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REMARK ON EXISTENCE AND UNIQUENESS FOR THE THERMISTOR PROBLEM UNDER MIXED BOUNDARY CONDITIONS*

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Abstract. The steady-state electrical heating of a solid conductor is studied with mixed boundary conditions. A theorem of existence, nonexistence, and uniqueness of solutions is given under general assumptions on the electrical and thermal conductivities. The basic tool of the proof is a transformation first proposed in [3] by H. Diesselhorst.

1. Introduction. Let us consider the solid conductor B represented by the open, bounded, and connected subset Ω of $\mathbf{R}^3$ with a C^2-boundary $\partial\Omega$. Let u be the temperature and ϕ the potential in B. Under steady conditions u and ϕ obey the equations

$$\nabla \cdot (\sigma(u)\nabla\phi) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$-\nabla \cdot (\kappa(u)\nabla u) = \sigma(u)|\nabla\phi|^2 \quad \text{in } \Omega. \tag{1.2}$$

The only thermoelectric effect considered in the energy equation (1.2) is the Joule effect. Thermal and electrical conductivities are denoted $\kappa(u)$, $\sigma(u)$, respectively, and are given functions of the temperature. Suppose the boundary $\partial\Omega$ is divided into three parts $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_3$ such that $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, $\partial\mathring{\Omega}_i \cap \partial\mathring{\Omega}_j = \emptyset$, $i, j = 1, 2, 3$, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$.[1] To describe a reasonably realistic situation we couple Eqs. (1.1) and (1.2) with the following boundary conditions:

$$u = u_0, \quad \phi = \phi_1 \quad \text{on } \partial\Omega_1, \qquad u = u_0, \quad \phi = \phi_2 \quad \text{on } \partial\Omega_2, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega_3. \tag{1.4}$$

Conditions (1.4) mean that the body is insulated on $\partial\Omega_3$ both electrically and thermally. $\partial\Omega_1$ and $\partial\Omega_2$ represent, on the other hand, the upper and lower electrodes to which the difference of potential $\phi_2 - \phi_1$ is applied. We suppose u_0, ϕ_1, ϕ_2 to be given constants and to satisfy

$$\phi_2 > \phi_1. \tag{1.5}$$

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[1] $\partial\mathring{\Omega}_i$ denotes the interior (relative to $\partial\Omega$) of $\partial\Omega_i$.

The mixed problem (1.1)–(1.4) was first proposed in [3, 4] by Kohlrausch and Diesselhorst together with the transformation

$$\theta = \frac{1}{2}\phi^2 + \int_{u_0}^{u} \frac{\kappa(t)}{\sigma(t)}\,dt \tag{1.6}$$

which considerably simplifies the problem. Diesselhorst gave in [4] a proof of uniqueness for problem (1.1)–(1.4). His proof however is incomplete. Using (1.6) we prove in Sec. 2 a result of existence, nonexistence, and uniqueness under very general hypotheses on $\sigma(u)$ and $\kappa(u)$. By a solution of problem (1.1)–(1.4) we intend here a classical solution.

2. The main result. Let $\kappa(t)$, $\sigma(t) \in C^2(\mathbf{R}^1)$ and satisfy

$$\kappa(t) > 0, \qquad \sigma(t) > 0 \qquad \text{for all } t. \tag{2.1}$$

THEOREM. Assume (2.1) holds true.
First case. Suppose

$$\int_{u_0}^{\infty} \frac{\kappa(t)}{\sigma(t)}\,dt = \alpha < \infty. \tag{2.2}$$

