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THERMODYNAMIC EFFICIENCY AND FREQUENCY SPECTRA*

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1. Introduction. Classical thermodynamics tells us that in order to maximize the efficiency of a heat engine it is necessary to employ a quasi-static cycle, i.e., one performed at an infinitesimal rate, for, it is asserted with some degree of plausibility, noninfinitesimal rates always entail a reduction in efficiency.

If the assertion is correct, it ought to be possible to use information about the frequency spectrum of a cycle in order to construct an upper limit on its efficiency.

The main purpose of the present article is to prove a result of this type by showing that when the spectrum is bounded and bounded away from zero—thus ruling out both arbitrarily large and arbitrarily small frequencies—the efficiency cannot exceed a certain fraction of the theoretical maximum. The fraction depends upon just the spectrum and the working material of the heat engine.

I am able to carry through the argument only within a very limited context, viz. one-dimensional linearised thermoelasticity with the effects of inertia ignored. Simplified and approximate though this theory is, it is nonetheless a genuine field theory of material behaviour, and one which takes account of heat conduction, thermomechanical coupling, and the possible inhomogeneity of the body.

It may be helpful to think of the body which forms the working material of the heat engine as occupying a slab $0 \leq x \leq 1$, of unit thickness, x being a Cartesian coordinate. The vectors of displacement and heat flux are taken to be parallel to the x-axis and, therefore, orthogonal to the faces $x = 0$ and $x = 1$. Henceforth, the body is identified with the unit interval [0,1] of the real line. It is supposed that there are no external body forces and no external sources of heat supply; the sole mechanism by which the body exchanges heat with its environment is conduction of heat across the faces of the slab.

The governing equations [Carlson, 1] are, in the absence of inertia, an approximate momentum equation

$$\sigma_x = 0,$$

an approximate energy equation

$$q_x = \theta_0 \eta_t,$$

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and the constitutive relations

$$\sigma = \beta(x)u_x - \mu(x)(\theta - \theta_0),$$
$$\eta = \mu(x)u_x + \frac{c(x)}{\theta_0}(\theta - \theta_0),$$
$$q = k\theta_x,$$

where $u(x,t)$ is the displacement, $\theta(x,t)$ is the absolute temperature, $\sigma(x,t)$ is the stress, $\eta(x,t)$ is the entropy density, and $q(x,t)$ is the heat flux. The positive constant θ_0 is the reference temperature, $\beta(x)$ is an elastic modulus, $\mu(x)$ is the stress-temperature modulus, $c(x)$ is the specific heat at constant strain, and $k(x)$ is the thermal conductivity.

Two minor questions of sign call for comment. First, $q(x,t)$ is the heat flux from $[x,1]$ *into* $[0,x]$; this convention removes a minus sign from the last of the constitutive relations (Fourier's law of heat conduction). Second, μ is the negative of what Carlson calls the stress-temperature modulus. Since μ can be expected to be positive there is a slight advantage in adopting our convention.

It will be necessary to restrict the coefficients $\beta(x)$, $\mu(x)$, $c(x)$, $k(x)$ and it will be enough, for the moment, to take each to be positive and continuous on [0,1].

According to the momentum equation, the stress is independent of x and may be written as $\sigma(t)$. The constitutive relation for the stress implies, as $\beta(x)$ vanishes nowhere, that the strain

$$u_x = \frac{\sigma(t)}{\beta(x)} + \frac{\mu(x)}{\beta(x)}(\theta - \theta_0),$$

and when we substitute for η, q, u_x into the energy equation we find the temperature to be a solution of the equation

$$(k(x)\theta_x)_x = \left(c(x) + \theta_0\frac{\mu(x)^2}{\beta(x)}\right)\theta_t + \theta_0\frac{\mu(x)}{\beta(x)}\dot{\sigma}(t).$$

It is assumed that the body is immersed in an environment whose temperature is spatially homogeneous at each instant but may vary with time. Thus, the temperature is required to satisfy boundary conditions of the form

$$\theta(0,t) = \theta(1,t) = \theta_0 + \phi(t) \quad (-\infty < t < \infty).$$

In what follows, we regard ϕ and σ as being at our disposal, i.e., we suppose that we can adjust both the temperature of the environment and the stress to which the body is subjected.

It will be convenient to work not with θ directly but with the difference

$$\psi(x,t) = \theta(x,t) - \theta_0 - \phi(t),$$

which is a solution of the differential equation

$$(k(x)\psi_x)_x = a(x)\psi_t + a(x)\dot{\phi}(t) + b(x)\dot{\sigma}(t), \tag{1.1}$$

the coefficients on the right-hand side being

$$a(x) = c(x) + \theta_0\frac{\mu(x)^2}{\beta(x)}, \qquad b(x) = \theta_0\frac{\mu(x)}{\beta(x)}.$$

Because $\beta(x)$, $\mu(x)$, $c(x)$ are positive and continuous on [0,1] the same is true of $a(x)$ and $b(x)$.

The boundary conditions on θ force ψ to satisfy the homogeneous boundary conditions

$$\psi(0,t) = \psi(1,t) = 0 \quad (-\infty < t < \infty). \tag{1.2}$$

The rate of working of the body upon its environment is

$$W = \sigma(t)u_t(0,t) - \sigma(t)u_t(1,t).$$

On writing W as

$$-\sigma \int_0^1 u_{xt}\, dx,$$

and noting that the strain-rate

$$u_{xt} = \frac{\dot{\sigma}}{\beta} + \frac{\mu}{\beta}\theta_t = \frac{\dot{\sigma}}{\beta} + \frac{1}{\theta_0}b\dot{\phi} + \frac{1}{\theta_0}b\psi_t,$$

we obtain the equation

$$W = -\sigma\dot{\sigma}\int_0^1 \frac{1}{\beta}\,dx - \frac{1}{\theta_0}\sigma\dot{\phi}\int_0^1 b\,dx - \frac{1}{\theta_0}\sigma\int_0^1 b\psi_t\,dx. \tag{1.3}$$

The net heat flux into the body from its environment is

$$\begin{aligned} Q &= q(1,t) - q(0,t) \\ &= k(1)\theta_x(1,t) - k(0)\theta_x(0,t) \\ &= k(1)\psi_x(1,t) - k(0)\psi_x(0,t), \end{aligned}$$

and, on integrating (1.1) with respect to x, we have

$$Q = \int_0^1 a\psi_t\,dx + \dot{\phi}\int_0^1 a\,dx + \dot{\sigma}\int_0^1 b\,dx. \tag{1.4}$$

The net heat flux can be expressed as the difference $Q = Q^+ - Q^-$, between the rate of absorption of heat, which is $Q^+ = \mathrm{Max}(Q,0)$, and the rate of emission of heat, which is $Q^- = -\mathrm{Min}(Q,0)$. Neither Q^+ nor Q^- can be negative, but Q may be positive, negative, or zero.

The usual discussions of heat engines operating in a cycle envisage that ϕ, σ, θ, u, ψ are all periodic in their dependence upon t, with a common period p say, and that ϕ is varied within certain limits:

$$m^- \le \phi(t) \le m^+ \quad (-\infty < t < \infty).$$

The problem is to choose ϕ and σ in such a way as to maximize the efficiency, which is the ratio

$$\int_0^p W\,dt \Big/ \int_0^p Q^+\,dt$$

of the work done in a period to the heat absorbed in a period.

