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DECAY ESTIMATES FOR THE CONSTRAINED ELASTIC CYLINDER OF VARIABLE CROSS SECTION*

By

J. N. FLAVIN (*University College, Galway, Ireland*)

R. J. KNOPS (*Heriot-Watt University, Edinburgh, U.K.,*)

AND

L. E. PAYNE (*Cornell University, Ithaca, New York*)

1. Introduction. We describe two methods, relevant to the study of Saint-Venant's principle, for the derivation of decay estimates in a linear isotropic homogeneous elastic nonprismatic cylinder loaded by prescribed end displacements and with fixed curved lateral surface. The results and associated calculations are expressed in terms of integrals taken over plane cross sections of the cylinder rather than averages over partial volumes as in many previous discussions. (See, for example, Toupin [25], Oleinik and Yosifian [23], and Fichera [6, 7]; other references to this and related issues may be found in the comprehensive survey by Horgan and Knowles [15].) A notable exception is the investigation by Biollay [1] who examines the three-dimensional semi-infinite prismatic beam with lateral sides held fixed and data specified over the base of the cylinder. (We also mention in this respect the study by Knowles [19] of the semi-infinite plane strip with lateral sides stress-free.) An exponentially decreasing decay rate is obtained which, of course, is a common feature of most studies of Saint-Venant's principle and whose underlying explanation is contained in the papers by Kirchgässner and Scheurle [17] and by Ladeveze [20].

The estimate, however, of Biollay also involves an amplitude function whose magnitude is related to the decay factor. The methods presented here do not suffer this defect and indeed the straightforward calculations together with the applicability to nonprismatic cylinders are seen as some of the advantages. The first method reduces the problem to a nonautonomous second-order differential inequality for the mean-square cross-sectional integral of the displacement and proves in part that this measure is a convex function of axial distance. (A similar technique has been used by Flavin and Knops [9, 10] in related two-dimensional problems.) Immediate criticisms of the method are that it is valid only for a restricted range of the elastic moduli and near the upper limit of this range produces a decay rate inferior to that given by Biollay. These deficiencies are absent from the second approach which relies upon a first-order differential inequality for a cross-sectional integral involving the

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displacement and its gradient. Both methods described yield estimates of decaying exponential type although more precisely when the cylinder is of semi-infinite length, the results state that either the measure grows faster than some function of axial distance, typically a growing exponential, or it decays faster than another function of axial distance, typically a decaying exponential.

In Sec. 2, we set down the basic equations and boundary conditions governing the solution to the problem under consideration. Section 3 contains the derivation and discussion of the second-order differential inequality which is valid for a cylinder of both finite and semi-infinite length. Since the comparison theorem reduces the analysis to the study of the associated differential equation whose theory is well known, we content ourselves with a brief indication of possible results, particularly emphasizing those yielding decay estimates. For the semi-infinite cylinder, such estimates are valid under various conditions on the asymptotic behaviour, including those imposed by Knops and Payne [18] and Galdi, Knops, and Rionero [11] in their discussions of the analogous nonlinear problem for the prismatic cylinder by means of weighted-energy arguments. These authors also derived mean-square cross-sectional estimates of exponentially decaying type.

Section 4 is devoted to the treatment of the first-order differential inequality for the cross-sectional measure involving the displacement and its gradient. This measure forms part of the work done by the external forces acting over the ends of the cylinder and the estimates obtained are again valid for the finite and semi-infinite cylinder. We also show how the estimate may be utilized to yield a second bound for the measure introduced in Sec. 3, which accordingly is not subject to either of the criticisms already noted. Finally, in this section, we briefly indicate how the method may be extended to a corresponding problem in linear elastodynamics.

In the concluding section, we discuss the axisymmetric problem and introduce a third mean-square measure, involving displacement gradients, which is shown to satisfy a second-order differential inequality similar to that already considered in Sec. 3. However, the advantage of this third measure is that it may be further employed to derive pointwise decay estimates for the displacement components which are valid up to the boundary.

Throughout the paper we adopt the comma notation for partial differentiation and also the summation convention with Latin symbols ranging over the values 1, 2, 3, while Greek symbols take the values 1, 2.

2. Notation, basic equations, and assumptions. We consider a nonprismatic cylinder B with plane ends and select a rectangular system of coordinates such that one end of the cylinder lies in the (x_1, x_2)-coordinate plane and contains the origin. We suppose that the length of the cylinder is L and that $D(x_3) \subset \mathbf{R}^2$ represents the bounded cross section at distance x_3 from the plane end. We distinguish between finite and infinite values of L and thus separately discuss cylinders that are respectively of finite and semi-infinite length. The boundary ∂D of each cross section is assumed sufficiently smooth to admit application of the divergence theorem in the plane of the cross section. The cylinder is occupied by a homogeneous isotropic compressible linear elastic material maintained in equilibrium by specified displacements over the plane ends with the lateral sides of the cylinder held fixed at zero displacement. No body-force acts.

We assume the existence of a sufficiently smooth displacement vector $u(x)$ satisfying the following well-known conditions of our problem:

$$u_{i,jj} + \alpha u_{j,ji} = 0, \qquad x \in B, \tag{2.1}$$

$$u_i(x) = 0, \qquad x \in \partial D(x_3), \quad x_3 \in [0, L], \tag{2.2}$$

$$u_i(x) = g_i(x), \qquad x \in D(0), \tag{2.3}$$

$$u_i(x) = h_i(x), \qquad x \in D(L). \tag{2.4}$$

Here, $u_i(x)$ are the components of the displacement with respect to the given cartesian axes, $g_i(x_1, x_2)$ and $h_i(x_1, x_2)$ are specified functions, and α is a constant which in terms of the Lamé constants λ and μ is given by

$$\alpha = (\lambda + \mu)/\mu. \tag{2.5}$$

For a positive-definite strain-energy, α lies in the range $1/3 < \alpha < \infty$. However, we suppose only that the Lamé constants are such that

$$\alpha > 0. \tag{2.6}$$

Let us also recall the following subsidiary result required in the subsequent calculations. Let $\lambda(x_3)$ denote the first eigenvalue in the two-dimensional clamped membrane problem for the cross section $D(x_3)$. That is, λ is the first eigenvalue corresponding to the boundary value problem for the function $v(x_1, x_2)$:

$$v_{,\beta\beta} + \lambda v = 0, \qquad x \in D(x_3), \tag{2.7}$$

$$v = 0, \qquad x \in \partial D(x_3). \tag{2.8}$$

We require the variational characterization of λ contained in the well-known inequality

$$\lambda(x_3) \int_{D(x_3)} \varphi^2 \, dx_1 \, dx_2 \le \int_{D(x_3)} \varphi_{,\beta} \varphi_{,\beta} \, dx_1 \, dx_2, \tag{2.9}$$

where $\varphi(x_1, x_2)$ is an arbitrary Dirichlet integrable function vanishing on ∂D. A lower bound for $\lambda(x_3)$ is provided by the Faber–Krahn estimate

$$\lambda(x_3) \ge \pi j_0^2 / A(x_3), \tag{2.10}$$

where $A(x_3)$ is the area of $D(x_3)$ and j_0 is the smallest positive zero of the Bessel function $J_0(y)$. Frequently we assume that

$$0 < \lambda_m < \lambda(x_3), \quad \forall x_3, \tag{2.11}$$

where λ_m is constant, although in several instances we are able to relax this condition.

3. A second-order differential inequality and related estimates. In this section we discuss decay estimates for the displacement vector measured by an L_2-norm over a plane cross section of the cylinder. As already mentioned in the Introduction, the method employed restricts the range of elastic moduli, but for those values of the moduli in the range of validity, the method leads to especially simple estimates with a natural form for the amplitude. We first derive a second-order differential inequality and then after integration derive several estimates and related properties. We deal with cylinders of both finite and semi-infinite length, the distinction appearing, of course, only in the integration of the differential inequality and not in its derivation. The next section describes an alternative approach valid for all physically acceptable elastic moduli, but which is comparatively less direct.

3a. *The differential inequality.* Let us consider the function $F(x_3)$ defined by

$$F(x_3) = \int_{D(x_3)} (u_\beta u_\beta + a u_3^2)\, dx_1\, dx_2, \tag{3.1}$$

where a is a positive constant to be chosen later. We wish to establish a differential inequality of the form

$$F''(x_3) \geq k^2(x_3) F(x_3) \tag{3.2}$$

whose integration will lead to the desired estimates for $F(x_3)$. Here, and throughout, differentiation with respect to the variable x_3 is denoted by a superposed prime. Obviously, (3.2) implies that $F(x_3)$ is convex on the interval of validity, and this property enables several elementary conclusions to be established regarding the evolutionary behaviour of F. (See Sec. 3b.)

