QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVI, NUMBER 4
DECEMBER 1988, PAGES 779-788

ON A NEW EXTENSION OF LIAPUNOV'S DIRECT METHOD TO DISCRETE EQUATIONS*

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Abstract. In this paper, a new procedure is given for applying Liapunov's direct method to autonomous discrete equations. This procedure is based on an idea that is closely related to Razumikhin's principle and it includes Liapunov's direct method as a special case. Examples are given.

I. Introduction. The direct (or second) method of Liapunov [1, 3, 7] is well established for ordinary differential equations. As of the second half of the century, it had been extended to functional differential equations and difference equations. The fundamental principles and results also have been established for general dynamical systems and processes evolving in abstract spaces [1,4,6]. In each of the above fields, the method presents one or more special features which cannot be entirely adapted to the other fields. One of these features is Razumikhin's idea [4,8], which is valid for retarded differential equations but cannot be used in ordinary differential equations or difference equations.

This paper is devoted to the presentation of a method of analysis of discrete equations that resembles the aforementioned Razumikhin's principle of retarded differential equations. It is shown that this idea leads to a substantial improvement of Liapunov's direct method.

In order to be more specific and to facilitate the comprehension of our results, we give below a brief account of the main principles of Liapunov's direct method to discrete equations as well as an abridged version of Razumikhin's principle.

Let $f\colon \mathbf{R}^m \to \mathbf{R}^m$ be a continuous map and consider the discrete equation

$$x_n = f(x_{n-1}), \qquad x_0 = 0, \quad n = 1, 2, \ldots, \tag{1.1}$$

where x, the initial condition, is a parameter of the problem. We denote the solution of (1.1), i.e., the sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies (1.1), by "$x_n(x)$". This solution exists, is unique and depends continuously on x due to the simple fact that f is a (well-defined) map [5,6].

*Received October 21, 1987.

A continuous map $V\colon \mathbf{R}^m \to \mathbf{R}$ is said to be positive-definite (positive-semidefinite) in a domain U containing the origin, when $V(0) = 0$, $V(x) > 0$ for $x \in U$, $x \neq 0$ ($V(x) \geq 0$). By $|x|$ we will denote the Euclidean norm of x. Suppose that $f(x) = x$. Then $x_n(x) = x$ is a solution of (1.1) called an "equilibrium". If $f(0) = 0$ the equilibrium $x_n(0)$ is called the "trivial equilibrium" and we assume from now on that $f(0) = 0$.

DEFINITION 1.1. The equilibrium $x_n(0)$ of Eq. (1.1) is said to be *stable* if, given $\varepsilon > 0$, one can find $\delta > 0$ such that $|x| < \delta$ implies that $|x_n(x)| < \varepsilon$ for $n > 0$.

DEFINITION 1.2. The trivial equilibrium of Eq. (1.1) is *asymptotically stable* if it is stable and, moreover, one can find $\gamma > 0$ such that $|x| < \gamma$ implies that $x_n(x) \to 0$ as $n \to \infty$.

DEFINITION 1.3. Given a function $V\colon \mathbf{R}^m \to \mathbf{R}$, let ΔV be the map defined by $\Delta V(x) = V(f(x)) - V(x)$. We call ΔV the *variation of V with respect to (or, along)* (1.1).

THEOREM 1.1. Suppose that there exists a positive-definite function V such that $-\Delta V$ is a positive-semidefinite (positive-definite) map in a neighborhood of the origin. Then, the trivial equilibrium of Eq. (1.1) is stable (asymptotically stable).

Usually, any function $V\colon \mathbf{R}^m \to \mathbf{R}$ such that $\Delta V(x) \leq 0$ in a set U is called a *Liapunov function on U*.

THEOREM 1.2 [6]. Suppose f is an $m \times m$ real matrix A. There exists a positive-definite Liapunov function which proves the (asymptotic) stability of the trivial equilibrium of Eq. (1.1) if, and only if, ($r(A) < 1$) $r(A) \leq 1$, where $r(A)$ denotes the spectral radius of the matrix A.

