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GEOMETRIC NONLINEARITY: POTENTIAL ENERGY, COMPLEMENTARY ENERGY, AND THE GAP FUNCTION*

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Abstract. Dual minimum principles for displacements and stresses are well established for linear variational problems and also for nonlinear (and monotone) constitutive laws. This paper studies the problem of geometric nonlinearity. By introducing a *gap function*, we recover complementary variational principles in the equilibrium problems of mathematical physics. When the gap function is nonnegative those become minimum principles. The theory is based on convex analysis, and the applications made here are to nonlinear mechanics.

1. Introduction. The equations of equilibrium are amazingly beautiful. They take a particular symmetric form which is repeated throughout the field equations of mathematical physics. We begin by expressing these equations in abstract terms, and at the same time identifying each variable as a familiar quantity in continuum mechanics. Then we move from material nonlinearities, which the form allows, to geometric nonlinearities, which require a modification. That modification of the equations, with its associated dual-complementary minimum principles, is the main goal of this paper.

The first unknown u belongs to the "configuration space" $\mathbf{U}$. In mechanics u represents the displacement; in physics it is the potential. The space $\mathbf{U}$ of all possible displacements or all possible potentials is the admissible space in minimizing potential energy, and it is paired with its dual, the source space $\mathbf{U}^*$. The pairing between a displacement u in $\mathbf{U}$ and a load u^* in $\mathbf{U}^*$ is given by an expression (u, u^*) that is linear in each variable. In applications u^* may specify a body force b in a domain Ω, and also a surface traction t on part of the boundary. Then $(u, u^*) = \int u_i b_i d\Omega + \int u_i t_i \, d\Gamma$ represents the *external work*.

An important mapping goes from $\mathbf{U}$ to a second space $\mathbf{E}$—the space of elongations or strains or potential gradients. That is the *geometric mapping* A, which we assume for the moment to be linear. It takes displacement to strain, u to $e = Au$. Then these elements e in $\mathbf{E}$ are paired with the elements e^* in the dual space $\mathbf{E}^*$ (the stress

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space) by a second bilinear form $\langle e, e^* \rangle$. That represents the *internal work*, and looks like $\int e_{ij} e^*_{ij} \, d\Omega$.

Because A is linear from $\mathbf{U}$ to $\mathbf{E}$, it leads automatically to another linear map from stresses in $\mathbf{E}^*$ to loads in $\mathbf{U}^*$. That is the adjoint A^*, the equilibrium mapping which is determined by the requirement

$$\langle Au, e^* \rangle = (u, A^* e^*) \quad \forall u \in \mathbf{U} \quad \text{and} \quad \forall e^* \in \mathbf{E}^*. \tag{1}$$

This relation is the continuum analogue of transposing a matrix. When the geometric mapping A is gradient-like, the equilibrium mapping A^* is divergence-like. Equation (1) becomes the Gauss–Green formula for integration by parts, and for brevity we refer to it as Gauss' law.

The final step is a *material mapping* from $\mathbf{E}$ to $\mathbf{E}^*$. It expresses the constitutive law $e^* = C(e)$. For reasonable materials C is a monotone operator, in which case $\langle e_1 - e_2, C(e_1) - C(e_2) \rangle$ is never negative. (Otherwise we have softening of the material, and instability.) For a linear material, C is positive definite—given by Hooke's law or Ohm's law.

These three mappings combine to yield the equilibrium equation. The external load u^* is prescribed in $\mathbf{U}^*$. Then (still in the case of linear A) the governing equations are

$$e = Au, \qquad e^* = C(e), \qquad u^* = A^* e^*. \tag{2}$$

They represent the connection of $\mathbf{U}$ to $\mathbf{E}$ to $\mathbf{E}^*$ to $\mathbf{U}^*$, and together they give a single equation for the unknown u:

$$A^* C(Au) = u^*. \tag{3}$$

That could reasonably be called the *fundamental equation of equilibrium* [1]. An equivalent statement is given by the principle of virtual work—which is the "weak form" of (3). Both sides are paired with an arbitrary v, a virtual displacement. Transforming the left side by Gauss' law yields an equation between internal and external work:

$$\langle Av, C(Au) \rangle = (v, u^*). \tag{4}$$

This weak form is to hold for all v in $\mathbf{U}$, and under suitable assumptions it leads back to the strong form (3). (That is the fundamental lemma in the calculus of variations.) When v is restricted to lie in a subspace of test functions, and u is sought in a corresponding space of trial functions, (4) becomes a "projection" of the full equation (3). This projection is the basis of Galerkin's method.