Successive differentiation of (3.1) yields

$$F'(x_3) = 2\int_{D(x_3)} (u_\beta u_{\beta,3} + a u_3 u_{3,3})\, dx_1\, dx_2, \tag{3.3}$$

and

$$F''(x_3) = 2\int_{D(x_3)} (u_{\beta,3} u_{\beta,3} + a u_{3,3}^2)\, dx_1\, dx_2 + 2\int_{D(x_3)} (u_\beta u_{\beta,33} + a u_3 u_{3,33})\, dx_1\, dx_2. \tag{3.4}$$

The differential equation (2.1) is used to replace the second derivatives appearing in the second integral of (3.4), and then an integration by parts leads to

$$\begin{aligned} F''(x_3) = {} & 2\int_{D(x_3)} \left[u_{\beta,3} u_{\beta,3} + \frac{\alpha a}{(1+\alpha)} u_{3,\beta} u_{\beta,3} + \frac{a}{(1+\alpha)} u_{3,\beta} u_{3,\beta} \right] dx_1\, dx_2 \\ & + 2\int_{D(x_3)} \left[a u_{3,3}^2 + \alpha u_{\beta,\beta} u_{3,3} + \alpha u_{\beta,\beta} u_{\gamma,\gamma} \right] dx_1\, dx_2 \\ & + 2\int_D u_{\beta,\gamma} u_{\beta,\gamma}\, dx_1\, dx_2. \end{aligned} \tag{3.5}$$

An equality of type (3.2) is clearly violated whenever $F''(x_3)$ fails to be positive-definite. However, as can be readily seen from (3.5), the positive-definiteness of the quadratic integrand on the right imposes a restriction on the choice of the coefficients a and α. Indeed, the last integral in (3.5) may be replaced using the relations

$$\int_{D(x_3)} u_{\beta,\gamma} u_{\beta,\gamma}\, dx_1\, dx_2 \geq \int_{D(x_3)} u_{\beta,\gamma} u_{\gamma,\beta}\, dx_1\, dx_2 = \int_{D(x_3)} u_{\beta,\beta} u_{\gamma,\gamma}\, dx_1\, dx_2, \tag{3.6}$$

and then standard conditions show that for positive-definiteness to hold it is necessary and sufficient for a and α to satisfy

$$\frac{\alpha^2}{4(1+\alpha)} \leq a < \frac{4(1+\alpha)}{\alpha^2} \tag{3.7}$$

which in turn restricts α to satisfy

$$\alpha^2 < 4(1+\alpha) \tag{3.8}$$

or, equivalently,

$$0 < \alpha < 2(1+\sqrt{2}) = 4.828. \tag{3.9}$$

In terms of Poisson's ratio, ν, condition (3.9) becomes

$$\nu < 0.396. \tag{3.10}$$

In order to establish inequality (3.2) subject to (3.7), we apply the arithmetic-geometric mean inequality to the second and fifth terms on the right of (3.5) to obtain:

$$\begin{aligned} F''(x_3) \geq & \left(2 - \frac{\alpha a c_1}{(1+\alpha)}\right) \int_{D(x_3)} u_{\beta,3} u_{\beta,3}\, dx_1\, dx_2 \\ & + \frac{a}{(1+\alpha)}\left(2 - \frac{\alpha}{c_1}\right) \int_{D(x_3)} u_{3,\beta} u_{3,\beta}\, dx_1\, dx_2 \\ & + (2a - \alpha c_2) \int_{D(jx_3)} u_{3,3}^2\, dx_1\, dx_2 \\ & + \left\{\alpha\left(2 - \frac{1}{c_2}\right) + c_3\right\} \int_{D(x_3)} u_{\beta,\beta} u_{\gamma,\gamma}\, dx_1\, dx_2 \\ & + (2 - c_3) \int_{D(x_3)} u_{\beta,\gamma} u_{\beta,\gamma}\, dx_1\, dx_2, \end{aligned} \tag{3.11}$$

where c_1, c_2, and c_3 are positive constants and the relations (3.6) have been used. We now set

$$c_1 = 2(1+\alpha)/\alpha a, \qquad c_2 = 2a/\alpha, \qquad c_3 = -\alpha\left(2 - \frac{1}{c_2}\right) \leq 2. \tag{3.12}$$

With this choice, which implies $c_2 \leq \frac{1}{2}$ and is consistent with (3.7), the inequality (3.11) becomes

$$\begin{aligned} F''(x_3) \geq & \frac{a}{2(1+\alpha)^2}[4(1+\alpha) - \alpha^2 a] \int_{D(x_3)} u_{3,\beta} u_{3,\beta}\, dx_1\, dx_2 \\ & + \frac{1}{2}\left[4(1+\alpha) - \frac{\alpha^2}{a}\right] \int_{D(x_3)} u_{\beta,\gamma} u_{\beta,\gamma}\, dx_1\, dx_2 \end{aligned} \tag{3.13}$$

$$\geq m^2 \int_{D(x_3)} u_{i,\beta} u_{i,\beta}\, dx_1\, dx_2, \tag{3.14}$$

where

$$m^2 = \min\left(\frac{1}{2(1+\alpha)^2}[4(1+\alpha) - \alpha^2 a],\ \frac{1}{2}\left[4(1+\alpha) - \frac{\alpha^2}{a}\right]\right). \tag{3.15}$$

Alternatively, we may equate the coefficients of the integrals in (3.13), so that

$$m^2 = \frac{1}{2(1+\alpha)^2}[4(1+\alpha) - \alpha^2 a], \tag{3.16}$$

where a is given by

$$a = \frac{2(1+\alpha)(2+\alpha)}{\alpha}\left[\sqrt{[1+\alpha^2/\{2(2+\alpha)\}^2]} - 1\right]. \tag{3.17}$$

In either case, α must satisfy (3.8).

An appeal to inequality (2.9) then leads at once to

$$F''(x_3) \geq m^2\lambda(x_3)F(x_3), \tag{3.18}$$

which is clearly of the desired form (3.2).

We note that the eigenvalue $\lambda(x_3)$ may be replaced by the lower bound given by the Faber–Krahn inequality (2.10). In particular, when the cross sections of the cylinder possess the same area but not necessarily the same shape or orientation, the coefficient of $F(x_3)$ in the resultant inequality reduces to a constant. Let us further observe that the coefficients of the integrals in (3.13), and therefore the respective values of m in (3.14), tend to zero as the modulus α tends to the maximum permitted by the range of validity (3.7) or (3.8).

3b. *Behaviour of $F(x_3)$. Decay estimates.* This subsection discusses the integration of the differential inequality (3.18) with the aim of establishing upper and lower bounds on the function $F(x_3)$ leading to a description of its evolutionary behaviour. Such properties, of course, may be derived from the standard comparison theorem (see, for example, [24]) combined with the well-known theory for the corresponding equation obtained from (3.18) when the inequality is replaced by equality. (See, for example, [5], [13], [14].) Nevertheless, we prefer to present what is perhaps a more direct treatment yielding estimates that are either apparently new or not readily available in the literature. We begin, however, by listing properties of $F(x_3)$ that easily follow from its convexity with respect to the variable x_3 inherent in inequality (3.18).

PROPOSITION 3.1. The function $F(x_3)$ satisfying (3.18) on the finite interval $(0, L)$ possesses the following properties:

(i) $F(x_3)$ cannot achieve a maximum on $(0, L)$;
(ii) $F(x_3)$ is nonoscillatory on $[0, L]$;
(iii) $F(0) = F(L) = 0$ implies that $F(x_3) = 0$, $x_3 \in [0, L]$;
(iv) $F'(0) = 0$ implies either (a) $F(x_3) = F(0)$, $x_3 \in [0, L]$, or (b) $F'(x_3) > 0$, $x_3 \in (0, L]$;
(v) $F'(0) > 0$ implies $F'(x_3) > 0$, $x_3 \in [0, L]$;
(vi) $F(L) < F(0)$ implies $F'(0) < 0$, $x_3 \in [0, L]$;
(vii) the function $F(x_3)$ has at most one extremum on $[0, L]$ or is identically constant.

Similar properties hold on $(0, \infty)$ provided suitable asymptotic behaviour is imposed on $F(x_3)$ for large values of its argument.

Further evolutionary properties depend upon consideration of a second inequality derived from (3.18) according to the following lemma.

LEMMA 3.1. Let $\theta(x_3)$ be a differentiable function, satisfying the inequality

$$m\lambda(x_3)(1-\theta^2(x_3)) + (\theta(x_3)\sqrt{\lambda(x_3)})' \geq 0. \tag{3.19}$$

Then $F(x_3)$ satisfies

$$\frac{d}{dx_3}\left\{\exp\left(-2m\int_0^{x_3}\beta(\eta)\,d\eta\right)\frac{d}{dx_3}\left(F(x_3)\exp\left(m\int_0^{x_3}\beta(\eta)\,d\eta\right)\right)\right\}\geq 0, \quad (3.20)$$

where

$$\beta(x_3)=\theta\sqrt{\lambda(x_3)}; \quad (3.21)$$

on introducing the new variable

$$\sigma(x_3)=\int_0^{x_3}\left(\exp\left(2m\int_0^{\tau}\beta(\eta)\,d\eta\right)\right)d\tau, \quad (3.22)$$

inequality (3.20) becomes

$$\frac{d^2}{d\sigma^2}\left(F(\sigma)\exp\left(m\int_0^{x_3}\beta(\eta)\,d\eta\right)\right)\geq 0. \quad (3.23)$$

Proof. It is easily shown from (3.18) and (3.19) that $F(x_3)$ satisfies the inequality

$$\left(\frac{d}{dx_3}-m\theta(x_3)\sqrt{\lambda(x_3)}\right)\left(\frac{dF(x_3)}{dx_3}+m\theta(x_3)\sqrt{\lambda(x_3)}\right)\geq 0, \quad (3.24)$$

which on rewriting leads immediately to (3.20) and hence (3.23).

Lemma 3.1, which is valid for the cylinder of either finite or semi-infinite length, shows that the exponentially weighted function $F(x_3)$ is convex with respect to the variable σ. We obtain our results by variously exploiting this property for selected values of the function $\theta(x_3)$. The reader will be able to establish further conclusions by means of the same general approach. We discuss in detail the case of the cylinder of finite length to illustrate the method, and then briefly indicate modifications required for the extension to the semi-infinite case. We conclude the subsection with some additional results.