Let M be a subset of $\mathbf{R}^m$ and denote by $\overline{M}$ the closure of M. The distance from x to M is $d(x, M) = \min\{|x - y|\colon y \in \overline{M}\}$ and "$x_n(x) \to M$" means that $d(x_n(x), M) \to 0$ as $n \to \infty$. For a given set U we put

$$E = \{x \in \overline{U}\colon \Delta V(x) = 0\}$$

and let M denote the largest invariant set in E, where, an invariant set S is a set such that $f(S) = S$. Then, we have:

THEOREM 1.3. Let V be a Liapunov function on U. Then every solution $x_n(x)$ of (1.1) which remains in U and is bounded is such that $x_n(x) \to M$.

This theorem is the essence of the so-called "Invariance Principle" [3, 6]. It is an extension of Liapunov's direct method in the sense that it does not require the positiveness of the function V in order to obtain important asymptotic information about the solutions of (1.1). It can be used to study either the stability or instability of sets of solutions of that equation. We say, for instance, that the equilibrium $x_n(0)$ of (1.1) is unstable when it is not stable. Thus, for example, if one proves that $M = \{0\}$ then Theorem 1.3 is indeed stating that the trivial equilibrium is asymptotically stable provided V is positive-definite near zero. If, on the other hand, $M = \varnothing$, then the solutions either leave U or are unbounded.

In the case of ordinary differential equations,

$$x'(t) = g(x(t)), \qquad x(0) = x_0, \tag{1.2}$$

where $g\colon \mathbf{R}^m \to \mathbf{R}^m$, $g(0) = 0$, and g is smooth enough to ensure existence in $[0, \infty)$ and uniqueness and continuity with respect to the initial condition of the solution $x(t, x_0)$ of (1.2), the following theorem due to Cetaev [1,3] holds.

THEOREM 1.4. Suppose U is an open set such that $0 \in \overline{U}$ and that V and $\dot{V}$ are positive-definite in $U \cap \Omega$, where Ω is a neighborhood of the origin. Suppose, moreover, that $V(x) = 0$ for x in the boundary of U with respect to Ω. It follows that the equilibrium $x(t, 0)$ is unstable.

Here,

$$\dot{V}(x_0) = \overline{\lim_{h \to 0^+}} [V(x(h, x_0) - V(x_0)]/h. \tag{1.3}$$

Now, given $\tau > 0$, we denote by C the space of continuous maps $\psi\colon [-\tau, 0] \to \mathbf{R}^m$ with $|\psi| = \sup\{|\psi(\theta)|\colon \theta \in [-\tau, 0]\}$. Given $x\colon \mathbf{R} \to \mathbf{R}^m$ we put $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Hence, if x is continuous then $x_t \in C$ for each $t \in \mathbf{R}$. With this notation, let $F\colon C \to \mathbf{R}^m$, $F(0) = 0$, be sufficiently smooth to guarantee the existence in $[0, \infty)$, uniqueness and continuity with respect to ψ of the solution $x(t, x_0)$ of the retarded functional differential equation

$$x'(t) = F(x_t), \qquad x_0 = \psi. \tag{1.4}$$

The natural extension of Liapunov's direct method to these equations requires the introduction of Liapunov functionals instead of Liapunov functions [4]. Razumikhin [4, 8] devised a means of using functions and not functionals in the analysis of the asymptotic behavior of the solutions of (1.4). It is as follows. Let $V\colon \mathbf{R}^m \to \mathbf{R}$ be a continuous function and put

$$\dot{V}(\psi(0)) = \overline{\lim_{h \to 0^+}} [V(\dot{x}(h, \psi) - V(\psi(0))]/h, \tag{1.5}$$

where $x(t, \psi)$ is the solution of (1.4) through $x_0 = \psi$.

THEOREM 1.5. If there exists a positive-definite function V in a neighborhood of the origin which satisfies the condition that $\dot{V}(\psi(0)) \leq 0$ whenever $V(\psi(\theta)) \leq V(\psi(0))$ for $\theta \in [-\tau, 0]$, then the equilibrium $x(t, 0)$ of (1.4) is stable. If, in addition to that, $\dot{V}(x) \neq 0$ for $x \neq 0$ in the above neighborhood, then the equilibrium $x(t, 0)$ is asymptotically stable.