I shall, in fact, consider a more general situation in which ϕ, σ, θ, u, ψ are almost periodic, but not necessarily periodic. Only the more elementary parts of the theory of almost periodic functions are required, and all are to be found in Bohr [2].

2. Almost periodic solutions. By a *spectrum* will be meant any strictly increasing sequence $\Omega = \{\omega_n\}_{n\geq 1}$ of positive real numbers ω_n; each ω_n is a *frequency.* $S(\Omega)$ will denote the collection of all real-valued trigonometric sums

$$f(t) = A_0 + \operatorname{Re}\sum_{n=1}^{\infty} A_n \exp(i\omega_n t),$$

where A_0 may be any real number, and the coefficients A_n may be any complex numbers which ensure the convergence of the series

$$\sum_{n=1}^{\infty} \omega_n^{\nu}|A_n| \quad (\nu = 0, 1, 2, 3, 4).$$

Certain implications of these definitions should be noted.

First, the series $\sum_{n=1}^{\infty} |A_n|$ converges (this being the case, $\nu = 0$) and hence $A_n \to 0$ when $n \to \infty$; thus, the series

$$\sum_{n=1}^{\infty} \omega_n^{\nu}|A_n|^2 \quad (\nu = 0, 1, 2, 3, 4)$$

must converge.

Next, the first four derivatives of f exist, and can be calculated by termwise differentiation, i.e.,

$$\dot{f}(t) = \operatorname{Re}\sum_{n=1}^{\infty}(i\omega_n)A_n \exp(i\omega_n t),$$
$$\ddot{f}(t) = \operatorname{Re}\sum_{n=1}^{\infty}(i\omega_n)^2 A_n \exp(i\omega_n t),$$

and so forth. Moreover, f and its first four derivatives are continuous and bounded on $-\infty < t < \infty$.

Again, f is the uniform limit, as $N \to \infty$, of the sequence of partial sums

$$A_0 + \operatorname{Re}\sum_{n=1}^{N} A_n \exp(i\omega_n t).$$

Each partial sum is almost periodic and, therefore, f itself must be almost periodic; likewise the first four derivatives of f are almost periodic.

We shall use M to denote the mean value operator of the theory of almost periodic functions, i.e.,

$$M(f) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt.$$

The formulae

$$M(f) = A_0,$$
$$M(f^2) = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} |A_n|^2,$$
$$M(\dot{f}^2) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n^2 |A_n|^2,$$
$$M(\ddot{f}^2) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n^4 |A_n|^2,$$
$$M(fg) = A_0 B_0 + \frac{1}{4}\sum_{n=1}^{\infty} (A_n \overline{B}_n + \overline{A}_n B_n),$$

are especially important. In the last, g belongs to $S(\Omega)$ and is the sum

$$B_0 + \operatorname{Re}\sum_{n=1}^{\infty} B_n \exp(i\omega_n t).$$

Each of the first four can be proved by verifying the corresponding result for an appropriate sequence of partial sums and then invoking the uniform convergence of the partial sums and the convergence of the series which appear in the formula. The last formula follows from the identity

$$M(fg) = \frac{1}{2}M((f+g)^2) - \frac{1}{2}M(f^2) - \frac{1}{2}M(g^2).$$

It will be convenient to write $U = \bigcup S(\Omega)$, the union being taken over all spectra Ω. If m^+ and m^- are any real numbers with $m^+ > m^-$, we write $S(\Omega, m^+, m^-)$ for the collection of trigonometric sums f which belong to $S(\Omega)$ and satisfy

$$m^- \le f(t) \le m^+ \quad (-\infty < t < \infty),$$

and we write $U(m^+, m^-) = \bigcup S(\Omega, m^+, m^-)$, the latter union also being taken over all spectra.

In order to be assured of an adequate supply of almost periodic solutions, we must record two features of the boundary value problem

$$(k(x)\Psi')' = i\omega a(x)\Psi + F(x), \tag{2.1}$$
$$\Psi(0) = \Psi(1) = 0, \tag{2.2}$$

in which the prime stands for the operator d/dx.

The first feature is existence and uniqueness of the solution, i.e., if ω is any positive real number, and if $F(x)$ is any complex-valued function which is continuous on $[0,1]$, the boundary value problem has one and only one complex-valued solution $\Psi(x)$.

The verification rests upon the fact that the boundary value problem is equivalent to an integral equation

$$\Psi(x) = -i\omega \int_0^1 K(x,y)a(y)\Psi(y)\,dy - \int_0^1 K(x,y)F(y)\,dy, \tag{2.3}$$

whose kernel $K(x, y)$ equals

$$\int_0^x \frac{d\xi}{k(\xi)} \int_y^1 \frac{d\xi}{k(\xi)} \Big/ \int_0^1 \frac{d\xi}{k(\xi)} \quad (0 \le x \le y \le 1),$$

$$\int_x^1 \frac{d\xi}{k(\xi)} \int_0^y \frac{d\xi}{k(\xi)} \Big/ \int_0^1 \frac{d\xi}{k(\xi)} \quad (0 \le y \le x \le 1),$$

and is, therefore, real-valued, continuous, and symmetric.

According to Fredholm's theory, the integral equation has exactly one solution—and the same is true of the boundary value problem—provided the homogeneous equation

$$\Psi(x) = -i\omega \int_0^1 K(x, y) a(y) \Psi(y)\, dy$$

has no solution other than $\Psi(x) = 0$. This is indeed the case, for the homogeneous equation implies that

$$\int_0^1 a(x)|\Psi(x)|^2\, dx = -i\omega \int_0^1 \int_0^1 a(x)a(y)K(x, y)\overline{\Psi(x)}\Psi(y)\, dx\, dy.$$

Taking the complex conjugate of each side yields

$$\begin{aligned}\int_0^1 a(x)|\Psi(x)|^2\, dx &= i\omega \int_0^1 \int_0^1 a(x)a(y)K(x, y)\Psi(x)\overline{\Psi(y)}\, dx\, dy \\ &= i\omega \int_0^1 \int_0^1 a(y)a(x)K(y, x)\Psi(y)\overline{\Psi(x)}\, dx\, dy \\ &= i\omega \int_0^1 \int_0^1 a(x)a(y)K(x, y)\overline{\Psi(x)}\Psi(y)\, dx\, dy \\ &= -\int_0^1 a(x)|\Psi(x)|^2\, dx.\end{aligned}$$

Hence

$$\int_0^1 a(x)|\Psi(x)|^2\, dx = 0$$

and, because $a(x)$ is positive, $\Psi(x)$ must vanish identically on [0,1]. Thus, the existence and the uniqueness of the solution of the boundary value problem are both guaranteed.

