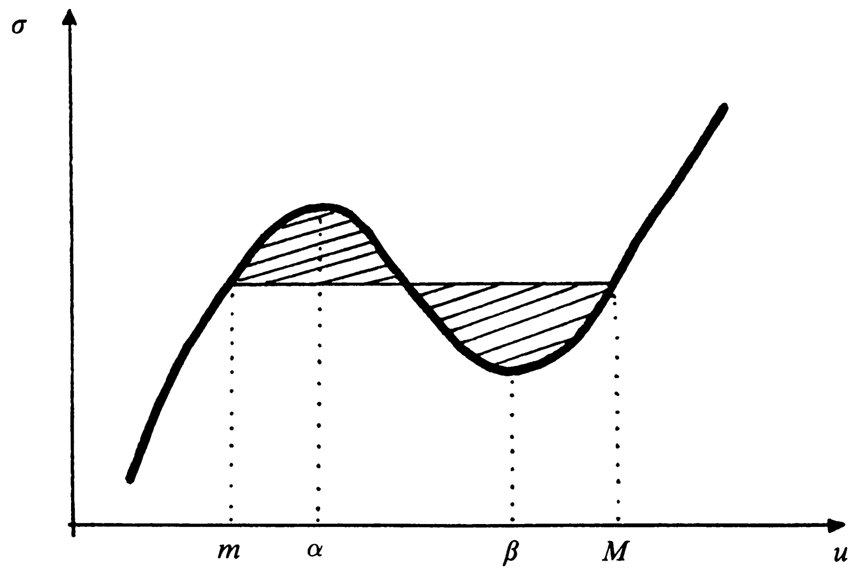

FIG. 1. Graph of σ and the Maxwell line

Some properties of shocks near the Maxwell line are presented in Sec. 2 and used in Sec. 3 to prove the main theorem. The crucial property concerns the monotonicity of a curve representing admissible phase jumps. The monotonicity of this curve allows an extension of the classical construction of solutions of Riemann problems [3] to be employed on either side of the elliptic region (where $\alpha < u < \beta$). I wish to thank Marshall Slemrod for helpful discussions and for suggesting that the results of this paper could be proved from the constructions in [8].

2. Wave curves. A shock wave

$$u(x,t) = \begin{cases} (u_1, v_1) & \text{if } x < st, \\ (u_2, v_2) & \text{if } x > st, \end{cases} \tag{2.1}$$

with shock speed s, is a weak solution of (1.1) if the Rankine–Hugoniot conditions are satisfied. It is convenient to write these conditions in the form

$$s^2 = (\sigma(u_2) - \sigma(u_1))/(u_2 - u_1), \tag{2.2}$$

$$v_2 - v_1 = -s(u_2 - u_1). \tag{2.3}$$

Throughout this paper, we take σ to be monotonically increasing except in an interval (α, β), as illustrated in Fig. 1. We say that the shock wave (2.1) is a phase jump if u_1, u_2 lie on opposite sides of the interval (α, β). The Maxwell line is a particular stationary phase jump, in which $u_1 = m$, $u_2 = M$ are related by

$$\sigma(m) = \sigma(M) \quad \text{and} \quad \int_m^M \{\sigma(u) - \sigma(m)\}\, du = 0. \tag{2.4}$$

Thus, from (2.2), (2.3), we have $s = 0$ and $v_1 = v_2$.

The Maxwell line is noncompressive in the sense that both families of characteristics

$$dx/dt = \lambda_j(u(x,t)), \qquad \lambda_j(u) = (-1)^j\{\sigma'(u(x,t))\}^{1/2}, \qquad j = 1,2, \tag{2.5}$$

pass through the shock from both sides. This situation contrasts with a compressive shock, in which one family of characteristics converges on the shock while the characteristics of the other family pass through the shock. Compressive shocks are discussed toward the end of this section.