If

$$(\phi_2 - \phi_1)^2 < 8\alpha, \tag{2.3}$$

problem (1.1)–(1.4) has one and only one solution. If

$$(\phi_2 - \phi_1)^2 \geq 8\alpha \tag{2.4}$$

then problem (1.1)–(1.4) has no solution.
Second case. If

$$\int_{u_0}^{\infty} \frac{\kappa(t)}{\sigma(t)}\,dt = \infty, \tag{2.5}$$

then problem (1.1)–(1.4) has one and only one solution.
Proof. Suppose (2.2) and (2.3). Define the strictly increasing function $F\colon [u_0, \infty) \to [0, \alpha)$ as follows:

$$F(u) = \int_{u_0}^{u} \frac{\kappa(t)}{\sigma(t)}\,dt. \tag{2.6}$$

Put

$$\theta = \tfrac{1}{2}\phi^2 + F(u). \tag{2.7}$$

Let the function $H\colon [\phi_1, \phi_2] \to [0, \frac{1}{8}(\phi_2 - \phi_1)^2]$ be defined in the following way:

$$H(\phi) = -\tfrac{1}{2}\phi^2 + \tfrac{1}{2}(\phi_2 + \phi_1)\phi - \tfrac{1}{2}\phi_1\phi_2. \tag{2.8}$$

By (2.2) and (2.3) the function

$$G(\phi) = F^{-1}(H(\phi)), \qquad \phi \in [\phi_1, \phi_2], \tag{2.9}$$

is well defined. Consider now the mixed problem for the Laplace equation:

$$\Delta\psi = 0 \quad \text{in } \Omega, \tag{2.10}$$

$$\psi = 0 \quad \text{on } \partial\Omega_1, \qquad \psi = \psi_2 \quad \text{on } \partial\Omega_2, \qquad \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega_3, \tag{2.11}$$

where

$$\psi_2 = \int_{\phi_1}^{\phi_2} \sigma(G(\phi))\, d\phi. \tag{2.12}$$

By well-known results, problem (2.10), (2.11) has one and only one solution $\psi(x)$ which, by the maximum principle, satisfies the inequalities

$$0 \le \psi(x) \le \psi_2 \qquad \text{in } \overline{\Omega}. \tag{2.13}$$

Define the one-to-one mapping of $[\phi_1, \phi_2]$ onto $[0, \psi_2]$ given by

$$L(\phi) = \int_{\phi_1}^{\phi} \sigma(G(\tau))\, d\tau. \tag{2.14}$$

By (2.13) the function

$$\phi(x) = L^{-1}(\psi(x)), \qquad x \in \overline{\Omega}, \tag{2.15}$$

is well defined. Let

$$u(x) = G(\phi(x)). \tag{2.16}$$

Since $\nabla\psi = \sigma(u)\nabla\phi$, we have by (2.10)

$$\nabla \cdot (\sigma(u)\nabla\phi) = 0. \tag{2.17}$$

Moreover, $\phi(x)$ satisfies the boundary conditions

$$\phi = \phi_1 \quad \text{on } \partial\Omega_1, \qquad \phi = \phi_2 \quad \text{on } \partial\Omega_2, \qquad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega_3. \tag{2.18}$$

By (2.9) we have the functional relation

$$H(\phi) = F(G(\phi)). \tag{2.19}$$

Therefore by (2.7)

$$\theta = \tfrac{1}{2}\phi^2 + H(\phi). \tag{2.20}$$

It follows by (2.8) that

$$\theta = \tfrac{1}{2}(\phi_1 + \phi_2)\phi - \tfrac{1}{2}\phi_1\phi_2. \tag{2.21}$$

Hence by (2.17)

$$\nabla \cdot (\sigma(u)\nabla\theta) = 0 \tag{2.22}$$

and by (2.18)

$$\theta = \frac{1}{2}\phi_1^2 \quad \text{on } \partial\Omega_1, \qquad \theta = \frac{1}{2}\phi_2^2 \quad \text{on } \partial\Omega_2, \qquad \frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega_3. \tag{2.23}$$

Since

$$\nabla\theta = \phi\nabla\phi + \frac{\kappa(u)}{\sigma(u)}\nabla u$$

we have by (2.22), taking into account (2.17),

$$-\nabla \cdot (\kappa(u)\nabla u) = \sigma(u)|\nabla\phi|^2. \tag{2.24}$$

By (2.23) u satisfies the boundary conditions

$$u = u_0 \quad \text{on } \partial\Omega_1 \cup \partial\Omega_2, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_3. \tag{2.25}$$

We conclude that $(u(x), \phi(x))$ gives a regular solution to problem (1.1)–(1.4).