The second feature of the boundary value problem is that there are positive constants α_1, α_2, α_3, depending upon the coefficients $k(x)$ and $a(x)$ only, such that

$$\operatorname{Max}|\Psi| \le \alpha_1 \operatorname{Max}|F|, \qquad \operatorname{Max}|\Psi'| \le (\alpha_2 + \alpha_3\omega) \operatorname{Max}|F|. \tag{2.4}$$

In order to obtain these estimates, we start by defining the positive constant λ to be the greatest lower bound of the ratio

$$\int_0^1 k(x)h'(x)^2\, dx \Big/ \int_0^1 a(x)h(x)^2\, dx, \tag{2.5}$$

in which the real-valued functions $h(x)$ are required to be continuous and piecewise continuously differentiable and to satisfy $h(0) = h(1) = 0$. We might equally well define λ as the least eigenvalue associated with the Sturm–Liouville problem

$$(k(x)h')' = -\lambda a(x)h, \qquad h(0) = h(1) = 0.$$

Next, we multiply (2.1) through by $\overline{\Psi}$ to obtain

$$(\overline{\Psi}k\Psi')' = k|\Psi'|^2 + i\omega a|\Psi|^2 + F\overline{\Psi},$$

and when we integrate and appeal to the boundary conditions (2.2) we find

$$\int_0^1 k|\Psi'|^2\,dx = -i\omega\int_0^1 a|\Psi|^2\,dx - \int_0^1 F\overline{\Psi}\,dx.$$

Hence

$$\begin{aligned}\int_0^1 k|\Psi'|^2\,dx &= -\frac{1}{2}\int_0^1 (F\overline{\Psi} + \overline{F}\Psi)\,dx \\ &\le \int_0^1 |F||\Psi|\,dx \le \left(\int_0^1 \frac{1}{a}|F|^2\,dx\right)^{1/2}\left(\int_0^1 a|\Psi|^2\,dx\right)^{1/2} \\ &\le \frac{1}{\lambda^{1/2}}\left(\int_0^1 \frac{1}{a}|F|^2\,dx\right)^{1/2}\left(\int_0^1 k|\Psi'|^2\,dx\right)^{1/2}\end{aligned}$$

and, therefore,

$$\int_0^1 k|\Psi'|^2\,dx \le \frac{1}{\lambda}\int_0^1 \frac{1}{a}|F|^2\,dx \le \frac{1}{\lambda}\int_0^1 \frac{1}{a}\,dx(\operatorname{Max}|F|)^2.$$

On the other hand, the boundary condition $\Psi(0) = 0$ implies that

$$|\Psi(x)| = \left|\int_0^x \Psi'(y)\,dy\right| \le \int_0^1 |\Psi'(y)|\,dy$$

and, therefore,

$$\begin{aligned}\operatorname{Max}|\Psi| &\le \left(\int_0^1 \frac{1}{k}\,dx\right)^{1/2}\left(\int_0^1 k|\Psi'|^2\,dx\right)^{1/2} \\ &\le \frac{1}{\lambda^{1/2}}\left(\int_0^1 \frac{1}{k}\,dx\right)^{1/2}\left(\int_0^1 \frac{1}{a}\,dx\right)^{1/2}\operatorname{Max}|F|,\end{aligned}$$

which proves the first of the inequalities (2.4).

To prove the second inequality we split each of the integrals in the integral equation (2.3) into a sum $\int_0^x + \int_x^1$, differentiate with respect to x, and cancel terms involving $K(x,x)$ to find that

$$\Psi'(x) = -\int_0^1 K_x(x,y)(i\omega a(y)\Psi(y) + F(y))\,dy.$$

Upon estimating the right-hand side and appealing to the first of (2.4) we arrive at the second.

Now let

$$\phi(t) = A_0 + \mathrm{Re}\sum_{n=1}^{\infty} A_n \exp(i\omega_n t), \tag{2.6}$$

$$\sigma(t) = B_0 + \mathrm{Re}\sum_{n=1}^{\infty} B_n \exp(i\omega_n t) \tag{2.7}$$

be any members of $S(\Omega)$, let $\Psi_n(x)$ be the solution of the boundary value problem

$$(k(x)\Psi_n')' = i\omega_n(a(x)\Psi_n + A_n a(x) + B_n b(x)), \tag{2.8}$$

$$\Psi_n(0) = \Psi_n(1) = 0, \tag{2.9}$$

and let

$$\psi(x,t) = \mathrm{Re}\sum_{n=1}^{\infty} \Psi_n(x) \exp(i\omega_n t). \tag{2.10}$$

The estimates (2.4), with $\omega = \omega_n$ and $F(x) = i\omega_n(A_n a(x) + B_n b(x))$, tell us that

$$\mathrm{Max}\,|\Psi_n| \le \alpha_1 \omega_n(|A_n|\,\mathrm{Max}\,|a| + |B_n|\,\mathrm{Max}\,|b|)$$

and so $\psi(x,t)$ is well-defined, being the sum of a series which converges uniformly with respect to both x and t.

Furthermore, the series

$$\mathrm{Re}\sum_{n=1}^{\infty} i\omega_n \Psi_n(x)\exp(i\omega_n t),$$

$$\mathrm{Re}\sum_{n=1}^{\infty} \Psi_n'(x)\exp(i\omega_n t),$$

$$\mathrm{Re}\sum_{n=1}^{\infty}(k(x)\Psi_n'(x))'\exp(i\omega_n t)$$

$$= \mathrm{Re}\sum_{n=1}^{\infty} i\omega_n(a(x)\Psi_n(x) + A_n a(x) + B_n b(x))\exp(i\omega_n t)$$

all converge uniformly with respect to x and t, and their sums are ψ_t, ψ_x, $(k\psi_x)_x$, respectively. It is now clear that the ψ we have constructed is a solution of the differential equation (1.1) and the boundary conditions (1.2).

The corresponding rate of working W and the net heat flux Q can be determined from (1.3) and (1.4). Since each of the integrals

$$\int_0^1 a\psi_t\,dx = \mathrm{Re}\sum_{n=1}^{\infty} i\omega_n\left(\int_0^1 a\Psi_n\,dx\right)\exp(i\omega_n t),$$

$$\int_0^1 b\psi_t\,dx = \mathrm{Re}\sum_{n=1}^{\infty} i\omega_n\left(\int_0^1 b\Psi_n\,dx\right)\exp(i\omega_n t)$$

is almost periodic, W and Q are almost periodic. Accordingly, $|Q|$ is almost periodic and, because

$$Q^+ = \tfrac{1}{2}(|Q| + Q), \qquad Q^- = \tfrac{1}{2}(|Q| - Q),$$

Q^+ and Q^- must be almost periodic.

The question now arises as to how the efficiency is to be defined. I propose to do this in essentially the same way as in my article [3], i.e., by defining the *efficiency* as the ratio

$$M(W)/M(Q^+)$$

of the mean rate of working to the mean rate of absorption of heat. The definition demands, of course, that $M(Q^+) > 0$.

When W and Q^+ are periodic, of period p say, the definition reduces to the usual one, for then

$$M(W) = \frac{1}{p}\int_0^p W\,dt, \qquad M(Q^+) = \frac{1}{p}\int_0^p Q^+\,dt.$$

3. Statement of the results. It is proposed to prove three results. The first two are related to the results of [4], but the arguments of that article are somewhat different for they take inertia into account and deal with periodic, rather than almost periodic, solutions.