Our first set of results on $(0,L)$ is based on the inequality

$$\begin{aligned}F(x_3)\leq{}&\exp\left(-m\int_0^{x_3}\beta(\eta)\,d\eta\right)\left\{\frac{\int_{x_3}^{L}\left(\exp\left(2m\int_0^{\tau}\beta(\eta)\,d\eta\right)\right)d\tau}{\int_0^{L}\left(\exp\left(2m\int_0^{\tau}\beta(\eta)\,d\eta\right)\right)d\tau}\right\}F(0)\\&+\exp\left(m\int_{x_3}^{L}\beta(\eta)\,d\eta\right)\left\{\frac{\int_0^{x_3}\left(\exp\left(2m\int_0^{\tau}\beta(\eta)\,d\eta\right)\right)d\tau}{\left(\int_0^{L}\left(\exp\left(2m\int_0^{\tau}\beta(\eta)\,d\eta\right)\right)d\tau\right)}\right\}F(L), \quad (3.25)\end{aligned}$$

which follows immediately from the convexity of $F(x_3)$ as given by (3.23). Note that (3.25) is exact at both $x_3=0$, $x_3=L$. Furthermore, when the displacement is specified zero on $x_3=L$, inequality (3.25) may be simplified to yield

$$F(x_3)\leq F(0)\exp\left(-m\int_0^{x_3}\beta(\eta)\,d\eta\right), \qquad x_3\in[0,L]. \quad (3.26)$$

In order to render either (3.25) or (3.26) precise, special choices depending on the behaviour of $\lambda(x_3)$ must be taken for $\theta(x_3)$ and hence $\beta(x_3)$. For example, we may establish the following properties.

PROPOSITION 3.2. Let

$$0 < \lambda_m = \min_{[0,L]} \lambda(x_3). \tag{3.27}$$

Then, for $x_3 \in [0, L]$,

$$F(x_3) \leq [F(0) \sinh m\sqrt{\lambda_m}(L - x_3) + F(L) \sinh x_3 m\sqrt{\lambda_m}]/ \sinh Lm\sqrt{\lambda_m}. \tag{3.28}$$

Proof. Set $\theta(x_3) = \sqrt{\lambda_m/\lambda(x_3)}$ in (3.25).

Assumption (3.27) includes the case of monotonically increasing (decreasing) $\lambda(x_3)$ with $\lambda(0) > 0$ (resp. $\lambda(L) > 0$). The last condition is removed in the next result.

PROPOSITION 3.3. Let $\lambda'(x_3) \geq 0$ ($\lambda'(x_3) \leq 0$). Then $F(x_3)$ on $[0, L]$ satisfies (3.25) with $\beta(x_3) = \sqrt{\lambda(x_3)}(-\sqrt{\lambda(x_3)})$.

Proof. Set $\theta(x_3) = 1(-1)$.

It is possible to derive an estimate when $\lambda(x_3)$ violates assumption (3.27) and is not monotonic. We have

PROPOSITION 3.4. Let

$$\gamma = \min_{[0,L]} \frac{\lambda'(x_3)}{2\lambda^{3/2}(x_3)}.$$

Then $F(x_3)$ on $[0, L]$ satisfies (3.25) with $\theta(x_3) = (\gamma + \sqrt{\gamma^2 + 4m^2})/(2m)$.

When $F(L) = 0$, the inequalities contained in Proposition (3.2)–(3.4) simplify to (3.26) with the respective values of $\beta(x_3)$.

We next integrate (3.23), first over $(0, x_3)$ and then over (x_3, L), to obtain

$$\begin{aligned} F(x_3) \leq F(L) \exp\left(m \int_{x_3}^{L} \beta(\eta)\, d\eta\right) &- [F'(0) + m\beta(0)F(0)] \\ \times \exp\left(-m \int_0^{x_3} \beta(\eta)\, d\eta\right) &\int_{x_3}^{L} \exp\left(2m \int_0^{\tau} \beta(\eta)\, d\eta\right) d\tau. \end{aligned} \tag{3.29}$$

We immediately conclude from (3.29) that when $F'(0) > 0$ then

$$F(x_3) \geq [F'(0) + m\beta(0)F(0)] \exp\left(-m \int_0^{x_3} \beta(\eta)\, d\eta\right) \int_0^{x_3} \exp\left(2m \int_0^{\tau} \beta(\eta)\, d\eta\right) d\tau. \tag{3.30}$$

On the other hand, when $F(L) = 0$ we must have $F'(0) < 0$, as already noted in Proposition 3.1(vi). In these circumstances, we may establish

PROPOSITION 3.5. Suppose that $\lambda(x_3)$ is nonincreasing so that $\lambda'(x_3) \leq 0$, $x_3 \in [0, L]$. Then with $F(L) = 0$ there holds

$$F(x_3) \leq [m\sqrt{\lambda(0)}F(0) - F'(0)](L - x_3) \exp\left(-m \int_0^{x_3} \sqrt{\lambda(\eta)}\, d\eta\right). \tag{3.31}$$

Proof. Set $\theta(x_3) = -1$, so that $\beta(x_3) = -\sqrt{\lambda(x_3)}$ in (3.29).

Clearly, similar upper bounds may be derived from (3.29) using the choices of $\theta(x_3)$ corresponding to the different conditions on $\lambda(x_3)$ stated in Propositions (3.2)–(3.4).

We now turn our attention to the semi-infinite interval and first derive a necessary condition for the function $F(x_3)$ to become unbounded for large values of x_3.

PROPOSITION 3.6. Suppose that

$$\lim_{x_3 \to \infty} \frac{\exp\left(m \int_0^{x_3} \beta(\eta)\, d\eta\right)}{\beta(x_3)} = \infty. \tag{3.32}$$

Then

$$\lim_{x_3 \to \infty} F(x_3) = 0 \tag{3.33}$$

unless

$$F'(0) + m\beta(0)F(0) \le 0. \tag{3.34}$$

Proof. By a Taylor series expansion, we obtain from (3.23) the inequality

$$\begin{aligned} F(x_3) \ge \exp&\left(-m \int_0^{x_3} \beta(\eta)\, d\eta\right) \\ &\times [F(0)\{F'(0) + m\beta(0)F(0)\}] \int_0^{x_3} \left(\exp 2m \int_0^{\tau} \beta(\eta)\, d\eta\right) d\tau \end{aligned} \tag{3.35}$$

from which the result follows on application of l'Hôpital's theorem.

By imposing conditions on the asymptotic behaviour of $F(x_3)$ we may prove that $F(x_3)$ must actually decay. These results are analogues of Propositions 3.2–3.4, and hence are easily derived. We therefore content ourselves with the statement and proof of only one such result which, it will be noticed, is akin to a Phragmèn–Lindelöf principle, reminiscent of several results previously obtained in the study of Saint-Venant's principle. Maz′ya and Plamenevskiĭ [22] establish a similar result for solutions that are separable obtaining the decay rate $\lambda_m/(\alpha + 1)^{1/2}$ which is faster than that given by (3.38). However, our result holds for all solutions and not only for those in the class considered in [22].

PROPOSITION 3.7. Let

$$0 < \lambda_m = \inf_{[0,\infty]} \lambda(x_3) \tag{3.36}$$

and

$$\lim_{x_3 \to \infty} F(x_3) \exp(-m\sqrt{\lambda_m} x_3) = 0. \tag{3.37}$$

Then

$$F(x_3) \le F(0) \exp(-m\sqrt{\lambda_m} x_3), \qquad x_3 \in [0, \infty). \tag{3.38}$$

Proof. We use inequality (3.25) together with $\theta(x_3) = \sqrt{(\lambda_m/\lambda(x_3))}$.

By imposing conditions on the asymptotic behaviour of both $F(x_3)$ and its derivative it is possible to obtain additional decay estimates based on the following inequality:

$$\begin{aligned} F(x_3) \le F(0) \exp&\left(-m \int_0^{x_3} \beta(\eta)\, d\eta\right) + [F'(L) + m\beta(L)F(L)] \\ &\times \exp\left(-m \int_{x_3}^{L} \beta(\eta)\, d\eta\right) \int_0^{x_3} \exp\left(2m \int_0^{\tau} \beta(\eta)\, d\eta\right) d\tau, \end{aligned} \tag{3.39}$$

which is a consequence of integrating (3.23) first over (x_3, L) and then over $(0, x_3)$. As an example of results of this type, we have the next two propositions.

PROPOSITION 3.8. Suppose $\lambda(x_3)$ is nondecreasing so that $\lambda'(x_3) \geq 0$ on $[0, \infty)$ and

$$\lim_{x_3 \to \infty} [F'(x_3) + m\sqrt{\lambda(x_3)}F(x_3)] \exp\left(-m \int_0^{x_3} \lambda(\eta)\, d\eta\right) = 0, \tag{3.40}$$

$$F(x_3) \leq F(0) \exp\left(-m \int_0^{x_3} \sqrt{\lambda(\eta)}\, d\eta\right). \tag{3.41}$$

Proof. Set $\theta(x_3) = 1$ and use (3.29).

