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SPHERICAL WAVES IN ODD-DIMENSIONAL SPACE*

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Abstract. The general solution is given of the $(2N+1)$-dimensional wave equation with spherical symmetry, $u_{tt} - u_{xx} - \frac{2N}{x} u_x = 0$, in terms of two arbitrary functions and their first N derivatives. Simple transformations then yield the general solutions to the Euler–Poisson–Darboux equation, $u_{xy} + \frac{N}{(x+y)}(u_x + u_y) = 0$, for integer N, and also the one-dimensional wave equation, $u_{tt} - c^2 u_{xx} = 0$, for certain variable wave speeds $c(x)$.

1. Introduction. The most well-known general solution of a linear partial differential equation in terms of arbitrary functions is the d'Alembert solution, $u = f(x+t) + g(x-t)$, of the one-dimensional wave equation $u_{tt} - u_{xx} = 0$. Synge [1] found a general solution to another class of wave equations $u_{tt} - c(x)^2 u_{xx} = 0$ (essentially when $c(x) = x^2$) which also involves arbitrary functions, namely $u = x\{f(t - x^{-1}) + g(t + x^{-1})\}$. Subsequently Seymour and Varley [2] found general solutions to a class of wave equations with wave speed varying in both space and time. Bluman [3] classified those second-order linear partial differential equations with two independent variables and variable coefficients which map into linear partial differential equations with constant coefficients and, in particular, those hyperbolic partial differential equations which map to the wave equation $u_{tt} - u_{xx} = 0$. Clearly the general solution to this class of hyperbolic equations may then be derived.

The author [4] found general solutions of the Stokes–Beltrami equations which involve not only arbitrary functions but also a finite number of the derived functions of these arbitrary functions. But the most famous of those partial differential equations whose general solution involves both an arbitrary function and its derivative is probably the nonlinear Liouville equation, $u_{xy} = e^u$, with general solution $u = \ln\left\{\frac{2f'(x)g'(y)}{(f(x)+g(y))^2}\right\}$.

Here the catalogue of such general solutions is augmented by the solution of the equation $u_{tt} - u_{xx} - \frac{2N}{x} u_x = 0$ for integer N. This is the equation for spherical waves (see e.g. Copson [5], page 90) in $2N+1$ dimensions. A simple transformation then enables the general solution to be found for the Euler–Poisson–Darboux equation, $u_{xy} + \frac{N}{(x+y)}(u_x + u_y) = 0$, for integer N. Finally, the result for spherical waves is used

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to find the general solution to the wave equation, $u_{tt} - c(x)^2 u_{xx} = 0$, for a range of functions $c(x)$ which include Synge's solution mentioned above.

2. Solutions. The solutions to the three partial differential equations referred to above are conveniently expressed as three theorems.

THEOREM I. The general solution of

$$u_{tt} - u_{xx} + \frac{2n}{x} u_x = 0, \tag{1}$$

where n is an integer, is

$$u = \sum_{i=0}^{n} (-1)^i \frac{(2n-i)!}{i!(n-i)!} 2^i x^i \{F^{(i)}(x+t) + G^{(i)}(x-t)\}, \quad n \geq 0, \tag{2}$$

$$u = \sum_{i=0}^{m-1} (-1)^i \frac{(2m-2-i)!}{i!(m-1-i)!} 2^i x^{i-2m+1} \{F^{(i)}(x+t) + G^{(i)}(x-t)\}, \quad m = -n, \ n < 0, \tag{3}$$

where F and G are arbitrary functions and $F^{(i)}$ and $G^{(i)}$ are the ith derived functions of F and G ($F^{(0)} = F$, $G^{(0)} = G$).

Notice that setting $u = x^{2n+1} u'$ in equation (1) gives

$$u'_{tt} - u'_{xx} - \frac{2(n+1)}{x} u'_x = 0, \tag{4}$$

so that, once (2) is established, (3) may be obtained by replacing n by $m-1$ in the right-hand side of (2) and multiplying that side by x^{-2m+1}.