The first result says that the efficiency is always strictly less than the bound $(m^+ - m^-)/\theta_0$: *if ϕ belongs to $U(m^+, m^-)$, if σ belongs to U, and if $M(Q^+) > 0$ then*

$$\theta_0 M(W) < (m^+ - m^-)M(Q^+). \tag{3.1}$$

The bound is nonetheless the least upper bound, as the second result tells us: *if ε is an arbitrarily small positive number, there is a ϕ belonging to $U(m^+, m^-)$ and there is a σ belonging to U, such that $M(Q^+) > 0$ and*

$$\theta_0 M(W) > (1 - \varepsilon)(m^+ - m^-)M(Q^+). \tag{3.2}$$

These conclusions are not quite what one would expect to be true, for the temperature of the environment, which is $\theta_0 + \phi(t)$, is not greater than $\theta_0 + m^+$ and is not less than $\theta_0 + m^-$. Classical thermodynamics leads us to suppose, therefore, that the correct least upper bound should be

$$\frac{(\theta_0 + m^+) - (\theta_0 + m^-)}{\theta_0 + m^+} = \frac{m^+ - m^-}{\theta_0 + m^+}$$

but, as I have pointed out in [4], the approximations involved in deriving the linearised theory of thermoelasticity distort the truth to the extent of replacing the denominator $\theta_0 + m^+$ by θ_0. (The distortion arises from having approximated a term $\theta\eta_t$ in the exact energy equation by the term $\theta_0\eta_t$.)

The third, and chief, result is of the type referred to in the Introduction in that it uses information about the spectrum in order to produce an improved upper bound on the efficiency.

The spectrum will be required to be bounded above; thus the sequence of frequencies must converge to a limit $\omega_\infty = \lim_{n\to\infty}\omega_n$ and the entire spectrum is confined to the compact interval $[\omega_1, \omega_\infty]$.

It will also be necessary to place a restriction on the body itself. The restriction arises from asking in what circumstances the temperature of the body coincides with the temperature of the environment, at every point x and at every instant t. Such a question is equivalent to asking in what circumstances $\psi = 0$ throughout the strip $[0, 1] \times (-\infty, \infty)$ in the (x, t)-plane. Our hypothesis is that this is the case only when the environmental temperature $\theta_0 + \phi$ and the stress σ are static, i.e., only when $\dot{\phi} = \dot{\sigma} = 0$ on $(-\infty, \infty)$.

A necessary and sufficient condition for the hypothesis to be satisfied is that the coefficients $a(x)$ and $b(x)$ be linearly independent on [0,1]. For, it is clear from the differential equation (1.1) that $\psi = 0$ only if the condition

$$a(x)\dot{\phi}(t) + b(x)\dot{\sigma}(t) = 0$$

is satisfied. If $a(x)$ and $b(x)$ are linearly dependent there is a positive constant α such that $b(x) = \alpha a(x)$ and the condition holds provided only that $\dot{\phi} + \alpha\dot{\sigma} = 0$. On the other hand, if $a(x)$ and $b(x)$ are linearly independent we can choose points x_1 and x_2 of the unit interval so that $a(x_1)b(x_2) - a(x_2)b(x_1) \neq 0$. The equations obtained by setting $x = x_1$ and $x = x_2$ then imply that $\dot{\phi} = \dot{\sigma} = 0$.

It should be noted that, in the context of our simplified and approximate theory at least, the hypothesis forces the body to be genuinely inhomogeneous. If the body were homogeneous the coefficients $\beta(x)$, $\mu(x)$, $c(x)$, $k(x)$ would all be constant and so $a(x)$ and $b(x)$ would be constant and, therefore, linearly dependent.

The third result is that: *if $a(x)$ and $b(x)$ are linearly independent on* $[0, 1]$, *and continuously differentiable, and if Ω is any bounded spectrum, there is a positive constant δ, depending only upon ω_1, ω_∞ and the coefficients $k(x)$, $a(x)$, $b(x)$ such that*

$$\theta_0 M(W) \leq (1-\delta)(m^+ - m^-)M(Q^+) \tag{3.3}$$

whenever ϕ belongs to $S(\Omega, m^+, m^-)$ and σ belongs to $S(\Omega)$.

4. Derivation of the upper bound. The proof of (3.1) rests upon the identities

$$M(Q) = 0, \tag{4.1}$$

$$\theta_0 M(W) + M\left(\int_0^1 k\psi_x^2\,dx\right) = M(\phi Q). \tag{4.2}$$

The first of these, which is equivalent to

$$M(Q^+) = M(Q^-),$$

is forced upon us by the approximations involved in the linearised theory and will not generally be valid within an exact theory. (Once again, the replacement of $\theta\eta_t$ by $\theta_0\eta_t$ is at the heart of the matter.) It is of interest to recall, though, that the truth of (4.1) was considered as axiomatic by Carnot [5, p. 19]—a point to which Lord Kelvin drew attention in his account of Carnot's theory [6, pp. 115–118].

In order to verify (4.1) it is enough to rewrite (1.4) as

$$Q = \frac{d}{dt}\left[\int_0^1 a\psi\,dx + \phi\int_0^1 a\,dx + \sigma\int_0^1 b\,dx\right].$$

The sum enclosed within square brackets is a bounded function of t, and on integrating both sides with respect to t over the interval $[0, T]$, dividing through by T, and taking the limit as $T \to \infty$, we arrive immediately at (4.1).

The initial step in the proof of (4.2) is to multiply the differential equation (1.1) through by ψ to obtain

$$(\psi k \psi_x)_x = k\psi_x^2 + a\psi\psi_t + (a\dot{\phi} + b\dot{\sigma})\psi.$$

Next, we integrate with respect to x, and take account of the boundary conditions (1.2), to find that

$$0 = \int_0^1 k\psi_x^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 a\psi^2\,dx + \int_0^1 (a\dot{\phi} + b\dot{\sigma})\psi\,dx.$$

On the other hand, (1.3) and (1.4) imply that

$$\phi Q - \theta_0 W = \int_0^1 (a\phi + b\sigma)\psi_t\,dx + \frac{1}{2}\frac{d}{dt}\left[\phi^2\int_0^1 a\,dx + 2\phi\sigma\int_0^1 b\,dx + \theta_0\sigma^2\int_0^1 \frac{1}{\beta}\,dx\right]$$

and, hence,

$$\phi Q - \theta_0 W - \int_0^1 k\psi_x^2\,dx = \frac{1}{2}\frac{d}{dt}\left[\int_0^1 a\psi^2\,dx + 2\int_0^1 (a\phi + b\sigma)\psi\,dx + \phi^2\int_0^1 a\,dx + 2\phi\sigma\int_0^1 b\,dx + \theta_0\sigma^2\int_0^1 \frac{1}{\beta}\,dx\right].$$

As before, the sum enclosed within square brackets is bounded, and when we integrate with respect to t over $[0, T]$, divide through by T, and let $T \to \infty$, we arrive at (4.2).