Thus, the idea of Razumikhin is that one should request that $V(x_t(0))$ be forced to decrease solely along the solutions $x(t)$ of (1.4) which tend to leave a given neighborhood of the origin if this solution had been in this same neighborhood for a time length equal to τ.

II. The new direct method.

DEFINITION 2.1. Given a function $V\colon \mathbf{R}^m \to \mathbf{R}$ and integers p and q, with $p > q \geq 0$, we define the (p, q)-variation of V with respect to (1.1) as being the map $\Delta_q^p V\colon \mathbf{R}^m \to \mathbf{R}$ such that

$$\Delta_q^p V(x) = V(x_p(x)) - V(x_q(x)). \tag{2.1}$$

We shall, nevertheless, sometimes use the notation "$\Delta_k V(x_n(x))$" to stand for what should be "$\Delta_{p-k}^p V(x_{n-p}(x))$". Thus, $\Delta_1 V(x_n(x)) = V(x_n(x)) - V(x_{n-1}(x))$, $\Delta_2 V(x_n(x)) = V(x_n(x)) - V(x_{n-2}(x))$, etc. These expressions are valid for $n > p \geq k \geq 1$. We have:

THEOREM 2.1. Suppose that there exist a positive-definite function V and constants $d > 0$ and $k > 1$, k an integer, such that whenever $|x| < d$ and $\Delta_{k-1}^{k} V(x) \geq 0$ one has $\Delta_0^k V(x) \leq 0$. It then follows that the trivial equilibrium of (1.1) is stable.

Proof. Given $\varepsilon > 0$ we choose $\delta, \zeta < \varepsilon$ such that $0 < \delta < \zeta < d$ and such that $|x| < \zeta$ implies that $|f(x)| < d$ and $|x| < \delta$ implies that $V(x_n(x)) < A$, $n = 0, 1, 2, \ldots, k$, where A is chosen so that $V(x) < A$ implies that $|x| < \zeta$. This procedure is always possible due to the continuity of f and V. We now claim that $V(x_n(x)) < A$ for $n \geq 0$ if $|x| < \delta$. This will readily imply the result of the theorem. Suppose then, on the contrary, that we could find an x such that $|x| < \delta$ and $V(x_n(x)) > A$ for some $n \geq k$. Then, letting n^* stand for the first index where this occurs, we would have $\Delta_1 V(x_{n^*}(x)) > 0$ with $|x_{n^*-k}(x)| < d$. So, the hypothesis of the theorem implies that $\Delta_k V(x_{n^*}(x)) \leq 0$, i.e., $V(x_{n^*}(x)) \leq V(x_{n^*-k}(x)) < A$, a contradiction. This finishes the proof of the theorem.

THEOREM 2.2. If, in addition to the hypotheses of the theorem above, the function V is such that $0 < |x| < d$ and $\Delta_{k-1}^{k} V(x) \geq 0$ imply that $\Delta_0^k V(x) < 0$, then the equilibrium $x_n(0)$ of Eq. (1.1) is asymptotically stable.

Proof. By Theorem 2.1 we already know that $x_n(0)$ is stable and we let δ, ζ, and A be as in the proof of that theorem. We pick $\gamma < \delta$ and show that if $|x| < \gamma$ then $V(x_n(x)) \to 0$ as $n \to \infty$. This implies that $x_n(x) \to 0$ as $n \to \infty$, the desired result. For each initial condition x, let the sequence $\{c_j\}_{j=0}^{\infty}$, $c_j = c_j(x)$, be given by

$$c_j = \max\{V(x_n(x)) \colon jk < n \leq (j+1)k\}, \qquad j = 0, 1, 2, \ldots.$$