The Maxwell line is an admissible shock according to the viscosity-capillarity criterion introduced by Slemrod [11]. This says that the shock wave (2.1) is admissible if there is a travelling wave solution (known as a viscous profile)

$$(u,v)(x,t) = (u,v)((x-st)/\varepsilon) \tag{2.6}$$

of the system

$$u_t - v_x = 0, \qquad v_t - \sigma(u)_x = \varepsilon v_{xx} - A\varepsilon^2 u_{xxx}, \tag{2.7}$$

that smooths the shock wave for small $\varepsilon > 0$. In (2.7), A is a constant, with $0 < A \leq \frac{1}{4}$. The travelling wave must satisfy appropriate boundary conditions:

$$(u,v)(-\infty) = (u_1,v_1), \qquad (u,v)(+\infty) = (u_2,v_2), \qquad (u',v')(\pm\infty) = (0,0). \tag{2.8}$$

A noncompressive shock is referred to as undercompressive if it has a viscous profile. Undercompressive shocks are discussed in [10] in a different context.

Substituting (2.6) into (2.7) and integrating once, using (2.8), leads to the equation

$$Au'' = -su' + \sigma(u_1) - s^2(u-u_1), \tag{2.9}$$

with boundary conditions

$$u(-\infty) = u_1, \qquad u(+\infty) = u_2, \qquad u'(\pm\infty) = 0. \tag{2.10}$$

We say that $u_1 \to u_2$ is a connection with speed s if the boundary value problem (2.9), (2.10) has a solution. From (2.2), we see that $s = 0$ if and only if $\sigma(u_1) = \sigma(u_2)$. Integrating (2.9) with $s = 0$ along a solution leads to (2.4). Thus, $m \to M$ and $M \to m$ are the only connections with speed zero. The shock wave (2.1) is admissible if and only if $u_1 \to u_2$ is a connection with speed s given by (2.2) and v_1, v_2 are related by (2.3).

In discussing solutions of (2.9), it is convenient to refer to the phase plane of ordered pairs (u,u'). We say a connection $u_1 \to u_2$ is a saddle-saddle connection if the phase plane equilibria $(u_1,0)$, $(u_2,0)$ are both saddle points. The stationary connections $m \to M$ and $M \to m$ are saddle-saddle connections. The following result is a restatement of Lemma 3.2 and Theorem 3.4 of [8].

THEOREM 2.1. Let σ be a C^2 function, with α, β, m, and M defined as above. There exist $\delta_1 > 0$ and C^1 functions $\tilde{s}$, $\tilde{u}_2$ defined on $[m-\delta_1, m]$ such that

(a) $\tilde{u}_2(m) = M$, $\tilde{s}(m) = 0$,

(b) $u_1 \to \tilde{u}_2(u_1)$ is a saddle-saddle connection with speed $\tilde{s}(u_1) \geq 0$,

(c) $\tilde{u}_2$ and $\tilde{s}$ are strictly monotonically increasing and decreasing, respectively.

Correspondingly, there exist $\delta_2 > 0$ and strictly monotonic C^1 functions $\hat{s}$, $\hat{u}_2$ defined on $[M, M+\delta_2]$ with $\hat{u}_2(M) = m$, $\hat{s}(M) = 0$, such that $u_1 \to \hat{u}_2(u_1)$ is a saddle-saddle connection. In particular, $m \to M$ is a saddle-saddle connection, and it is embedded in a one-parameter family of saddle-saddle connections obtained by combining $\tilde{u}_2$ and $\hat{u}_2$.

COROLLARY 2.2. There exist $\varepsilon > 0$ and continuous functions s^*, u_2^*, defined on $[m-\varepsilon, m+\varepsilon]$ such that $u_2^*(m) = M$, u_2^* is strictly increasing, and if $|u_1 - m| < \varepsilon$, then $u_1 \to u_2^*(u_1)$ is a saddle-saddle connection with speed $s = s^*(u_1)$, which is positive for $u_1 < m$ and negative for $u_1 > m$.