It is now necessary to examine the mean value $M(\phi Q)$ more closely. Since $Q = Q^+ - Q^-$, it must be that

$$\int_0^T \phi Q\,dt = \int_0^T \phi Q^+\,dt - \int_0^T \phi Q^-\,dt$$
$$= m^+\int_0^T Q^+\,dt - m^-\int_0^T Q^-\,dt - \int_0^T (m^+ - \phi)Q^+\,dt - \int_0^T (\phi - m^-)Q^-\,dt.$$

Because ϕ belongs to $U(m^+, m^-)$, each of the integrands $(m^+ - \phi)Q^+$ and $(\phi - m^-)Q^-$ is nonnegative and, therefore,

$$\int_0^T \phi Q\,dt \le m^+\int_0^T Q^+\,dt - m^-\int_0^T Q^-\,dt.$$

On dividing through by T, letting $T \to \infty$, and invoking (4.1), we arrive at the conclusion

$$M(\phi Q) \le m^+ M(Q^+) - m^- M(Q^-) = (m^+ - m^-)M(Q^+).$$

Hence (4.2) implies the inequality

$$\theta_0 M(W) + M\left(\int_0^1 k\psi_x^2\,dx\right) \le (m^+ - m^-)M(Q^+). \tag{4.3}$$

The second term on the left-hand side of (4.3) being nonnegative it follows that

$$\theta_0 M(W) \le (m^+ - m^-) M(Q^+)$$

and, therefore, the efficiency cannot exceed the bound $(m^+ - m^-)/\theta_0$.

To show that the efficiency is always strictly less than the bound, suppose it were possible to have both $M(Q^+) > 0$ and

$$\theta_0 M(W) = (m^+ - m^-) M(Q^+).$$

It would then follow from (4.3) that the mean value

$$M\left(\int_0^1 k\psi_x^2\,dx\right) = 0.$$

However, an elementary argument enables us to deduce from (2.10) that

$$M\left(\int_0^1 k\psi_x^2\,dx\right) = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 k|\Psi_n'|^2\,dx.$$

Since $k(x)$ is positive it must be that $\Psi_n' = 0$ $(n = 1, 2, 3, \dots)$ and, hence,

$$\psi_x = \mathrm{Re}\sum_{n=1}^{\infty}\Psi_n'(x)\exp(i\omega_n t) = 0.$$

It now follows that the heat flux $q = k\theta_x = k\psi_x = 0$, that the net heat flux $Q(t) = q(1, t) - q(0, t) = 0$, that the rate of absorption of heat $Q^+(t) = 0$, and, finally, that $M(Q^+) = 0$. This conclusion, though, is in conflict with the hypothesis $M(Q^+) > 0$ and so (3.1) is correct.

5. Impossibility of reducing the bound in every case. Although it is envisaged that the series which define ϕ, σ, ψ need be neither finite sums nor periodic in t, it is enough, for the purpose of proving (3.2), to take ϕ, σ, ψ to be finite sums which share a common period p. The frequencies of the underlying spectrum will be

$$\omega_n = (4n - 2)\pi/p. \tag{5.1}$$

At a later stage in the argument we shall be led to consider what happens in the limit as $p \to \infty$; one implication of taking this limit is that the lowest frequency $\omega_1 = 2\pi/p \to 0$.

We start by showing that if ϕ, σ, ψ are periodic in t, with period p, then

$$\lambda^2\int_0^p\int_0^1 a\psi_t^2\,dx\,dt \le \int_0^p\int_0^1 \frac{1}{a}(a\ddot{\phi} + b\ddot{\sigma})^2\,dx\,dt, \tag{5.2}$$

where λ is the eigenvalue introduced earlier, i.e., the greatest lower bound of the ratio (2.5).

To verify this, let us differentiate (1.1) throughout with respect to t to obtain

$$(k\psi_{xt})_x = a\psi_{tt} + a\ddot{\phi} + b\ddot{\sigma}.$$

On multiplying through by ψ_t, we find

$$(\psi_t k\psi_{xt})_x = k\psi_{xt}^2 + a\psi_t\psi_{tt} + (a\ddot{\phi} + b\ddot{\sigma})\psi_t,$$

and, on integrating with respect to x and remembering that $\psi_t(0,t) = \psi_t(1,t) = 0$, we get

$$0 = \int_0^1 k\psi_{xt}^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 a\psi_t^2\,dx + \int_0^1 (a\ddot{\phi} + b\ddot{\sigma})\psi_t\,dx.$$

By virtue of periodicity, a further integration with respect to t over the interval $[0, p]$ yields the conclusion

$$\int_0^p \int_0^1 k\psi_{xt}^2\,dx\,dt = -\int_0^p \int_0^1 (a\ddot{\phi} + b\ddot{\sigma})\psi_t\,dx\,dt.$$

According to the definition of λ, the left-hand side cannot be less than

$$\lambda \int_0^p \int_0^1 a\psi_t^2\,dx\,dt,$$

and, as the Schwarz inequality implies, the right-hand side cannot be greater than

$$\left(\int_0^p \int_0^1 \frac{1}{a}(a\ddot{\phi} + b\ddot{\sigma})^2\,dx\,dt\right)^{1/2} \left(\int_0^p \int_0^1 a\psi_t^2\,dx\,dt\right)^{1/2}.$$

Thus, (5.2) is correct.

Next, let $f(s)$ and $g(s)$ be real, finite, trigonometric sums which are periodic in s, with unit period; specific choices of f and g will be made later. Let p be any positive number and let $\phi(t) = f(t/p)$, $\sigma(t) = g(t/p)$. Clearly, $\phi(t)$ and $\sigma(t)$ are real, finite, trigonometric sums of period p.

According to (1.3), the rate of working

$$W(t) = -\frac{1}{p}g(t/p)\dot{g}(t/p)\int_0^1 \frac{1}{\beta}\,dx - \frac{1}{\theta_0 p}g(t/p)\dot{f}(t/p)\int_0^1 b\,dx - \frac{1}{\theta_0}g(t/p)\int_0^1 b\psi_t\,dx.$$

Upon integrating with respect to t over $[0, p]$, and making the change of variable $t = ps$ in the first two integrals on the right-hand side, we see that the work done

$$\int_0^p W(t)\,dt = -\int_0^1 g(s)\dot{g}(s)\,ds \int_0^1 \frac{1}{\beta}\,dx - \frac{1}{\theta_0}\int_0^1 g(s)\dot{f}(s)\,ds \int_0^1 b\,dx$$
$$- \frac{1}{\theta_0}\int_0^p \int_0^1 g(t/p)b(x)\psi_t(x,t)\,dx\,dt.$$

Because $g(s)$ has unit period, the first term on the right-hand side vanishes and, therefore,

$$\int_0^p W(t)\,dt = -\frac{1}{\theta_0}\int_0^1 g(s)\dot{f}(s)\,ds \int_0^1 b\,dx - \frac{1}{\theta_0}\int_0^p \int_0^1 g(t/p)b(x)\psi_t(x,t)\,dx\,dt. \tag{5.3}$$