Thus, in order to show that $V(x_n(x)) \to 0$ as $n \to \infty$, it suffices to show that the sequence $\{c_j\}$ is such that $c_j \to 0$ as $j \to \infty$. If $|x| < \gamma$, we claim that the sequence $\{c_j\}$ is a strictly decreasing sequence. Indeed, let $c_{j+1} = V(x_{n'}(x))$. If $n' = (j+1)k+1$ and $\Delta_1 V(x_{n'}(x)) < 0$, we obtain

$$c_{j+1} < V(x_{n'-1}(x)) \leq c_j,$$

as desired. On the other hand, if $n' = (j+1)k + 1$ and we have $\Delta_1 V(x_{n'}(x)) \geq 0$, then, according to the hypothesis of the statement, we must have $\Delta_k V(x_{n'}(x)) < 0$ which implies that

$$c_{j+1} = V(x_{n'}(x)) < V(x_{n'-k}(x)) \leq c_j,$$

as it should be. Finally, if $n' > (j+1)k + 1$, then we must have $\Delta_1 V(x_{n'}(x)) \geq 0$ and, again, the hypothesis of the theorem implies that $\Delta_k V(x_{n'}(x)) < 0$ and the result follows just as in the previous case. Thus, the only way c_j can now fail to converge to zero is the existence of a positive constant $c < c_j$ such that $c_j \to c$ as $j \to \infty$. Since $c < d$ and the only point x with $|x| < d$ where $\Delta_{k-1}^{k} V$ and $\Delta_0^k V$ vanish simultaneously is $x = 0$, the standing hypotheses this time guarantee the existence of a positive constant $\rho(c)$ such that either $\Delta_{k-1}^{k} V(x) < -\rho(c)$ or $\Delta_0^k V(x) < -\rho(c)$ for all x, $c < |x| < d$. As a consequence, we must have

$$c_{j+1} - c_j < -\rho(c), \qquad j = 0, 1, 2, \ldots,$$

which leads us to conclude that $c_j \to -\infty$, a contradiction. So, $c = 0$ and the theorem is proved.

The above two theorems show that it is possible to prove Liapunov-like theorems for discrete equations by using more than one variation of a positive-definite function V, even though this function is not a Liapunov function. Even more, as is shown in the next theorem, this method includes Liapunov's direct method to these equations.

DEFINITION 2.2. We shall say that a function $V: \mathbf{R}^m \to \mathbf{R}$ is a dichotomic map (with respect to Eq. (1.1)) if V is continuous and there are a positive constant d and an integer $k \geq 2$ such that whenever $\Delta_{k-1}^k V(x) \geq 0$ and $|x| < d$ it follows that $\Delta_0^k V(x) \leq 0$. A strict dichotomic map is a dichotomic map for which $\Delta_{k-1}^k V(x) \geq 0$, $0 < |x| < d$, implies that $\Delta_0^k V(x) < 0$.

THEOREM 2.3. If there exists a positive-definite Liapunov function that proves the (asymptotic) stability of the trivial equilibrium of Eq. (1.1), then there exists a positive-definite dichotomic map that also proves this (asymptotic) stability.

Proof. The proof is transparent. One only needs to observe that the existence of a Liapunov function V for Eq. (1.1) signifies that $\Delta_0^1 V(x) = \Delta V(x) \leq 0$ for all points x in an appropriate neighborhood of the origin. Thus, the set of points x in this neighborhood where $\Delta_0^1 V(x) \geq 0$ is precisely the set of those points where $\Delta_0^1 V(x) = 0$. As a consequence of this and of the continuity of V and f, we may shrink the neighborhood, if necessary, so that if x is in it, then we also have $\Delta_0^1 V(x_1(x)) \leq 0$. Hence, for all points x in this (perhaps smaller) neighborhood we will have $\Delta_0^2 V(x) = \Delta_0^1 V(x) + \Delta_0^1 V(x_1(x)) \leq 0$, which proves that V is also a dichotomic map for (1.1) with $k = 2$. The result concerning stability then follows from Theorem 2.1. In the case of asymptotic stability, we have that $x = 0$ is the only point where $\Delta_0^1 V(x) \geq 0$. Hence, the given V is a strict dichotomic map for (1.1) already for $k = 2$ as above, and the result follows from Theorem 2.2.