PROPOSITION 3.9. Suppose that $\lambda(x_3)$ satisfies the condition

$$1 < \gamma < \inf_{x_3 \in [0,\infty)} \left[\frac{\lambda'(x_3)}{4m\lambda^{3/2}(x_3)} + \sqrt{\left(1 + \left\{\frac{\lambda'(x_3)}{4m\lambda^{3/2}(x_3)}\right\}^2\right)} \right], \tag{3.42}$$

where γ is constant, and let $F(x_3)$ possess the asymptotic behaviour

$$\lim_{x_3 \to \infty} [F'(x_3) + \gamma\sqrt{\lambda(x_3)}F(x_3)] \exp\left(-m\gamma \int_0^{x_3} \sqrt{\lambda(\eta)}\, d\eta\right) = 0. \tag{3.43}$$

Then

$$F(x_3) \leq F(0) \exp\left(-\gamma m \int_0^{x_3} \sqrt{\lambda(\eta)}\, d\eta\right). \tag{3.44}$$

Proof. Set $\theta(x_3) = \gamma$ and use (3.39).

We conclude this section with some remarks.

REMARK 3.1. Inequality (3.18) clearly also holds on cylinders of infinite length, which immediately enables us to establish an analogue of Liouville's theorem.

PROPOSITION 3.10. Assume that (3.18) holds on $(-\infty, \infty)$. Then either the displacement as measured by $F(x_3)$ is unbounded as $x_3 \to +\infty$ or $x_3 \to -\infty$, or the displacement is identically zero.

The proof of the Proposition, included here for completeness, is an easy application of the properties of a convex function. The same approach has been employed in a more general context by, for example, Brezis and Goldstein [4], Levine [21], and Goldstein and Lubin [12].

Proof. We suppose that $F'(x_3) \neq 0$ at some fixed point $x_3 = t$. Then, by (3.18), $F''(x_3) \geq 0$, $x_3 \in (-\infty, \infty)$, and hence $F(x_3)$ is a nonnegative, nonconstant convex function on $(-\infty, \infty)$ and is therefore unbounded. On the other hand, when $F(x_3)$ is everywhere bounded, we must have $F'(x_3) = 0$, for all x_3. Thus, $F''(x_3) = 0$, for all x_3 and so by (3.18), we have $F(x_3) = 0$ on $(-\infty, \infty)$. The proposition is therefore proved.

A similar result involving the strain $(u_{i,j} + u_{j,i})$ has been discussed by Kinderlehrer [16].

REMARK 3.2. It is possible to derive a modification of inequality (3.18) which produces improved rates of decay. We return to (3.11) and set

$$c_1 = (2-\delta)(1+\alpha)/(\alpha a), \qquad c_2 = (2-\delta)a/\alpha, \qquad c_3 = -\alpha\left(2 - \frac{1}{c_2}\right) \leq 2,$$

where $0 < \delta < 2$. Then instead of (3.18) we obtain

$$F''(x_3) \geq \delta \int_{D(x_3)} (u_{\alpha,3}u_{\alpha,3} + a u_{3,3}^2)\, dx_1\, dx_2 + m^2 \lambda(x_3) F(x_3), \tag{3.45}$$

where

$$\frac{\alpha^2}{2(1+\alpha)(2-\delta)} < a < \frac{2(1+\alpha)(2-\delta)}{\alpha^2}$$

and

$$m^2 = \min\left(\frac{1}{(2-\delta)(1+\alpha)^2}[2(2-\delta)(1+\alpha) - a\alpha^2],\ [2(1+\alpha) - \alpha^2/(a(2-\delta))]\right)$$

or, alternatively, on equating the coefficients of the last two integrals in (3.13),

$$m^2 = \frac{1}{(2-\delta)(1+\alpha)^2}[2(2-\delta)(1+\alpha) - a\alpha^2]$$

where a is now given by

$$a = \frac{(2-\delta)(1+\alpha)(2+\alpha)}{\alpha}\left[\sqrt{1 + \left\{\frac{\alpha}{(2-\delta)(2+\alpha)}\right\}^2} - 1\right].$$

An application of Schwarz's inequality (3.67) then yields

$$FF'' - \frac{\delta}{4}(F')^2 \geq m^2 \lambda(x_3) F^2$$

which, on setting $P(x_3) = F^{\varepsilon}(x_3)$, $\varepsilon = 1 - \delta/4 > 0$, as $\delta < 2$, leads to

$$P''(x_3) \geq \varepsilon m^2 \lambda(x_3) P(x_3).$$

We thus recover an inequality of the same form as (3.18) and all the previous arguments are immediately applicable. For example, the analogue of (3.38) becomes

$$F(x_3) \leq F(0) \exp(-x_3 m \sqrt{(\lambda_m/\varepsilon)}), \qquad x_3 \in [0, \infty),$$

which, since $\varepsilon < 1$, represents an improved decay rate. Note that now, however, the range of α is determined additionally by choice of δ and is always included in that for $\delta = 0$.

REMARK 3.3. While the basic inequality (3.18) has been established only for a limited range of elastic moduli, nevertheless the various estimates that have been derived are valid for *all* $x_3 \geq 0$. The decay estimates hold arbitrarily close to the base load region and are not invalidated within some boundary layer.

REMARK 3.4. It is also worth emphasizing that the rate of decay in most other available estimates deteriorates as the incompressibility limit is approached. This may thus be a further factor contributing to the limitation on the moduli apparently essential to the present approach.

REMARK 3.5. The Faber–Krahn inequality (2.10) may be used to replace $\lambda(x_3)$ by $A^{-1}(x_3)$ in the basic inequality (3.18). Then Propositions (3.2)–(3.9) continue to hold but with conditions on $\lambda(x_3)$ replaced by appropriate conditions on $A(x_3)$.

4. A first-order differential inequality and related estimates. We now describe a second method for discussing the behaviour of the solution to the boundary value problem (2.1)–(2.4). Unlike the approach considered in the previous section, the arguments used, while in passing involve volume integrals, do not restrict the elastic moduli. Nor is it essential to require any *a priori* asymptotic decay assumption of the solution in the case of a semi-infinite cylinder. Our method yields an alternative theorem of Phragmèn-Lindelöf type.

Thus, let us introduce the function

$$H(x_3) = \int_{D(x_3)} (u_i u_{i,3} + \alpha u_{j,j} u_3)\, dx_1\, dx_2, \tag{4.1}$$

which may be rewritten as

$$H(x_3 + h) = H(x_3) + \int_{x_3}^{x_3+h} \int_{D(\eta)} (u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j})\, dx_1\, dx_2\, d\eta \tag{4.2}$$

for $0 \le h \le L - x_3$. Hence, in particular, when

$$u_i(x_\alpha, L) = 0, \quad \text{or} \quad u_{i,3} + \alpha u_{j,j} \delta_{i,3} = 0 \quad \text{at} \quad x_3 = L, \tag{4.3}$$

we have $H(x_3) \le 0$, but in general $H(x_3)$ is nondecreasing and its (nonnegative) derivative is given explicitly by

$$H'(x_3) = \int_{D(x_3)} (u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j})\, dx_1\, dx_2. \tag{4.4}$$

Let us note the relationship between the function $H(x_3)$ and the corresponding integral for the work done by external forces across the cross section $D(x_3)$, defined by

$$V(x_3) = \int_{D(x_3)} \sigma_{i,3} u_i\, dx_1\, dx_2.$$

This immediately yields

$$V(x_3 + h) = V(x_3) + \int_{x_3}^{x_3+h} \int_{D(\eta)} \sigma_{ij} u_{i,j}\, dx_1\, dx_2\, d\eta.$$

But we have

$$\begin{aligned}\int_{D(\eta)} \sigma_{ij} u_{i,j}\, dx_1\, dx_2 &= H'(x_3) + \mu \int_{D(x_3)} (u_{i,j} u_{j,i} - u_{j,j} u_{k,k})\, dx_1\, dx_2 \\ &= \mu H'(x_3) + \mu \int_{D(x_3)} (u_{i,j} u_{i,j} - 2\omega_{ij}\omega_{ij} - u_{j,j} u_{k,k})\, dx_1\, dx_2 \\ &\le 2\mu H'(x_3),\end{aligned}$$

where

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}).$$

Hence, it follows that

$$V'(x_3) \le 2\mu H'(x_3),$$

and

$$V(x_2 + h) - V(x_3) \le 2\mu[H(x_3 + h) - H(x_3)].$$

Furthermore, condition $(4.3)_2$ implies that on $X_3 = L$ the stress satisfies

$$\sigma_{\alpha 3}(L) = \mu u_{3,\alpha}(L),$$
$$\sigma_{33}(L) = -\mu u_{\alpha,\alpha}(L),$$

and these stress components hence vanish only when $u_i(x_\alpha, L)$ is constant.

We now wish to compute a bound of the form

$$|H(x_3)| \leq n(x_3)H'(x_3). \tag{4.5}$$

Once (4.5) is established, it leads immediately to the inequalities

$$H(x_3) \leq n(x_3)H'(x_3) \tag{4.6}$$

and

$$-H(x_3) \leq n(x_3)H'(x_3), \tag{4.7}$$

which, on integration, yield the desired information on the behaviour of the solution as measured by $H(x_3)$. Let us first derive (4.5).