The inequality (5.2) implies the estimate

$$\lambda^2 \int_0^p \int_0^1 a\psi_t^2\,dx\,dt \le \frac{1}{p^3}\int_0^1 \int_0^1 \frac{1}{a(x)}(a(x)\ddot{f}(s) + b(x)\ddot{g}(s))^2\,dx\,ds$$

and, hence, the second term on the right-hand side of (5.3) must satisfy

$$\left(\int_0^p \int_0^1 g(t/p)b(x)\psi_t(x,t)\,dx\,dt\right)^{1/2}$$
$$\leq \int_0^p \int_0^1 g(t/p)^2 \frac{b(x)^2}{a(x)}\,dx\,dt \cdot \int_0^p \int_0^1 a\psi_t^2\,dx\,dt$$
$$= p\int_0^1 g(s)^2\,ds \cdot \int_0^1 \frac{b^2}{a}\,dx \cdot \int_0^p \int_0^1 a\psi_t^2\,dx\,dt = O\left(\frac{1}{p^2}\right).$$

In short, (5.3) tells us that, when $p \to \infty$,

$$\int_0^p W(t)\,dt \to -\frac{1}{\theta_0}\int_0^1 g(s)\dot{f}(s)\,ds \cdot \int_0^1 b\,dx$$

or, by means of an integration by parts and an appeal to the periodicity of $f(s)$, that

$$\int_0^p W(t)\,dt \to \frac{1}{\theta_0}\int_0^1 f(s)\left(\dot{f}(s)\int_0^1 a\,dx + \dot{g}(s)\int_0^1 b\,dx\right)ds. \tag{5.4}$$

The integral of the product $f\dot{f}$ vanishes, of course, but it is convenient to retain it in (5.4).

According to (1.4), the net heat flux

$$Q(t) = \int_0^1 a\psi_t\,dx + \frac{1}{p}\dot{f}(t/p)\int_0^1 a\,dx + \frac{1}{p}\dot{g}(t/p)\int_0^1 b\,dx. \tag{5.5}$$

Since the right-hand side is

$$\frac{d}{dt}\left[\int_0^1 a\psi\,dx + f(t/p)\int_0^1 a\,dx + g(t/p)\int_0^1 b\,dx\right],$$

and the sum enclosed within square brackets is periodic in t, of period p, the integral

$$\int_0^p Q(t)\,dt = 0.$$

This conclusion, which is the counterpart to (4.1) in the present context, can be used to determine the limiting value of

$$\int_0^p Q^+(t)\,dt$$

as $p \to \infty$. For, we have $Q^+ = \frac{1}{2}(|Q| + Q)$ and, therefore,

$$\int_0^p Q^+(t)\,dt = \tfrac{1}{2}\int_0^p |Q(t)|\,dt.$$

However, (5.5) implies that

$$\left||Q(t)| - \left|\frac{1}{p}\dot{f}(t/p)\int_0^1 a\,dx + \frac{1}{p}\dot{g}(t/p)\int_0^1 b\,dx\right|\right| \leq \left|\int_0^1 a\psi_t\,dx\right|$$

and, hence, that

$$\left|\int_0^p |Q(t)|\,dt - \int_0^p \left|\frac{1}{p}\dot{f}(t/p)\int_0^1 a\,dx + \frac{1}{p}\dot{g}(t/p)\int_0^1 b\,dx\right| dt\right|$$

$$\le \int_0^p \left|\int_0^1 a\psi_t\,dx\right| dt \le \int_0^p \int_0^1 a|\psi_t|\,dx\,dt$$

$$\le \left(\int_0^p \int_0^1 a\,dx\,dt\right)^{1/2} \left(\int_0^p \int_0^1 a\psi_t^2\,dx\,dt\right)^{1/2}$$

$$= p^{1/2}\left(\int_0^1 a\,dx\right)^{1/2} O\left(\frac{1}{p^{3/2}}\right) = O\left(\frac{1}{p}\right),$$

where (5.2) has been used again. Since

$$\int_0^p \left|\frac{1}{p}\dot{f}(t/p)\int_0^1 a\,dx + \frac{1}{p}\dot{g}(t/p)\int_0^1 b\,dx\right| dt = \int_0^1 \left|\dot{f}(s)\int_0^1 a\,dx + \dot{g}(s)\int_0^1 b\,dx\right| ds,$$

we have proved that, when $p \to \infty$,

$$\int_0^p Q^+(t)\,dt \to \frac{1}{2}\int_0^1 \left|\dot{f}(s)\int_0^1 a\,dx + \dot{g}(s)\int_0^1 b\,dx\right| ds. \tag{5.6}$$

It is now possible to complete the proof of (3.2). Let N be any *odd* positive integer, and define $f(s)$ and $g(s)$ by

$$f(s) = \tfrac{1}{2}(m^+ + m^-) + \tfrac{1}{2}(m^+ - m^-)\cos(2\pi s),$$

$$f(s)\int_0^1 a\,dx + g(s)\int_0^1 b\,dx = \frac{1}{2^N}\sum_{n=0}^{N}\frac{1}{(2n-N)\pi}\binom{N}{n}\sin((2n-N)2\pi s).$$

A number of points should be noted.

First, $g(s)$ is well-defined, because the integral $\int_0^1 b\,dx$ cannot vanish. Also, each of $f(s)$ and $g(s)$ is periodic, with unit period, and the choice of g ensures that

$$\dot{f}(s)\int_0^1 a\,dx + \dot{g}(s)\int_0^1 b\,dx = \frac{1}{2^N}\sum_{n=0}^{N}\binom{N}{n}\cos((2n-N)2\pi s) = \cos^N(2\pi s).$$

If we set $\phi(t) = f(t/p)$ and $\sigma(t) = g(t/p)$ and use the identities $\cos(\omega t) = \mathrm{Re}[\exp(i|\omega|t)]$, $\sin(\omega t) = \mathrm{Re}[-i\,\mathrm{sgn}\,\omega\exp(i|\omega|t)]$, we see that ϕ and σ are indeed sums of the form (2.6) and (2.7), with the frequencies ω_n drawn from the spectrum (5.1). The sums are finite in this case, for $A_n = 0$ if $n > 1$, and $B_n = 0$ if $n > (N+1)/2$. Hence, ϕ and σ belong to U. In fact, ϕ belongs to $U(m^+, m^-)$ since m^+ and m^- are, respectively, the maximum and minimum values that ϕ attains.