The argument of the above proof is enough to prove that the invariance principle of Theorem 1.3 is also an invariance principle for dichotomic maps. We shall now, nevertheless, present a special version of the Invariance Principle for positive-semidefinite dichotomic maps that is important in the applications.

THEOREM 2.4. Suppose that V is a positive-semidefinite strictly dichotomic map in a neighborhood U of the origin. Consider the set $E = \{x \in U: V(x) = 0\}$. Then, every bounded solution $x_n(x)$ of Eq. (1.1) which remains in U is such that $x_n(x) \to M$, where M is the largest invariant set of (1.1) in E.

Proof. Let the sequence $\{c_j\}$, $c_j = c_j(x)$ be defined just as in the proof of Theorem 2.2. In spite of V being just positive-semidefinite, we can use the same kind of argument as in that proof in order to show that $c_j \to 0$. Hence, $V(x_n(x)) \to 0$. On the other hand, $x_n(x) \to \Omega(x)$, where $\Omega(x)$ denotes the positive limit-set of x [6], which is an invariant set of (1.1). The continuity of V then implies the result.

We shall give examples of applications of this version of the Invariance Principle in Section 3.

Now, we are in position to make a few pertinent remarks on the results thus far obtained. First, one clearly notes that, as said in the introduction of this paper, it is impossible to literally transpose Razumikhin's principle to discrete equations on the basis of the traditional extension of Liapunov's direct method to these equations.

In fact, this transposition would, for instance, in the case of asymptotic stability, request that $\Delta V(x) < 0$ whenever $\Delta V(x) \geq 0$, an evidently impossible assumption. Secondly, it is important to mention that Hale [4, p. 132] observes that it is possible to obtain sharper results on the stability of Eq. (1.4) if one retards Razumikhin's condition that $\dot{V}(\psi(0)) \leq 0$ whenever $V(\psi(\theta)) \leq V(\psi(0))$, $\theta \in [-\tau, 0]$, so that it is applied after the equation has been integrated for k delay intervals. In other words, one should request that

$$\tilde{V}(\psi) = \sup\{V(x(\theta, \psi)) \colon \theta \in [-\tau, (k-1)\tau]\}$$

be decreasing along the solutions of Eq. (1.4). The above functional $\tilde{V}$ resembles our sequence $\{c_j(x)\}$ of Theorem 2.2. The unfavorable side of the idea in the present case of functional differential equations is that one is, except for special cases, obliged to integrate the equation for a few steps in order to obtain the stability conditions. This also occurs with our method, but (at least theoretically) the integration (iteration) of discrete equations is relatively easy and appropriate for automatic computation. The third comment we want to make is that Cetaev's theorem (Theorem 1.4) also cannot be adapted to discrete equations on the basis of the traditional extension of Liapunov's direct method to these equations for the simple fact that we have shown that it is possible to have even asymptotic stability of the trivial equilibrium of (1.1) even though we have a positive-definite function V with $\Delta V(x) = \Delta_0^1 V(x)$ positive-definite in an open set U such that $0 \in \overline{U}$. But, we can also prove instability results for (1.1) in the spirit of the present method. For instance, we have:

THEOREM 2.5. Suppose there are a positive-definite function V and an integer $k > 1$ such that if $x \neq 0$ is in a neighborhood of the origin and $\Delta_0^k V(x) \leq 0$, then also $\Delta_0^{k+1}(x) > 0$. Under these assumptions, every solution of (1.1) with initial condition in this neighborhood of the origin will either leave it or will approach its boundary as $n \to \infty$.

Proof. For each x in the neighborhood, let $c_j(x) = c_j$ be as in Theorem 2.2. It is not difficult to see that $\{c_j\}$ is a strictly increasing sequence if $x \neq 0$. Hence, the result follows.