We claim that this solution is unique. By contradiction let $(u'(x), \phi'(x))$ be a second regular solution. With the usual transformation

$$\theta' = \frac{1}{2}\phi'^2 + \int_{u_0}^{u'} \frac{\kappa(t)}{\sigma(t)}\,dt$$

we obtain

$$\nabla \cdot (\sigma(u')\nabla\theta') = 0 \quad \text{in } \Omega, \tag{2.26}$$

$$\theta' = \frac{1}{2}\phi_1^2 \quad \text{on } \partial\Omega_1, \qquad \frac{\partial\theta'}{\partial n} = 0 \quad \text{on } \partial\Omega_3, \qquad \theta' = \frac{1}{2}\phi_2^2 \quad \text{on } \partial\Omega_2. \tag{2.27}$$

Now the function

$$\Phi' = \tfrac{1}{2}(\phi_1 + \phi_2)\phi' - \tfrac{1}{2}\phi_1\phi_2$$

is a solution to the problem

$$\nabla \cdot (\sigma(u')\nabla\Phi') = 0 \quad \text{in } \Omega, \tag{2.28}$$

$$\Phi' = \theta' \quad \text{on } \partial\Omega_1 \cup \partial\Omega_2, \qquad \frac{\partial\Phi'}{\partial n} = 0 \quad \text{on } \partial\Omega_3. \tag{2.29}$$

By (2.1) we have $\sigma(u'(x)) \geq a_0 > 0$. It follows by standard results that the mixed problems (2.26), (2.27) and (2.28), (2.29) have the same solution, i.e., $\Phi' = \theta'$. This implies, recalling the definition of θ', that u' and ϕ' are related by the same functional relation as u and ϕ. That is, we have

$$F(u') = H(\phi'). \tag{2.30}$$

Solving (2.30) with respect to u', we obtain $u' = F^{-1}(H(\phi')) = G(\phi')$. Now

$$\psi'(x) = \int_{\phi_1}^{\phi'(x)} \sigma(G(t))\,dt$$

is a solution to problem (2.10), (2.11). However (2.10), (2.11) has a unique solution and we conclude that $\psi(x) = \psi'(x)$. Because $L^{-1}(\psi'(x)) = L^{-1}(\psi(x))$, we also have $G(\phi(x)) = G(\phi'(x))$; thus $(u(x), \phi(x)) = (u'(x), \phi'(x))$. This completes the proof of the uniqueness of the solution.

Suppose now (2.2) and (2.4). We claim that under these hypotheses problem (1.1)–(1.4) has no solution. Suppose the contrary, and let $(u(x), \phi(x))$ be a solution. We find again

$$F(u(x)) = H(\phi(x)), \qquad x \in \overline{\Omega}. \tag{2.31}$$

By the maximum principle we have

$$\phi_1 \leq \phi(x) \leq \phi_2. \tag{2.32}$$

Hence there exists $\overline{x} \in \Omega$ such that

$$\phi(\overline{x}) = (\phi_1 + \phi_2)/2.$$

Therefore

$$H(\phi(\overline{x})) = \tfrac{1}{8}(\phi_2 - \phi_1)^2.$$

On the other hand, $F(u(\overline{x})) < \alpha$. This contradicts (2.4), and we conclude that no regular solution exists. The second case when (2.5) holds can be discussed in a similar manner if we note that by (2.5) we have

$$H([\phi_1, \phi_2]) \subset F([u_0, \infty)).$$

Note 2.1. If the material obeys the Wiedemann-Franz law, i.e.,

$$\kappa/\sigma = Lu, \qquad L \text{ a positive constant}, \tag{2.33}$$

condition (2.5) is certainly satisfied, and in this important case the solution to problem (1.1)–(1.4) always exists and is unique.

Note 2.2. With slightly more complicated calculations the above discussion applies equally well when the boundary conditions for the temperature are

$$u = u_1 \quad \text{on } \partial\Omega_1, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_3, \qquad u = u_2 \quad \text{on } \partial\Omega_2,$$

where u_1 and u_2 are given constants.

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