By means of Schwarz's inequality combined with inequality (2.9) we obtain

$$\begin{aligned} |H(x_3)| \leq & \left(\int_{D(x_3)} u_i u_i \, dx_1 \, dx_2 \int_{D(x_3)} u_{i,3} u_{i,3} \, dx_1 \, dx_2\right)^{1/2} \\ & + \alpha \left(\int_{D(x_3)} u_i u_i \, dx_1 \, dx_2 \int_{D(x_3)} u_{i,i} u_{j,j} \, dx_1 \, dx_2\right)^{1/2} \end{aligned} \tag{4.8}$$

$$\begin{aligned} & \leq [\lambda(x_3)]^{-1/2} \left[\int_{D(x_3)} u_{i,\beta} u_{i,\beta} \, dx_1 \, dx_2\right]^{1/2} \\ & \times \left[\left(\int_{D(x_3)} u_{i,3} u_{i,3} \, dx_1 \, dx_2\right)^{1/2} + \left(\int_{D(x_3)} u_{i,i} u_{j,j} \, dx_1 \, dx_2\right)^{1/2}\right] \end{aligned} \tag{4.9}$$

$$\begin{aligned} \leq \frac{1}{2}[\lambda(x_3)]^{-1/2} \Bigg[& (c_3 + c_4) \int_{D(x_3)} u_{i,\beta} u_{i,\beta} \, dx_1 \, dx_2 \\ & + \frac{1}{c_3} \int_{D(x_3)} u_{i,3} u_{i,3} \, dx_1 \, dx_2 \\ & + \frac{\alpha^2}{c_4} \int_{D(x_3)} u_{i,i} u_{j,j} \, dx_1 \, dx_2 \Bigg], \end{aligned} \tag{4.10}$$

where in the last line we have used the arithmetic-geometric mean inequality with positive constants c_3, c_4. On setting

$$c_4 = \alpha c_3 = \alpha(1+\alpha)^{-1/2},$$

we find that (4.10) becomes

$$|H(x_3)| \leq \frac{1}{2}[\lambda(x_3)]^{-1/2}(1+\alpha)^{1/2} H'(x_3), \tag{4.11}$$

which is of the required form (4.5) with

$$n(x_3) = \frac{1}{2}[(1+\alpha)/\lambda(x_3)]^{1/2}. \tag{4.12}$$

We have already seen that under the end-conditions (4.3), we must have $H(x_3) \le 0$, for $x_3 \in [0, L]$, and in view of the nonnegativeness of $H'(x_3)$, it follows that inequality (4.7) holds. Integration of (4.7) leads to

$$\begin{aligned} -H(x_3) &\le -H(0)\exp\left(-\int_0^{x_3} \kappa^{-1}(\eta)\,d\eta\right) \\ &= -H(0)\exp\left(-\kappa\int_0^{x_3} \lambda^{1/2}(\eta)\,d\eta\right) && (4.13) \\ &\le -H(0)\exp\left(-\omega\int_0^{x_3} A^{-1/2}(\eta)\,d\eta\right), && (4.14) \end{aligned}$$

where

$$\kappa = 2(1+\alpha)^{-1/2}, \qquad \omega = 2(1+\alpha)^{-1/2} j_0 \pi^{1/2} \tag{4.15}$$

and we have used (4.12) together with the Faber–Krahn inequality (2.10) for $\lambda(x_3)$. Since $H(L) = 0$, inequality (4.14) may be rewritten as

$$E(x_3) \le E(0)\exp\left(-\omega\int_0^{x_3} A^{-1/2}(\eta)\,d\eta\right) \tag{4.16}$$

where from (4.2),

$$E(x_3) = \int_{x_3}^{L}\int_{D(\eta)} (u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j})\,dx\,d\eta. \tag{4.17}$$

We next turn to the case of a cylinder of semi-infinite length and impose no assumption on the asymptotic behaviour of the solution as $x_3 \to \infty$. It follows as before that whenever $H(x_3)$ becomes positive at some point $t_1 \in [0, \infty)$, then $H(x_3)$ is a positive increasing function for $x_3 \ge t_1$. Indeed, in these circumstances, it follows from integration of (4.6) that

$$H(x_3) \ge H(t_1)\exp\left(\kappa\int_{t_1}^{x_3} \lambda^{1/2}(\eta)\,d\eta\right) \tag{4.18}$$

which, on using (4.2), becomes

$$I(t_1, x_3) \ge H(t_1)\left[\exp\left(\kappa\int_{t_1}^{x_3} \lambda^{1/2}(\eta)\,d\eta\right) - 1\right], \tag{4.19}$$

where

$$I(t_1, x_3) = \int_{t_1}^{x_3}\int_{D(\eta)} (u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j})\,dx_1\,dx_2\,d\eta. \tag{4.20}$$

We now suppose that

$$\lim_{x_3\to\infty}\int_{t_1}^{x_3} \lambda^{1/2}(\eta)\,d\eta = \infty \tag{4.21}$$

so that the left side of (4.18) becomes unbounded as $x_3 \to \infty$. More precisely, by means of l'Hôpital's theorem, we obtain from (4.19) the inequality

$$\lim_{x_3\to\infty} K(x_3)\exp\left(-\kappa\int_{t_1}^{x_3} \lambda^{1/2}(\eta)\,d\eta\right) \ge \kappa H(t_1) \tag{4.22}$$

where

$$K(x_3) = \dot{\lambda}^{-1/2}(x_3) \int_{D(x_3)} [u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j}]\, dx_1\, dx_2.$$

Hence, on imposing the asymptotic condition

$$\lim_{x_3 \to \infty} K(x_3) \exp\left(-\kappa \int_0^{x_3} \lambda^{1/2}(\eta)\, d\eta\right) = 0, \tag{4.23}$$

we see that with (4.21) we are led to a contradiction and accordingly (4.23) and (4.21) imply that

$$H(x_3) \le 0, \qquad x_3 \in [0, \infty). \tag{4.24}$$

Finally, on appealing to (4.7), we obtain the estimate

$$-H(x_3) \le -H(0) \exp\left(-\kappa \int_0^{x_3} \lambda^{1/2}(\eta)\, d\eta\right) \tag{4.25}$$

or

$$E(x_3) \le E(0) \exp\left(-\kappa \int_0^{x_3} \lambda^{1/2}(\eta)\, d\eta\right) \tag{4.26}$$

where now, in the definition (4.17) of $E(x_3)$, the upper limit extends to infinity. Let us note that (4.21) and (4.25) imply the boundedness of $E(0)$, which therefore does not require separate postulation.

We have thus proved the following:

PROPOSITION 4.1. For the semi-infinite cylinder, whose cross sections give rise to (4.21), the solution measured by $H(x_3)$ either violates the asymptotic behaviour indicated by (4.23) or is bounded above by a decaying exponential function of x_3 given by (4.25).

Of course, in each of the previous inequalities, $\lambda(x_3)$ may be replaced by its lower bound (2.10).

In the special case of constant cross-sectional area, we observe that the Proposition shows that $H(x_3)$ either grows at least exponentially in x_3 or it possesses at most exponential decay. When the cylinder is a frustrum of a cone so that $A(x_3) = \pi\sigma(x_3 + \gamma)^2$, where σ, γ are positive constants, then $H(x_3)$ either grows at least like $|(x_3 + \gamma)/\gamma|^{1/\sqrt{\pi\sigma}}$ or decays at most like $[\gamma/(x_3 + \gamma)]^{\omega/\sqrt{\pi\sigma}}$.

Proposition 4.1 is analogous to a Phragmèn–Lindelöf principle for the present problem and yields decay estimates similar to those obtained by Maz′ya and Plamenevskiĭ [22] who, however, use entirely different arguments and adopt a different measure for the solution.

The estimate (4.25) is not of practical use until the quantity $E(0)$ is bounded in terms of the prescribed data $u_i(x_\alpha, 0)$. For simplicity, we assume the cylinder is prismatic, and observe that by Dirichlet's principle, we have

$$E(0) \le \tilde{E}(0), \tag{4.27}$$

where $\tilde{E}(0)$ is obtained from $E(0)$ by replacing each derivative of the displacement $u_i(x_\alpha, x_3)$ by the respective derivatives of the harmonic function $h_i(x_\alpha, x_3)$ satisfying

$$\begin{aligned} h_i(x_\alpha, 0) &= u_i(x_\alpha, 0), \\ h_i(x_\alpha, x_3) &= 0, \qquad x \in \partial D \times [0, \infty), \\ \lim_{x_3 \to \infty} h_i(x_\alpha, x_3) &= 0, \qquad x \in D. \end{aligned}$$

But

$$\tilde{E}(0) \leq (1 + 3\alpha) \int_0^\infty \int_{D(\eta)} h_{i,j} h_{i,j} \, dx_1 \, dx_2 \, d\eta \tag{4.28}$$

and, on using Dirichlet's principle again, we obtain

$$\int_0^\infty \int_{D(\eta)} h_{i,j} h_{i,j} \, dx_1 \, dx_2 \, d\eta \leq \int_0^\infty \int_{D(\eta)} w_{i,j} w_{i,j} \, dx_1 \, dx_2 \, d\eta \tag{4.29}$$

where $w_i(x_\alpha, x_3)$ is any sufficiently smooth function with the same boundary values as h_i. In particular, let us set

$$w_i(x_\alpha, x_3) = u_i(x_\alpha, 0) e^{-\gamma x_3}, \tag{4.30}$$

for some positive constant γ to be chosen. On inserting (4.30) into (4.29) we find that

$$\begin{aligned} E(0) \leq (1 + 3\alpha) \bigg\{ \frac{1}{2\gamma} \int_D & u_{i,\beta}(x_\alpha, 0) u_{i,\beta}(x_\alpha, 0) \, dx_1 \, dx_2 \\ & + \frac{\gamma}{2} \int_D u_i(x_\alpha, 0) u_i(x_\alpha, 0) \, dx_1 \, dx_2 \bigg\}, \end{aligned} \tag{4.31}$$

so on selecting the optimal value for γ we conclude that

$$\begin{aligned} E(0) \leq (1 + 3\alpha) \bigg[\bigg(\int_D & u_{i,\beta}(x_\alpha, 0) u_{i,\beta}(x_\alpha, 0) \, dx_1 \, dx_2 \bigg) \\ & \times \bigg(\int_D u_i(x_\alpha, 0) u_i(x_\alpha, 0) \, dx_1 \, dx_2 \bigg) \bigg]^{1/2}, \end{aligned} \tag{4.32}$$

which is the desired estimate involving the data.