Another pertinent remark is that the above results are of local nature [4]. The global results are easily obtainable and, indeed, most examples in the next section refer to global instances. Besides that, it seems clear that, not to mention the possibility of generalizing the procedure of comparing not only $\Delta_{k-1}^k V$ and $\Delta^k V$, but several $\Delta_j^k V$'s, there are the possibilities of extending it to general dynamical systems and processes. The method is clearly related to the possibility of introducing a Minkowski type metric in the phase space. But, this is beyond the scope of this paper.

III. Examples. We give in this section a few examples which cover all the situations described in the previous section. They range from linear to nonlinear equations which, for ease of computation, are all in $\mathbf{R}^2$ and f is of the form

$$f(x, y) = (g(x, y), x), \tag{3.1}$$

where $g \colon \mathbf{R}^2 \to \mathbf{R}$ is continuous.

(1) Consider the equation

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \qquad x_0 = x,\ y_0 = y,\ n = 1, 2, \dots, \tag{3.2}$$

where a and b are given constants. It is known (or one easily checks directly) that at $a = 1$ and $b = -1$, the trivial equilibrium of (3.2) is stable (but not asymptotically) [2] and that $V(x, y) = x^2 + y^2$ is not a Liapunov function for (3.2). This last assertion is easy to check since $\Delta V(x, y) = (x - y)^2 - y^2$, which does not have a definite sign in any neighborhood of the origin (we are picking $a = 1$ and $b = -1$). We show that, nevertheless, V is a (positive-definite) dichotomic map for (3.2), which, therefore, proves the stability of the trivial equilibrium. In fact, we have:

$$\begin{aligned} \Delta_2^3 V(x, y) &= x^2 + y^2 - (x - y)^2 - y^2 = x^2 - (x - y)^2 \\ \Delta_1^3 V(x, y) &= x^2 + y^2 - (x - y)^2 - x^2 = y^2 - (x - y)^2 \\ \Delta_0^3 V(x, y) &\equiv 0. \end{aligned}$$

Since the positive-definiteness of V is immediate and the region where $\Delta_1^3 V \le 0$ does not cover the region where $\Delta_2^3 V \ge 0$ but the region where $\Delta_0^3 V \le 0$ (the whole plane) evidently covers this region, we see that the hypothesis of Theorem 2.1 is satisfied. Hence, the trivial equilibrium of Eq. (3.2) is stable, as we had to prove. The reader can now verify by himself that $|a| + |b| < 1$ implies that the conditions of Theorem 2.2 are satisfied by this same function V, proving thus the asymptotic stability of the said equilibrium [2]. The above example is one where we have global stability and asymptotic stability, respectively.

(2) Let

$$\begin{cases} x_n = \sqrt{\mu |x_{n-1}|\,|y_{n-1}|}, & x_0 = x, \\ y_n = x_{n-1}, \quad y_0 = y, & n = 1, 2, \dots, \end{cases} \tag{3.3}$$

where $\mu > 0$ is a given parameter and, again, take the positive-definite function $V(x, y) = x^2 + y^2$. Then, letting (to simplify the notation) $x_n = x_n(x)$, we have:

$$\begin{aligned} \Delta_1^2 V(x, y) &= x_2^2 + y_2^2 - x_1^2 - y_1^2 = |x|(\mu\sqrt{\mu|xy|} - |x|), \\ \Delta_0^2 V(x, y) &= x_2^2 + y_2^2 - x^2 - y^2 = \mu|x_1|\,|y_1| + x_1^2 - x^2 - y^2 \\ &= \sqrt{\mu|xy|}(\mu|x| + \sqrt{\mu|xy|}) - x^2 - y^2. \end{aligned}$$

Note that $\Delta_1^2 V(x, y) \ge 0$ if, and only if , $|x| \le \mu^3 |y|$.