Proof. From Theorem 2.1, let $\varepsilon_1 > 0$ be defined by $\hat{u}_2([M, M+\delta_2]) = [m, m+\varepsilon_1]$, and let $\overline{u}_2 : [m, m+\varepsilon_1] \to \mathbf{R}$ be the inverse of $\hat{u}_2$. Then $\overline{u}_2$ is continuous and strictly monotonic, and $\overline{u}_2(M) = m$. Now set $\varepsilon = \min(\delta_1, \varepsilon_1)$ and define

$$u_2^*(u_1) = \begin{pmatrix} \tilde{u}_2(u_1) & \text{if } m-\varepsilon \le u_1 \le m, \\ \overline{u}_2(u_1) & \text{if } m \le u_1 \le m+\varepsilon. \end{pmatrix}$$

Since $u_1 \to \overline{u}_2(u_1)$ is a connection with speed $-\hat{s}(u_1)$, when $m \le u_1 \le m+\varepsilon$, define

$$s^*(u_1) = \begin{pmatrix} \tilde{s}(u_1) & \text{if } m-\varepsilon \le u_1 \le m, \\ -\hat{s}(u_1) & \text{if } m \le u_1 \le m+\varepsilon. \end{pmatrix}$$

Then $u_1 \to u_2^*(u_1)$ is a saddle-saddle connection with speed $s^*(u_1)$ as required, for each $u_1 \in [m-\varepsilon, m+\varepsilon]$.

Next, I show that all weak shocks satisfying the Lax admissibility condition are also admissible according to the viscosity-capillarity criterion. It should be noted that not all strong Lax admissible shocks satisfy the viscosity-capillarity criterion [7]. On the other hand, if σ has just a single inflection point, then all Lax admissible shocks satisfy condition (E) of Liu [4] (see also [12]). The shock wave (2.1) (satisfying the Rankine–Hugoniot conditions (2.2), (2.3)) is Lax admissible if

$$\lambda_j(u_2) < s < \lambda_j(u_1) \quad \text{for } j = 1 \text{ or for } j = 2. \tag{2.11}$$

If (2.11) holds, we say the shock is a j-shock, and write $(u_2, v_2) \in S_j(u_1, v_1)$. Now, for $k = 1, 2$, the critical point $u = u_k$, $u' = 0$ of (2.8) has eigenvalues

$$\mu_\pm = \frac{1}{2A}\left\{-s \pm \sqrt{4A\sigma'(u_k) + (1-4A)s^2}\right\}. \tag{2.12}$$

Since $0 \le 1 - 4A < 1$, we conclude the following from (2.11), (2.12). For a 1-shock, $\mu_\pm(u_1) > 0$ and $\mu_-(u_2) < 0 < \mu_+(u_2)$, so that $(u, u') = (u_1, 0)$ is an unstable node and $(u, u') = (u_2, 0)$ is a saddle point for Eq. (2.8). Similarly, a 2-shock involves a saddle point at $(u_1, 0)$ and a stable node at $(u_2, 0)$. In either case, if u_1, u_2 are sufficiently close, then there is a trajectory joining the critical point $(u_1, 0)$ to the critical point $(u_2, 0)$. This trajectory, $\{(u(\xi), u'(\xi)) : |\xi| < \infty\}$, corresponds to a viscous profile (2.6), with $\xi = (x - st)/\varepsilon$, and $v(\xi)$ given in terms of $u(\xi)$ by

$$v(u) = v_1 - s(u - u_1).$$

The set $S_j(u_1, v_1)$, $j = 1$ or 2, representing Lax admissible j-shocks is (near (u_1, v_1)) an arc having (u_1, v_1) as an end point. It is straightforward to check the following description of the shock curves $S_j(u_1, v_1)$ in a neighborhood N of (u_1, v_1).

$$N \cap S_1(u_1, v_1) = \{(u, v) \in N : v = -s(u - u_1) + v_1, \\ s^2 = (\sigma(u_1) - \sigma(u_2))/(u_1 - u_2), s < 0, u < u_1\},$$

$$N \cap S_2(u_1, v_1) = \{(u, v) \in N : v = -s(u - u_1) + v_1, \\ s^2 = (\sigma(u_1) - \sigma(u_2))/(u_1 - u_2), s > 0, u > u_1\},$$

if $u_1 < \alpha$, and there are similar formulas when $u_1 > \beta$.