A result similar to (4.32) may be established for more general regions by means of methods presented, for example, by Bramble and Payne [3].

We employ Proposition 4.1 to obtain under appropriate conditions an estimate on the behaviour of the displacement as measured by the function $F(x_3)$ introduced in Sec. 3. We have

PROPOSITION 4.2. Let the cylinder be of semi-infinite length and let the eigenvalue $\lambda(x_3)$ satisfy

$$0 < \lambda_m = \inf_{[0,\infty)} \lambda(x_3). \tag{4.33}$$

We suppose the solution possesses the asymptotic behaviour (4.23). Then the function $F(x_3)$ defined by (3.1) with $a = 1$ satisfies the inequality

$$F(x_3) \leq \frac{1}{2} \left(\frac{1 + \alpha}{\lambda_m} \right)^{1/2} E(0) \exp \left(-\kappa \int_0^{x_3} \sqrt{\lambda(\eta)} \, d\eta \right) \tag{4.34}$$

or, in terms of $A(x_3)$ and ω defined by (4.15),

$$F(x_3) \leq 4\left(\frac{1+\alpha}{\lambda_m}\right)^{1/2} E(0)\exp\left(-\omega\int_0^{x_3} A^{-1/2}(\eta)\,d\eta\right). \tag{4.35}$$

Proof. We first show that $F(x_3)$ is bounded above by $H'(x_3)$. Thus, with $F(x_3)$ defined by

$$F(x_3) = \int_{D(x_3)} u_i u_i \, dx_1\, dx_2, \tag{4.36}$$

it follows from (2.9) that

$$\begin{aligned} F(x_3) &\leq \lambda^{-1}(x_3)\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,dx_1\,dx_2 \\ &\leq \lambda^{-1}(x_3)\int_{D(x_3)} (u_{i,j}u_{i,j} + \alpha u_{i,i}u_{j,j})\,dx_1\,dx_2 \qquad (4.37)\\ &\leq \lambda^{-1}(x_3)(1+\alpha)H'(x_3) \qquad (4.38)\end{aligned}$$

where the last two inequalities rely upon the positivity of α and $H'(x_3)$ is given by (4.4).

We next relate $F(x_3)$ with $H(x_3)$. Now, we conclude from (4.1) that

$$H(x_3) = \frac{1}{2}F'(x_3) + \alpha\int_{D(x_3)} u_{j,j}u_3\,dx_1\,dx_2,$$

so that by Schwarz's inequality and (2.9) we have

$$\begin{aligned} H(x_3) &\leq \frac{1}{2}F'(x_3) + \alpha\lambda^{-1/2}(x_3)\left(\int_{D(x_3)} u_{i,i}u_{j,j}\,dx_1\,dx_2\right)^{1/2}\left(\int_{D(x_3)} u_{3,\alpha}u_{3,\alpha}\,dx_1\,dx_2\right)^{1/2} \\ &\leq \frac{1}{2}F'(x_3) + \lambda^{-1/2}(x_3)\left[\frac{\alpha^2}{2c}\int_{D(x_3)} u_{i,i}u_{j,j}\,dx_1\,dx_2 + \frac{c}{2}\int_{D(x_3)} u_{3,i}u_{3,i}\,dx_1\,dx_2\right],\end{aligned}$$

where the arithmetic-geometric mean inequality has been employed with $c \geq 0$. On taking $c = \alpha(1+\alpha)^{-1/2}$, it then follows that

$$\begin{aligned} H(x_3) &\leq \frac{1}{2}F'(x_3) + \frac{1}{2}\lambda^{-1/2}(x_3)\left[\alpha(1+\alpha)^{1/2}\int_{D(x_3)} u_{i,i}u_{j,j}\,dx_1\,dx_2 \right.\\ &\qquad\qquad \left. + \alpha(1+\alpha)^{-1/2}\int_{D(x_3)} u_{i,j}u_{i,j}\,dx_1\,dx_2\right] \qquad (4.39)\\ &\leq \frac{1}{2}F'(x_3) + \frac{1}{2}(1+\alpha)^{1/2}\lambda^{-1/2}(x_3)H'(x_3).\end{aligned}$$

where again we have appealed to the positivity of α.

With the help of (4.38), we may express (4.39) as

$$\begin{aligned}0 &\le \frac{1}{8}F'(x_3) + \frac{1}{8}(1+\alpha)^{1/2}\lambda^{-1/2}(x_3)H'(x_3) - \frac{1}{4}H(x_3)\\ &\le \frac{1}{8}(F'(x_3) - 2(1+\alpha)^{-1/2}\lambda^{1/2}(x_3)F(x_3))\\ &\quad + \frac{3}{8}(1+\alpha)^{1/2}\lambda^{-1/2}(x_3)H'(x_3) - \frac{1}{4}H(x_3)\\ &\le \frac{1}{8}(F'(x_3) - 2(1+\alpha)^{-1/2}\lambda^{1/2}(x_3)F(x_3))\\ &\quad + \frac{1}{2}(1+\alpha)^{1/2}\lambda^{-1/2}(x_3)[H'(x_3) - 2(1+\alpha)^{-1/2}\lambda^{1/2}(x_3)H(x_3)], \end{aligned} \tag{4.40}$$

since by hypothesis $H(x_3)$ is negative and we already know from (4.4) that $H'(x_3)$ is nonnegative. Finally, on recalling (4.33), we may write (4.40) in the form

$$\begin{aligned}0 &\le \left[\exp\left(-2(1+\alpha)^{-1/2}\int_0^{x_3}\sqrt{\lambda(\eta)}\,d\eta\right)F(x_3)\right]'\\ &\quad + 4(1+\alpha)^{1/2}\lambda_m^{-1/2}\left[\exp\left(-2(1+\alpha)^{-1/2}\int_0^{x_3}\sqrt{\lambda(\eta)}\,d\eta\right)H(x_3)\right]',\end{aligned}$$

and then integrate the last inequality over $[x_3, \infty)$ to obtain

$$F(x_3) \le -4(1+\alpha)^{1/2}\lambda_m^{-1/2}H(x_3), \tag{4.41}$$

where the terms at infinity vanish by virtue of (4.36), (4.23), (4.25). The result now follows on using (4.26).

We observe that replacement of the asymptotic condition (4.23) by the stronger requirement that

$$\lim_{x_3\to\infty} F(x_3) = 0,$$

enables a shorter proof to be given of Proposition 4.2. Thus, on using (4.17), we have

$$\begin{aligned}F(x_3) &\le 2\left|\int_{x_3}^{\infty}\int_{D(\eta)} u_i u_{i,3}\,dx_1\,dx_2\,d\eta\right|\\ &\le 2\lambda_m^{-1/2}\left[\int_{x_3}^{\infty}\int_{D(\eta)} u_{i,\alpha}u_{i,\alpha}\,dx_1\,dx_2\,d\eta\int_{x_3}^{\infty}\int_{D(\eta)} u_{i,3}u_{i,3}\,dx_1\,dx_2\,d\eta\right]^{1/2}\\ &\le 2\lambda^{-1/2}E(x_3),\end{aligned}$$

and the result again follows from (4.26).

It is possible to derive decay estimates for $F(x_3)$ without requiring $\lambda(x_3)$ to satisfy condition (4.33). However, $\lambda(x_3)$ must have restricted asymptotic behaviour and be monotonically decreasing. When $\lambda(x_3)$ is monotonically increasing, the argument of Proposition 4.2 remains applicable provided $\lambda(0) > 0$, since in this case we have $\lambda(0) \le \lambda(x_3)$.

The proof of the decay estimate is based on the following lemmas, the first of which does not require monotonicity.

LEMMA 4.3. Let us suppose that (4.21) and (4.23) hold together with

$$\lim_{x_3\to\infty}\left\{\lambda^{-1/2}(x_3)\exp\left(-\kappa\int_0^{x_3}\lambda^{1/2}(\eta)\,d\eta\right)\right\}=0. \tag{4.42}$$

Then, as $x_3\to\infty$,

$$\text{(i) } H'(x_3)=O\left(\lambda^{1/2}(x_3)\exp\left(-\kappa\int_0^{x_3}\lambda^{1/2}(\eta)\,d\eta\right)\right), \tag{4.43}$$

$$\text{(ii) } F(x_3)=O\left(\lambda^{1/2}(x_3)\exp\left(-\kappa\int_0^{x_3}\lambda^{1/2}(\eta)\,d\eta\right)\right), \tag{4.44}$$

$$\text{(iii) } \lim_{x_3\to\infty}F(x_3)=0, \tag{4.45}$$

where $H'(x_3)$ and $F(x_3)$ are given by (4.4) and (4.36) respectively.

Proof. We have seen that (4.21) and (4.23) imply inequality (4.26) which by l'Hôpital's theorem in turn implies

$$\lim_{x_3\to\infty}\left\{\kappa^{-1}\lambda^{-1/2}(x_3)H'(x_3)\exp\left(-\kappa\int_0^{x_3}\lambda^{1/2}(\eta)\,d\eta\right)\right\}\le E(0),$$

and hence (4.43) is established. The relation (4.44) now follows on appeal to (4.37), and then (4.45) is immediate on noting (4.42).