The result (2) is proved by induction on n. For $n = 0$, (1) reduces to the one-dimensional wave equation with unit wave propagation speed and (2) reduces to $u = F(x+t) + G(x-t)$, the d'Alembert solution. Suppose (2) is indeed the general solution of (1) for a given n and consider the equation for v, say, corresponding to $n+1$:

$$v_{tt} - v_{xx} + \frac{2(n+1)}{x} v_x = 0. \tag{5}$$

Define w so that

$$x w_t = v_x, \tag{6}$$

$$x w_x - (2n+1) w = v_t. \tag{7}$$

This is possible because the integrability condition on w is precisely equation (5). Eliminating v now gives

$$w_{tt} - w_{xx} + \frac{2n}{x} w_x = 0, \tag{8}$$

so that, from the induction hypothesis, w can be written as

$$w = \sum_{i=0}^{n} (-1)^i \frac{(2n-i)!}{i!(n-i)!} 2^i x^i \{-2f^{(i+1)}(x+t) + 2g^{(i+1)}(x-t)\}, \tag{9}$$

where f and g are arbitrary functions. v may now be found from w by integrating equations (6) and (7) and is given by

$$v = \sum_{i=0}^{n+1}(-1)^i \frac{(2n+2-i)!}{i!(n+1-i)!} 2^i x^i \{f^{(i)}(x+t) + g^{(i)}(x-t)\}. \tag{10}$$

The arbitrary constant introduced in the integration of (6) and (7) may be absorbed into either $f(x+t)$ or $g(x-t)$. Equation (10) is precisely the same form as (2) with n replaced by $n+1$ and the result is proved.

THEOREM II. The general solution of the Euler–Poisson–Darboux equation,

$$u_{xy} - \frac{n}{(x+y)}(u_x + u_y) = 0, \tag{11}$$

where n is an integer, is

$$u = \sum_{i=0}^{n}(-1)^i \frac{(2n-i)!}{i!(n-i)!}(x+y)^i \{F^{(i)}(x) + G^{(i)}(y)\}, \quad n \geq 0, \tag{12}$$

$$u = \sum_{i=0}^{m-1}(-1)^i \frac{(2m-2-i)!}{i!(m-1-i)!} 2^{2m-1}(x+y)^{i-2m+1}\{F^{(i)}(x) + G^{(i)}(y)\}, \quad m = -n, \ n < 0, \tag{13}$$

where, again, F and G are arbitrary functions.

The Euler–Poisson–Darboux equation (11) may be obtained from (1) by making the transformation $x' = x + t$, $y' = x - t$ and then dropping the primes. Solutions (12) and (13) then follow immediately from (2) and (3).

THEOREM III. The general solution of

$$u_{tt} - x^{4n/(2n+1)} u_{xx} = 0, \tag{14}$$

where n is an integer, is

$$u = \sum_{i=0}^{n}(-1)^i \frac{(2n-i)!}{i!(n-i)!} 2^i (2n+1)^i x^{i/(2n+1)} \times \{F^{(i)}((2n+1)x^{1/(2n+1)} + t) + G^{(i)}((2n+1)x^{1/(2n+1)} - t)\}, \quad n \geq 0, \tag{15}$$

$$u = \sum_{i=0}^{m-1}(-1)^i \frac{(2m-2-i)!}{i!(m-1-i)!} 2^i (-2m+1)^{i-2m+1} x^{(i-2m+1)/(-2m+1)} \times \{F^{(i)}((-2m+1)x^{1/(-2m+1)} + t) + G^{(i)}((-2m+1)x^{1/(-2m+1)} - t)\}, \quad m = -n, \ n < 0, \tag{16}$$

where F and G are arbitrary functions. If x' is defined by $x' = (2n+1)x^{1/(2n+1)}$, (14) becomes

$$u_{tt} - u_{x'x'} + \frac{2n}{x'} u_{x'} = 0,$$

so that results (2) and (3) with x replaced by $(2n+1)x^{1/(2n+1)}$ give (15) and (16) as required.

When $n = 0$, (14) reduces to $u_{tt} - u_{xx} = 0$ and (15) gives the general solution $u = F(x+t) + G(x-t)$, the d'Alembert solution.

When $n = -1$, Theorem III gives the general solution of

$$u_{tt} - c^2 u_{xx} = 0, \qquad c(x) = x^2,$$

to be

$$u = -x\{F(-x^{-1} + t) + G(-x^{-1} - t)\}.$$

This is the general solution given by Synge [1] in his "interesting case".

Cases $n = 1$ and $n = -2$ give general solutions involving first derivatives of arbitrary functions. For example, when $n = -2$, Theorem III shows that the general solution of

$$u_{tt} - c^2 u_{xx} = 0, \qquad c(x) = x^{4/3},$$

is

$$u = x\{f(t - 3x^{-1/3}) + g(t + 3x^{-1/3})\} + 3x^{2/3}\{f'(t - 3x^{-1/3}) + g'(t + 3x^{-1/3})\},$$

where f and g are arbitrary functions.

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