Now let us make use of the limits (5.4) and (5.6). Because N is odd,

$$\lim_{p\to\infty}\int_0^p W(t)\,dt = \frac{1}{\theta_0}\int_0^1 (\tfrac{1}{2}(m^+ + m^-) + \tfrac{1}{2}(m^+ - m^-)\cos(2\pi s))\cos^N(2\pi s)\,ds$$

$$= \frac{(m^+ - m^-)}{2\theta_0}\int_0^1 \cos^{N+1}(2\pi s)\,ds$$

$$= \frac{2(m^+ - m^-)}{\theta_0}\int_0^{1/4} \cos^{N+1}(2\pi s)\,ds,$$

$$\lim_{p\to\infty}\int_0^p Q^+(t)\,dt = \tfrac{1}{2}\int_0^1 |\cos^N(2\pi s)|\,ds = 2\int_0^{1/4}\cos^N(2\pi s)\,ds. \tag{5.7}$$

However,

$$\int_0^{1/4}\cos^{N+1}(2\pi s)\,ds < \int_0^{1/4}\cos^N(2\pi s)\,ds < \int_0^{1/4}\cos^{N-1}(2\pi s)\,ds$$

and, by a familiar reduction formula,

$$(N+1)\int_0^{1/4}\cos^{N+1}(2\pi s)\,ds = N\int_0^{1/4}\cos^{N-1}(2\pi s)\,ds.$$

Thus,

$$\lim_{p\to\infty}\frac{\int_0^p W(t)\,dt}{\int_0^p Q^+(t)\,dt} > \frac{N}{N+1}\cdot\frac{(m^+ - m^-)}{\theta_0}$$

and by choosing $N > (2-\varepsilon)/\varepsilon$ we arrange that

$$\lim_{p\to\infty}\frac{\int_0^p W(t)\,dt}{\int_0^p Q^+(t)\,dt} > (1 - \tfrac{1}{2}\varepsilon)\frac{(m^+ - m^-)}{\theta_0}. \tag{5.8}$$

Having chosen N, (5.7) and (5.8) enable us to choose p so as to satisfy both of the inequalities

$$\int_0^p Q^+(t)\,dt > \int_0^{1/4}\cos^N(2\pi s)\,ds,$$

$$\frac{\int_0^p W(t)\,dt}{\int_0^p Q^+(t)\,dt} > (1-\varepsilon)\frac{(m^+ - m^-)}{\theta_0}.$$

These choices of N and p ensure that

$$M(Q^+) = \frac{1}{p}\int_0^p Q^+(t)\,dt > \frac{1}{p}\int_0^{1/4}\cos^N(2\pi s)\,ds > 0,$$

$$\frac{M(W)}{M(Q^+)} = \frac{\frac{1}{p}\int_0^p W(t)\,dt}{\frac{1}{p}\int_0^p Q^+(t)\,dt} = \frac{\int_0^p W(t)\,dt}{\int_0^p Q^+(t)\,dt} > (1-\varepsilon)\frac{(m^+ - m^-)}{\theta_0},$$

and so the proof of (3.2) is complete.

It should be observed that, when p is large, the net heat flux $Q(t)$, calculated from (5.5), is approximately equal to $\frac{1}{p}\cos^N(\frac{2\pi t}{p})$. On the other hand, the environmental temperature is $\theta_0 + \frac{1}{2}(m^+ + m^-) + \frac{1}{2}(m^+ - m^-)\cos(\frac{2\pi t}{p})$. Thus, if N is a large odd integer, the bulk of the graph of $Q(t)$ is concentrated into thin peaks of height $1/p$, concentrated near the values of t at which the environmental temperature attains its maximum value m^+, and thin troughs of height $-1/p$, concentrated near the values of t at which the environmental temperature attains its minimum value m^-. In consequence, absorption of heat takes place at temperatures close to the maximum, and emission of heat takes place at temperatures close to the minimum. These considerations explain why the method of proof is effective for they show that our choices of ϕ and σ cause the body to execute an approximate Carnot cycle.

6. The case of bounded spectra. In order to prove (3.3) we begin by noting that (4.1) and (4.2) imply the identity

$$\theta_0 M(W) + M\left(\int_0^1 k\psi_x^2\,dx\right) = M((\phi - M(\phi))Q). \tag{6.1}$$

It will be necessary to examine the second and third terms in some detail.

If ϕ and σ are the sums (2.6) and (2.7) then, as we have seen already,

$$M\left(\int_0^1 k\psi_x^2\,dx\right) = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 k|\Psi_n'|^2\,dx, \tag{6.2}$$

where Ψ_n is the solution of the boundary value problem (2.8), (2.9). Furthermore,

$$\phi(t) - M(\phi) = \operatorname{Re}\sum_{n=1}^{\infty} A_n \exp(i\omega_n t), \tag{6.3}$$

and the net heat flux, calculated from (1.4), proves to be the sum

$$Q(t) = \operatorname{Re}\sum_{n=1}^{\infty} Q_n \exp(i\omega_n t), \tag{6.4}$$

whose coefficients are

$$\begin{aligned} Q_n &= k(1)\Psi_n'(1) - k(0)\Psi_n'(0) \\ &= i\omega_n\left(\int_0^1 a\Psi_n\,dx + A_n\int_0^1 a\,dx + B_n\int_0^1 b\,dx\right). \end{aligned} \tag{6.5}$$

It is a consequence of (6.3) and (6.4) that

$$M((\phi - M(\phi))Q) = \frac{1}{4}\sum_{n=1}^{\infty}(A_n\overline{Q}_n + \overline{A}_n Q_n). \tag{6.6}$$

The key part of our argument will be to establish an inequality

$$\int_0^1 k|\Psi_n'|^2\,dx \geq \tfrac{1}{2}\delta(A_n\overline{Q}_n + \overline{A}_n Q_n) \quad (n = 1, 2, 3, \dots) \tag{6.7}$$

between individual terms in the sums (6.2) and (6.6), with a constant δ which is independent of n and lies in $0 < \delta \leq 1$. Whatever the value of δ, the inequality is trivially true if $A_n = 0$ and so it is enough to establish (6.7) for those n at which $A_n \neq 0$.

Three preliminary steps are needed to prove (6.7); the first relies upon the linear independence of $a(x)$ and $b(x)$.

Let $F(x)$ be any function which is real-valued and continuous on $[0,1]$ and is positive on $(0,1)$. We shall show that there is a positive constant γ_1 such that

$$\int_0^1 F(x)|a(x)A+b(x)B|^2\,dx \geq \gamma_1(|A|^2+|B|^2) \tag{6.8}$$

for all complex numbers A and B.

For, the strict inequality

$$\left(\int_0^1 Fab\,dx\right)^2 < \int_0^1 Fa^2\,dx \int_0^1 Fb^2\,dx \tag{6.9}$$

must hold if $a(x)$ and $b(x)$ are linearly independent. Moreover, the integral

$$\int_0^1 F|aA+bB|^2\,dx = \int_0^1 Fa^2\,dx|A|^2 + \int_0^1 Fab\,dx(A\overline{B}+\overline{A}B) + \int_0^1 Fb^2\,dx|B|^2$$

and is bounded from below by

$$\int_0^1 Fa^2\,dx|A|^2 - 2\int_0^1 Fab\,dx|A||B| + \int_0^1 Fb^2\,dx|B|^2,$$

the lower bound being a quadratic form in $|A|$ and $|B|$. In view of (6.9), the quadratic form is positive definite and must itself be bounded from below by $\gamma_1(|A|^2+|B|^2)$, where γ_1 is an appropriately chosen positive constant. Thus, (6.8) is correct.