LEMMA 4.4. Let us suppose that (4.21), (4.23), and (4.42) hold and that additionally $\lambda(x_3)$ satisfies

$$\text{(i) } \lambda'(x_3) \text{ for } x_3\ge Z>0, \text{ where } Z \text{ is constant}, \tag{4.46}$$

$$\text{(ii) } \lim_{x_3\to\infty}\left\{x_3^{1+\varepsilon}\exp\left(-\kappa\int_0^{x_3}\lambda^{1/2}(\eta)\,d\eta\right)=0 \text{ for any } \varepsilon>0\,. \tag{4.47}$$

Then the function $P(x_3)$, defined by

$$P(x_3)=\int_{x_3}^{\infty}\int_{D(\eta)}\lambda^{-1/2}(\eta)H'(\eta)\,d\eta, \tag{4.48}$$

satisfies the inequality

$$P(x_3)\le\kappa E(0)\int_{x_3}^{\infty}\exp\left(-\kappa\int_0^{\tau}\lambda^{1/2}(\eta)\,d\eta\right)d\tau \tag{4.49}$$

for $x_3\ge Z$, and consequently decays for sufficiently large x_3.

Proof. Inequality (4.25), which is valid by virtue of (4.21) and (4.23), implies that $H(x_3)\to 0$ as $x_3\to\infty$. Hence, an integration by parts in (4.48) enables us to write

$$P(x_3)=-\lambda^{-1/2}(x_3)H(x_3)-\int_{x_3}^{\infty}\int_{D(\eta)}[\lambda^{-1/2}]'H(\eta)\,dx_1\,dx_2\,d\eta, \tag{4.50}$$

from which (4.49) follows on using (4.26) and a further integration by parts. Condition (4.47) is sufficient to guarantee decay of $P(x_3)$.

A decay estimate for $F(x_3)$ may now be easily established. We have

Proposition 4.5. Under the conditions of Lemma 4.4 it follows that

$$F(x_3) \leq 2\kappa E(0) \int_{x_3}^{\infty} \exp\left(-\kappa \int_0^{\tau} \lambda^{1/2}(\eta)\, d\eta\right) d\tau, \qquad x_3 \geq Z. \tag{4.51}$$

Proof. We have from (4.45) and Schwarz's inequality that

$$\begin{aligned} F(x_3) &= -2\int_{x_3}^{\infty} \int_{D(\eta)} u_i u_{i,\eta}\, dx_1\, dx_2\, d\eta \\ &\leq 2\left[\int_{x_3}^{\infty} \int_{D(\eta)} \lambda^{-1/2} u_{i,\eta} u_{i,\eta}\, dx_1\, dx_2\, d\eta \int_{x_3}^{\infty} \int_{D(\eta)} \lambda^{1/2} u_i u_i\, dx_1\, dx_2\, d\eta\right]^{1/2} \\ &\leq P(x_3), \end{aligned}$$

where the last inequality follows on using (2.9). On combining with (4.49) we arrive at (4.51).

As a simple illustration of these results, we consider the circular cone given by $(x_1^2 + x_2^2)^{1/2} = \beta x_3$, where β is a positive constant. Then $\lambda(x_3) = j_0^2 \beta^{-2} x_3^{-2}$ and so (4.21) is satisfied while (4.42) and (4.47) require $\beta < j_0 \kappa$. Then (4.51) reduces to

$$F(x_3) \leq 2\kappa E(0) \beta [\kappa j_0 - \beta]^{-1} x_3^{-(\kappa j_0 - 1)\beta^{-1}},$$

valid for $x_3 > 0$.

Remark 4.1. The previous analysis can be easily modified to treat an analogous dynamical problem in which the plane end $x_3 = 0$ of the cylinder is subjected to a displacement of the form $g_i(x_\alpha) e^{i\Omega t}$ where Ω is real and t denotes time.

The steady state displacement field $u_i(x_\alpha, x_3)$, apart from the exponential time factor $e^{i|\Omega|t}$, is easily seen to satisfy the system

$$u_{i,jj} + \alpha u_{j,ji} + (\rho \Omega^2/\mu) u_i = 0, \qquad x \in B, \tag{4.52}$$

with

$$u_i(x_\alpha, 0) = g_i(x_\alpha), \qquad x_\alpha \in D(0), \tag{4.53}$$

$$u_i(x_\alpha, x_3) = 0, \qquad x \in (\partial D(x_3),\ x_3 \in (0, L)) \cup D(L), \tag{4.54}$$

where ρ is the constant density and μ the rigidity modulus. A more detailed discussion of such a dynamical system, including damping effects, is given by Flavin and Knops [8].

On defining $H(x_3)$ as in (4.1) it can easily be shown that

$$H(x_3) = H(0) + \int_0^{x_3} \int_{D(\eta)} [u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j} - (\rho \Omega^2/\mu) u_i u_i]\, dx_1\, dx_2\, d\eta. \tag{4.55}$$

We suppose that the excitation frequency Ω satisfies

$$1 - \rho \Omega^2 (\mu \lambda(x_3))^{-1} \geq 0, \tag{4.56}$$

or

$$1 - \rho \Omega^2 (\mu \lambda_m)^{-1} \geq 0, \tag{4.57}$$

and then, on noting from (2.9) that

$$\int_{D(x_3)} u_i u_i\, dx_1\, dx_2 \leq \lambda^{-1}(x_3) \int_{D(x_3)} (u_{i,j} u_{i,j} + \alpha u_{i,i} u_{j,j})\, dx_1\, dx_2, \tag{4.58}$$

we see that the second term on the left in (4.55) is positive-definite. Furthermore, provided (4.56) holds, it follows that

$$\int_0^{x_3}\int_{D(\eta)}(u_{i,j}u_{i,j}+\alpha u_{i,i}u_{j,j})\,dx_1\,dx_2\,d\eta$$
$$\leq [1-\rho\Omega^2(\mu\lambda_m)^{-1}]^{-1}\int_0^{x_3}\int_{D(\eta)}(u_{i,j}u_{i,j}+\alpha u_{i,i}u_{j,j}-\rho\Omega^2\mu^{-1}u_iu_i)\,dx_1\,dx_2\,d\eta. \tag{4.59}$$

A repetition of the previous analysis, utilizing (4.52) and subject to (4.56) or (4.57), then leads to the estimate

$$E(x_3)\leq E(0)\exp\left\{-\omega\int_0^{x_3}\left[1-\frac{\rho\Omega^2 A(\eta)}{\pi j_0^2}\right]A^{-1/2}(\eta)\,d\eta\right\} \tag{4.60}$$

where the Faber–Krahn inequality (2.10) has been used and once again $E(x_3) = -H(x_3)$. The estimate (4.60) continues to be valid for a semi-infinite cylinder provided that the condition stipulating the vanishing of $u_i(x_\alpha, x_3)$ at $x_3 = L$ is replaced by an asymptotic condition corresponding to (4.23)

It is clear that the upper bound for Ω implied by (4.56) is a lower bound for the "cut-off" frequency. Explicit upper bounds for $E(0)$ may be derived along lines already discussed using a modified version of Dirichlet's principle.

5. The axisymmetric problem. In this final section, we consider the axisymmetric form of the boundary value problem discussed in Secs. 3 and 4 and obtain decay estimates for a different cross-sectional mean-square measure involving only derivatives of the displacements. The reason for restricting attention to axisymmetry is that the appropriate Sobolev embedding inequalities enable pointwise estimates to be derived for the displacement which are valid up to the boundary. A similar approach has been adopted by Flavin and Knops [9] for the corresponding two-dimensional problem.

We consider a homogeneous isotropic compressible elastic material occupying a region of revolution R. It is assumed that R is either hollow or solid and that it has plane annular circular ends S_0 and S_L, separated by a distance L. We introduce cylindrical polar coordinates (r,θ,z) with origin 0 in the end S_0 and such that the z-axis coincides with the axis of symmetry of the body. We suppose that R is given by

$$R=\{(r,\theta,z)\colon r_1(z)<r<r_2(z),\ 0\leq\theta\leq 2\pi,\ 0<z<L\},$$

so that when R is hollow we have $r_1(z)>0$, while when R is solid we put $r_1(z) = 0$. The functions $r_\gamma(z)$, $\gamma = 1,2$, describing the shape of the curved boundary are assumed to be twice continuously differentiable. Further restrictions are imposed later.

We suppose, as before, that the body is deformed by specified displacements on the plane ends S_0, S_L, and that on the lateral surfaces the displacement is zero. On denoting the radial and axial components of the displacement by $u(r,z)$ and $w(r,z)$ respectively, we further suppose that, for the solid body, we have

$$u(r,z)=O(r),\qquad w_r(r,z)=O(r)\quad \text{as } r\to 0 \tag{5.1}$$

where a subscript indicates partial differentiation.

We recall that the axisymmetric forms of the equilibrium equations (2.1) when expressed in cylindrical polar coordinates become

$$(1+\alpha)[r^{-1}(ru)_r]_r + u_{zz} + \alpha w_{rz} = 0, \tag{5.2}$$

$$\alpha r^{-1}(ru)_{rz} + r^{-1}(rw_r)_r + (1+\alpha)w_{zz} = 0, \tag{5.3}$$

where α, given by (2.5), is assumed positive. We are again neglecting body-force and assuming the existence of a classical solution.

We now mention a lemma which plays a central role in the analysis.