We shall also need rarefaction curves through (u_1, v_1). These consist of intermediate states $(u, v) = (u(x/t), v(x/t))$ in a rarefaction wave fan. If we fix (u_1, v_1) as the state on the left of the wave, so that x/t increases, then u and v are related by

$$v = v_1 \pm \int_{u_1}^{u} c(z)\, dz,$$

with $x = \mp c(u)t$, respectively. Here,

$$c(u) = \sqrt{\sigma'(u)}$$

is the characteristic wave speed. Since x/t increases through the fan, the range of values of u in (2.12) is restricted. The set of possible states (u, v) on the right of a rarefaction fan is given by two rarefaction curves $R_j(u_1, v_1)$, $j = 1, 2$, which have a C^2 connection with the corresponding shock curves $S_j(u_1, v_1)$ at (u_1, v_1). (See [3] for details.) We have

$$\frac{dv}{du} = c(u) \quad \text{on } R_1(u_1, v_1), \qquad \frac{dv}{du} = -c(u) \quad \text{on } R_2(u_1, v_1).$$

The weak wave curves $W_j(u_1, v_1)$, $j = 1, 2$, for a given (u_1, v_1) consist of states (u_2, v_2) to which (u_1, v_1) may be joined by a weak shock or rarefaction wave. Thus

$$W_j(u_j, v_j) = (R_j(u_1, v_1) \cup S_j(u_1, v_1)) \cap N,$$

for some neighborhood N of (u_1, v_1). It is easy to check directly that $dv/du > 0$ on $W_1(u_1, v_1)$, and $dv/du < 0$ on $W_2(u_1, v_1)$, if $u_1 \notin [\alpha, \beta]$. If $(u, v) \in W_j(u_1, v_1)$, then we say that (u_1, v_1) is joined to (u, v) by a weak j-wave.

3. Solution of the Riemann problem. In this section we consider Riemann problems with initial data near the Maxwell line. The existence of solutions is described by the following theorem, whose proof uses the constructions of Sec. 2.

THEOREM 3.1. Let $v_0 \in \mathbf{R}$ be fixed. There exist neighborhoods N of (m, v_0) and N' of (M, v_0) such that if $(u_L, v_L) \in N$ and $(u_R, v_R) \in N'$, then the Riemann problem consisting of equations (1.1), with initial data

$$(u, v)(x, 0) = \begin{cases} (u_L, v_L) & \text{if } x < 0, \\ (u_R, v_R) & \text{if } x > 0, \end{cases} \tag{3.1}$$

has a solution consisting of a weak 1-wave, a phase jump, and a weak 2-wave, separated in the (x, t)-plane by constant values of (u, v).

Proof. Let $u_2^*(m - \varepsilon) = M - \gamma_1$, $u_2^*(m + \varepsilon) = M + \gamma_2$. Consider (u_L, v_L) with $|u_L - m| < \varepsilon$, and let $\Gamma = W(u_L, v_L) \cap \{(u_L, v_L) : |u_L - m| < \varepsilon\}$. Now let

$$\Gamma' = \{(u_2, v_2) : u_2 = u_2^*(u_1), v_2 = v_1 - s(u_2 - u_1), (u_1, v_1) \in \Gamma\}.$$

Finally, define $D(u_L, v_L)$ to be the union of the sets $W_2(u_2, v_2)$ with $(u_2, v_2) \in \Gamma'$. For each $(u_R, v_R) \in D(u_L, v_L)$ we have constructed a solution of the Riemann problem (3.1), involving the combination of waves specified in the theorem.

For $(u_L, v_L) = (m, v_0)$, the set $D(u_L, v_L)$ is a neighborhood of (M, v_0). Since $D(u_L, v_L)$ depends continuously on (u_L, v_L), by construction, we can choose a neighborhood N of (m, v_0) such that the set

$$\bigcap_{(u_L, v_L) \in N} D(u_L, v_L)$$

contains a neighborhood N' of (M, v_0). Then for each $(u_L, v_L) \in N$, $(u_R, v_R) \in N'$, the Riemann problem has a solution as claimed in the theorem.

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