Suppose that $\Delta_1^2 V(x, y) \ge 0$ and substitute $|x|$ by $\mu^3|y|$ in the first term of $\Delta_0^2 V(x, y)$ in order to obtain

$$\Delta_0^2 V(x, y) \le (\mu^6 + \mu^4 - 1)y^2 - x^2.$$

Hence, we have $\Delta_0^2 V(x, y) \le 0$ if, and only if, either $\mu^6 + \mu^4 \le 1$ or $\mu^6 + \mu^4 \ge 1$ and $|x| \ge |y|\sqrt{\mu^6 + \mu^4 - 1}$. The analysis of these inequalities show that we can appropriately choose μ so that either Theorem 2.1 or Theorem 2.2 will hold. Moreover, whatever be this choice, V will not be a Liapunov function for (3.3).

(3) Consider the equation

$$\begin{cases} x_n = \dfrac{a x_{n-1}^2 y_{n-1}}{x_{n-1}^2 + b y_{n-1}^2}, & x_0 = x, \\ y_n = x_{n-1}, \quad y_0 = y, & n = 1, 2, \dots, \end{cases} \tag{3.4}$$

where a and b are given positive constants.

Let $V(x, y) = x^2$, a positive-semidefinite function. We have:

$$\Delta_1^2 V(x, y) = \frac{(ax_1^2 y_1)^2}{(x_1^2 + by_1^2)^2} - x_1^2 = W_1(x_1, y_1)$$

and

$$\Delta_0^2 V(x, y) = \frac{(ax_1^2 y_1)^2}{(x_1^2 + by_1^2)^2} - y_1^2 = W_0(x_1, y_1),$$

where, as before, we are putting $x_1(x) = x_1$, $y_1(x) = y_1$.

Note that $\Delta_1^2 V(x, y) \geq 0$ if, and only if, $W_1(x_1, y_1) \geq 0$ and $\Delta_0^2 V(x, y) < 0$ if, and only if, $W_0(x_1, y_1) < 0$. Taking advantage of this fact, we see that the region in the (x_1, y_1)-plane where $W_1 \geq 0$ is the shaded area of Fig. 1, where the lines R_1, R_2, R_3, and R_4 are given by the equations

$$\begin{aligned} R_1&: \quad 2x_1 = (a + \sqrt{a^2 - 4b})y_1, \\ R_2&: \quad 2x_1 = (a - \sqrt{a^2 - 4b})y_1, \\ R_3&: \quad 2x_1 = -(a + \sqrt{a^2 - 4b})y_1, \\ R_4&: \quad 2x_1 = -(a - \sqrt{a^2 - 4b})y_1. \end{aligned}$$

In particular, we see that we must have $a^2 > 4b$. On the other hand, the region where $W_0 < 0$ is the interior of the shaded area of Fig. 2. The bordering lines of this region are given by (we must have $a > 1$)

$$\begin{aligned} L_1&: \quad x_1 = \sqrt{b/(a-1)}\,y_1, \\ L_2&: \quad x_1 = -\sqrt{b/(a-1)}\,y_1. \end{aligned}$$

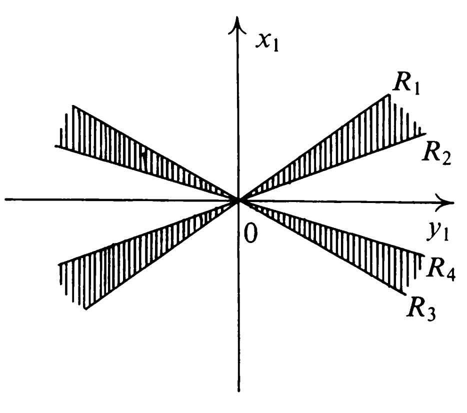

FIG. 1.

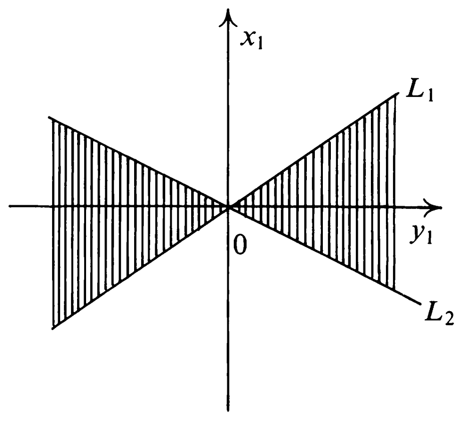

FIG. 2.