LEMMA 5.1. Let $p(r,z) \in C^3(R)$ and let $p(r,z)$ vanish on the lateral surface of R. Then the function $P(z)$, defined by

$$P(z) = \int_{r_1(z)}^{r_2(z)} r p_r^2 \, dr, \tag{5.4}$$

where the integral is over the straight line in a semi-meridianal cross section, satisfies

$$P'(z) = 2\int_{r_1}^{r_2(z)} r p_r p_r' \, dr + [r^2 p_r^2 r']_{r_1(z)}^{r_2(z)}, \tag{5.5}$$

$$P''(z) = 2\int_{r_1(z)}^{r_2(z)} (r p_r'^2 - (rp_r)_r p'')\, dr - [p_r^2(rr'' - r'^2)]_{r_1(z)}^{r_2(z)}. \tag{5.6}$$

Here, a superposed prime indicates partial differentiation with respect to the variable z.

The proof of the lemma follows immediately from the elementary identities:

$$\begin{aligned} \frac{d}{dz} g(r(z), z) &= \frac{\partial g}{\partial z}(r,z) + \frac{\partial g}{\partial r} r', \\ \frac{d}{dz}\int_{r_1(z)}^{r_2(z)} \phi(r,z)\, dr &= \int_{r_1(z)}^{r_2(z)} \phi'(r,z)\, dr + [\phi(r,z) r'(z)]_{r_1(z)}^{r_2(z)}. \end{aligned} \tag{5.7}$$

We now consider the behaviour of the function $Q(z)$ defined by

$$Q(z) = \int_{r_1(z)}^{r_2(z)} (r^{-1}[(ru)_r]^2 + brw_r^2)\, dr, \tag{5.8}$$

where b is a positive constant to be chosen later. An application of Lemma 5.1 together with some easy manipulations yields

$$Q''(z) = 2\int \{r^{-1}[(ru)_r']^2 - [r^{-1}(ru)_r]_r ru'' + b[rw_r'^2 - (rw_r)_r w'']\}\, dr - [A(r,z)]_{r_1(z)}^{r_2(z)} \tag{5.9}$$

where

$$A(r,z) = u_r^2(rr'' + r'^2) + bw_r^2(rr'' - r'^2), \tag{5.10}$$

and henceforth the limits of integration are implied.

An appeal to (5.2), (5.3), then reduces (5.9) to

$$\begin{aligned} Q''(z) = 2\int \{ & r^{-1}[(ru)_r']^2\, dr + (1+\alpha) r([r^{-1}(ru)_r]_r)^2 \\ & + \alpha r[r^{-1}(ru)_r]_r w_r' + brw_r'^2 \\ & + (1+\alpha)^{-1} b r^{-1}([(rw_r)_r]^2 + \alpha (rw_r)_r (ru)_r')\}\, dr - [A(r,z)]_{r_1(z)}^{r_2(z)}. \end{aligned} \tag{5.11}$$

Successive use of the arithmetic-geometric mean inequality then gives

$$
\begin{aligned}
Q''(z) \geq (2 - b\alpha(1+\alpha)^{-1}c_5^{-1}) \int r^{-1}[(ru)'_r]^2\,dr + (2b - \alpha c_6^{-1}) \int r w'^2_r\,dr \\
+ (2(1+\alpha) - \alpha c_6) \int r([r^{-1}(ru)_r]_r)^2\,dr \\
+ (1+\alpha)^{-1}(2 - \alpha c_5) b \int r^{-1}[(rw_r)_r]^2\,dr - [A(r,z)]_{r_1(z)}^{r_2(z)}
\end{aligned}
\tag{5.12}
$$

for arbitrary positive constants c_5, c_6. Let us set

$$
c_5 = \frac{1}{2}\alpha b(1+\alpha)^{-1}, \qquad c_6 = \frac{\alpha}{2b}. \tag{5.13}
$$

Then the right side of (5.12) is positive-definite in the integral terms provided

$$
\frac{\alpha^2}{4(1+\alpha)} < b < \frac{4(1+\alpha)}{\alpha^2} \tag{5.14}
$$

and hence

$$
0 \leq \alpha < 2(1+\sqrt{2}), \tag{5.15}
$$

which is the restriction previously derived in (3.23).

As in Sec. 3, we wish to show that $Q(z)$ satisfies a differential inequality of the type (3.2), but to achieve this we must first dispose of the term $A(r,z)$ appearing in (5.12). Accordingly, we suppose that the lateral surfaces are such that for the solid body

$$
[r_2^2(z)]'' \leq 0, \tag{5.16}
$$

while for the hollow body

$$
[r_2^2(z)]'' \leq 0, \qquad [\log r_1(z)]'' \geq 0. \tag{5.17}
$$

Restrictions on the shape of the lateral surface are to be expected since portions resembling re-entrant angles must be excluded. At such angles, the displacement gradients are likely to be singular. These last two conditions, together with (5.1) and (5.14), then show that

$$
Q''(z) \geq 0. \tag{5.18}
$$

Hence, $Q(z)$ is convex and so we may note in passing the bound,

$$
Q(z) \leq Q(0) + [Q(L) - Q(0)]z/L. \tag{5.19}
$$

However, to derive a stronger differential inequality of type (3.2), we require the following two inequalities which are immediate consequences of the standard variational inequalities.

Lemma 5.2. The displacement components $u(r,z)$, $w(r,z)$ satisfy

$$
\lambda_1 \int r^{-1}[(ru)_r]^2\,dr \leq \int \{[r^{-1}(ru)_r]_r\}^2 r\,dr, \tag{5.20}
$$

$$
\lambda_2 \int r w_r^2\,dr \leq \int r^{-1}[(rw_r)_r]^2\,dr, \tag{5.21}
$$

where in (5.20) $\lambda_1(z)$ is the lowest positive root of the equation which in the hollow region is given by

$$J_1(\lambda^{1/2}r_1(z))Y_1(\lambda^{1/2}r_2(z)) - J_1(\lambda^{1/2}r_2(z))Y_1(\lambda^{1/2}r_1(z)) = 0 \tag{5.22}$$

and in the solid region is

$$J_1(\lambda^{1/2}r_2(z)) = 0; \tag{5.23}$$

while in (5.21), $\lambda_2(z)$ is the lowest positive root of the equation which for the hollow region is given by

$$J_0(\lambda^{1/2}r_1(z))Y_0(\lambda^{1/2}r_2(z)) - J_0(\lambda^{1/2}r_2(z))Y_0(\lambda^{1/2}r_1(z)) = 0 \tag{5.24}$$

and in the solid region is given by

$$J_0(\lambda^{1/2}r_2(z)) = 0. \tag{5.25}$$

Here, J_n, Y_n denote the nth-order Bessel functions of the first and second kind.

We then have

Proposition 5.1. The function $Q(z)$, defined by (5.8) on the smooth solution to the boundary value problem (5.2), (5.3) subject to (5.1), zero lateral displacement, elastic moduli in the range (5.15), and the region whose lateral boundaries are governed by (5.16), (5.17), satisfies the inequality

$$Q''(z) \geq m^2\lambda(z)Q(z) \tag{5.26}$$

where m^2 is given by either (3.15) or (3.16), with b defined by (3.17), and

$$\lambda(z) = \min(\lambda_1(z), \lambda_2(z)). \tag{5.27}$$

Proof. Inequality (5.26) easily follows from (5.12) on using Lemma 5.2, with

$$c_5 = \frac{\alpha b}{2(1+\alpha)}, \qquad c_6 = \frac{\alpha}{2b}. \tag{5.28}$$

The choice (5.28) reduces the right side of (5.12) to a quadratic form similar to that on the right side of (3.13) which immediately leads to the stated values of m^2.

We may now discuss (5.26) along the lines of Sec. 3(b) and obtain decay estimates similar to those derived before. Thus, for example, when the displacement at $z = L$ is specified to be zero, we recover an estimate analogous to (3.26). However, the principal aim of this section is to derive pointwise estimates for the displacement components. We require the inequalities contained in the next lemma (cf. [2]).

Lemma 5.3. (i) Suppose $f(r) \in C_0^1(r_1, r_2)$, where r_1, r_2 are constants satisfying $0 < r_1 < r_2$. Then

$$f^2(\xi) \leq \frac{\log(\xi/r_1)\log(r_2/\xi)}{\log(r_2/r_1)} \int_{r_1}^{r_2} r\left(\frac{df}{dr}\right)^2 dr,$$

and

$$f^2(\xi) \leq \frac{(r_2^2-\xi^2)(\xi^2-r_1^2)}{2(r_2^2-r_1^2)\xi^2} \int_{r_1}^{r_2} \left[r\left(\frac{df}{dr}\right)^2 + r^{-1}f^2\right] dr,$$

where $r_1 < \xi < r_2$ and ξ is independent of r.

(ii) Suppose $f \in C^1(0, r_2)$, where $f(r_2) = 0$. Then for ξ independent of r, we have

$$f^2(\xi) \le \log\left(\frac{r_2}{\xi}\right) \int_{r_1}^{r_2} r \left(\frac{df}{dr}\right)^2 dr, \tag{5.29}$$

and

$$f^2(\xi) \le \frac{1}{2}\left(1 - \frac{\xi^2}{r_2^2}\right) \int_{r_1}^{r_2} \left[r \left(\frac{df}{dr}\right)^2 + r^{-1} f^2 \right] dr,$$

where $0 < \xi < r_2$.

Pointwise bounds for u, w which reflect position and which are valid up to the boundary for both the hollow and solid region, follow upon using Lemma 5.3 in conjunction with, say, the estimate analogous to (3.26), and the readily verifiable inequalities

$$\int (r u_r^2 + r^{-1} u^2)\, dr \le Q(z),$$

$$b \int r w_r^2\, dr \le Q(z).$$

In the case of the solid cylinder, however, the relevant bound for w degenerates on the axis of symmetry in accordance with (5.29).

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