The second step involves restricting $F(x)$ further and requiring it to be continuously differentiable and to satisfy $F(0)=F(1)=0$. We multiply the differential equation (2.8) through by $i(a\overline{A}_n+b\overline{B}_n)F$, thus obtaining the equation

$$\begin{aligned} i((a\overline{A}_n+b\overline{B}_n)Fk\Psi_n')' = {} & i((aF)'\overline{A}_n+(bF)'\overline{B}_n)k\Psi_n' \\ & - \omega_n F(a\overline{A}_n+b\overline{B}_n)a\Psi_n - \omega_n F|aA_n+bB_n|^2. \end{aligned}$$

When we integrate with respect to x we find that

$$\begin{aligned} \omega_n\int_0^1 F|aA_n+bB_n|^2\,dx = {} & i\int_0^1((aF)'\overline{A}_n+(bF)'\overline{B}_n)k\Psi_n'\,dx \\ & - \omega_n\int_0^1 F(a\overline{A}_n+b\overline{B}_n)a\Psi_n\,dx. \end{aligned}$$

Hence

$$\begin{aligned} \omega_n & \int_0^1 F|aA_n+bB_n|^2\,dx \\ & \leq (|A_n|^2+|B_n|^2)^{1/2}\left[\int_0^1 k(|(aF)'|^2+|(bF)'|^2)^{1/2}|\Psi_n'|\,dx \right. \\ & \qquad \left. + \omega_n\int_0^1 Fa(a^2+b^2)^{1/2}|\Psi_n|\,dx\right]. \end{aligned}$$

On estimating the right-hand side with the aid of the Schwarz inequality, noting that the definition of the eigenvalue λ implies that

$$\lambda\int_0^1 a|\Psi_n|^2\,dx \leq \int_0^1 k|\Psi_n'|^2\,dx,$$

and remembering that $\omega_1 \le \omega_n < \omega_\infty$ we arrive at the conclusion

$$\omega_1 \int_0^1 F|aA_n + bB_n|^2\,dx \le (|A_n|^2 + |B_n|^2)^{1/2}(\gamma_2 + \gamma_3\omega_\infty)\left(\int_0^1 k|\Psi_n'|^2\,dx\right)^{1/2}, \quad (6.10)$$

where γ_2 and γ_3 are the positive constants

$$\gamma_2 = \left(\int_0^1 k(|(aF)'|^2 + |(bF)'|^2)\,dx\right)^{1/2},$$

$$\gamma_3 = \frac{1}{\lambda^{1/2}}\left(\int_0^1 F^2 a(a^2 + b^2)\,dx\right)^{1/2}.$$

Since only the case in which $A_n \neq 0$ (and therefore $|A_n|^2 + |B_n|^2 > 0$) need be considered, it follows from (6.10) and (6.8) that

$$\gamma_1\omega_1(|A_n|^2 + |B_n|^2)^{1/2} \le (\gamma_2 + \gamma_3\omega_\infty)\left(\int_0^1 k|\Psi_n'|^2\,dx\right)^{1/2}. \quad (6.11)$$

This completes the second step in the proof of (6.7).

To take the third step, let $G(x)$ be any real-valued function which is continuously differentiable in [0,1] and satisfies

$$G(0) = G(1) = 1, \qquad \int_0^1 b(x)G(x)\,dx = 0.$$

On multiplying (2.8) through by G, we obtain

$$(Gk\Psi_n')' = kG'\Psi_n' + i\omega_n(aG\Psi_n + A_n aG + B_n bG),$$

and when we integrate with respect to x and appeal to the properties of G we find

$$Q_n = \int_0^1 kG'\Psi_n'\,dx + i\omega_n\int_0^1 aG\Psi_n\,dx + i\omega_n A_n\int_0^1 aG\,dx.$$

If we use this expression to calculate $A_n\overline{Q}_n + \overline{A}_n Q_n$, the third term makes no contribution and we have

$$A_n\overline{Q}_n + \overline{A}_n Q_n = A_n\int_0^1 kG'\overline{\Psi_n'}\,dx + \overline{A}_n\int_0^1 kG'\Psi_n'\,dx + i\omega_n\left(-A_n\int_0^1 aG\overline{\Psi_n}\,dx + \overline{A}_n\int_0^1 aG\Psi_n\,dx\right).$$

The right-hand side can be estimated as in the proof of (6.11) to yield the conclusion

$$|A_n\overline{Q}_n + \overline{A}_n Q_n| \le 2|A_n|(\gamma_4 + \gamma_5\omega_\infty)\left(\int_0^1 k|\Psi_n'|^2\,dx\right)^{1/2},$$

where γ_4 and γ_5 are the positive constants

$$\gamma_4 = \left(\int_0^1 kG'^2\,dx\right)^{1/2}, \qquad \gamma_5 = \frac{1}{\lambda^{1/2}}\left(\int_0^1 aG^2\,dx\right)^{1/2}.$$

In fact, the weaker, but more convenient, conclusion

$$|A_n\overline{Q}_n + \overline{A}_n Q_n| \leq 2(|A_n|^2 + |B_n|^2)^{1/2}(\gamma_4 + \gamma_5\omega_\infty)\left(\int_0^1 k|\Psi_n'|^2\,dx\right)^{1/2} \tag{6.12}$$

must hold, and this last is the third step in the proof of (6.7).

If we combine (6.11) with (6.12) we find that

$$\begin{aligned}\int_0^1 k|\Psi_n'|^2\,dx &\geq \frac{\gamma_1\omega_1(|A_n|^2+|B_n|^2)^{1/2}}{\gamma_2+\gamma_3\omega_\infty}\cdot\frac{|A_n\overline{Q}_n+\overline{A}_nQ_n|}{2(|A_n|^2+|B_n|^2)^{1/2}(\gamma_4+\gamma_5\omega_\infty)}\\ &= \frac{\gamma_1\omega_1}{2(\gamma_2+\gamma_3\omega_\infty)(\gamma_4+\gamma_5\omega_\infty)}|A_n\overline{Q}_n+\overline{A}_nQ_n|.\end{aligned}$$

Thus, if δ is the number

$$\operatorname{Min}\left(1,\ \frac{\gamma_1\omega_1}{2(\gamma_2+\gamma_3\omega_\infty)(\gamma_4+\gamma_5\omega_\infty)}\right),$$

δ lies in the interval $0<\delta\leq 1$ and

$$\int_0^1 k|\Psi_n'|^2\,dx \geq \tfrac{1}{2}\delta|A_n\overline{Q}_n+\overline{A}_nQ_n| \geq \tfrac{1}{2}\delta(A_n\overline{Q}_n+\overline{A}_nQ_n),$$

which proves (6.7).

It is an immediate consequence of (6.7) that

$$\frac{1}{2}\sum_{n=1}^{\infty}k|\Psi_n'|^2\,dx \geq \frac{1}{4}\delta\sum_{n=1}^{\infty}(A_n\overline{Q}_n+\overline{A}_nQ_n),$$

i.e.,

$$M\left(\int_0^1 k\psi_x^2\,dx\right) \geq \delta M((\phi-M(\phi))Q),$$

and this estimate and the identity (6.1) yield the inequality

$$\theta_0 M(W) \leq (1-\delta)M((\phi-M(\phi))Q)$$

in which $1-\delta\geq 0$. Finally, we argue as we did in proving (3.1) to obtain

$$M((\phi-M(\phi))Q) = M(\phi Q) \leq (m^+-m^-)M(Q^+)$$

whenever ϕ belongs to $S(\Omega, m^+, m^-)$, and so the proof of (3.3) is complete.

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