Thus, we will have $\Delta_0^2 V(x, y) < 0$ whenever $\Delta_1^2 V(x, y) \geq 0$ if, and only if, we can choose a and b such that the slope of L_1 is bigger than the slope of R_1 and the slope of L_2 is smaller than the slope of R_3, that is, if and only if $a > 1$, $a^2 > 4b$, $b > 0$, and

$$4b/(a-1) > \left(a + \sqrt{a^2 - 4b}\right)^{-2}.$$

This condition is equivalent to $b > a - 1$. Hence, the region in the plane (a, b) where we do have $\Delta_0^2 V(x, y) < 0$ whenever $\Delta_1^2 V(x, y) \geq 0$ is given by

$$\{(a, b)\colon 0 < b < (a/2)^2,\ 1 < a < 2,\ b > a - 1\},$$

which is shown in Fig. 3.

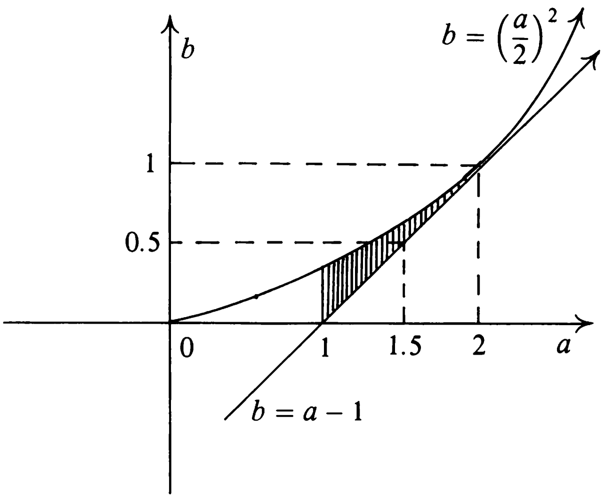

FIG. 3.

We are now in position to apply the invariance principle of Theorem 2.4. The only invariant set of (3.4) in the set $E = \{(0, y)\colon y \in R\} = V^{-1}(0)$ is $(0, 0)$ itself when (a, b) is inside the above described region. Thus, the trivial equilibrium of (3.4) is asymptotically stable in this case.

(4) Consider the equation

$$\begin{cases} x_n = y_{n-1}\sqrt{|x_{n-1}|}, & x_0 = x, \\ y_n = x_{n-1}, & y_0 = y, \quad n = 1, 2, \ldots, \end{cases} \tag{3.6}$$

and take $V(x, y) = x^2$. We have:

$$\begin{aligned} \Delta_1^2 V(x, y) &= x_2^2 - x_1^2 = x^2|y\sqrt{|x|}| - y^2|x| \\ &= |x|\,|y|(|x|\sqrt{|x|} - |y|), \\ \Delta_0^2 V(x, y) &= x_2^2 - x^2 = x^2|y\sqrt{|x|}| - x^2 \\ &= x^2(|y|\sqrt{|x|} - 1). \end{aligned}$$

Hence, $\Delta_1^2 V(x, y) = 0$ if and only if $x = 0$, $y = 0$, or $y = \pm|x|\sqrt{|x|}$, and $\Delta_1^2 V(x, y) > 0$ if and only if $|y| < |x|\sqrt{|x|}$. On the other hand, $\Delta_0^2 V(x, y) = 0$ if and only if $x = 0$ or $y = \pm\sqrt{|x|}/|x|$, and $\Delta_0^2 V(x, y) < 0$ if and only if $|y| < \sqrt{|x|}/|x|$. Let U be an open ball with center in the origin and radius less than $\sqrt{2}$. Then, within U we have that $\Delta_0^2 V(x, y) < 0$ whenever $\Delta_1^2 V(x, y) \geq 0$. Hence, we are again in position to apply the invariant principle of Theorem 2.4. It is clear that the only invariant set of $V^{-1}(0) \cap U$ is the origin itself. Hence, the trivial equilibrium of (3.6) is asymptotically stable.

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