Figure 1 shows the relationship of the four spaces. When C is monotone, or when it is linear and positive definite, the same is true of A^*CA. In applications to partial differential equations, A^*CA is an elliptic operator (in divergence form). Typical cases are $-d/dx(c(d/dx))$ for a rod with elastic stiffness c, or $d^2/dx^2(c(d^2/dx^2))$ for a beam, or $-\mathrm{div}(c \ \mathrm{grad})$ for a membrane. (With $c = 1$ we have the Laplacian, and (3) is Poisson's equation.) The space $\mathbf{U}$ may include a requirement that $u = 0$ on a part Γ_u of the boundary, and then u^* specifies the surface tractions on the complementary part Γ_t.

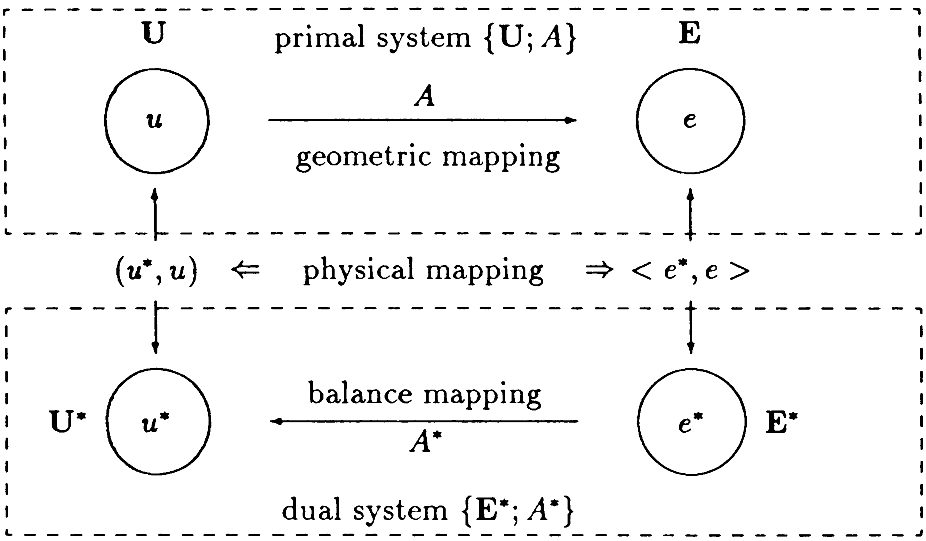

FIG. 1. General structure of physical systems.

2. Minimum principles. We stay briefly with the familiar case of a linear geometric mapping A and a (possibly) nonlinear material mapping C. In a variational problem, when u minimizes an appropriate energy, the constitutive law $e^* = C(e)$ comes from a *stored energy* (or *superpotential*) W. It is the internal strain energy associated with e, and C is its derivative:

$$e^* = C(e) \quad \text{becomes } e^* \in \partial W(e). \tag{5}$$

The latter notation correctly suggests a greater generality than the former. If the superpotential W is a smooth function, then C is its gradient. If W is convex, then C is also monotone. But beyond that, there may be "corners" in the graph of W, where the notation $\partial W(e)$ indicates the set of all possible derivatives at e—the slopes of all tangent planes that lie below the graph of W:

$$\partial W(e) = \{e^* \in \mathbf{E}^* | \langle e - \varepsilon, e^* \rangle \geq W(e) - W(\varepsilon) \ \forall \varepsilon \in \mathbf{E}\}.$$

Then e^* can be any one of those derivatives, and C is multiple-valued. That is illustrated by a rigid-perfectly plastic material in Fig. 2, in which e and e^* are scalars.

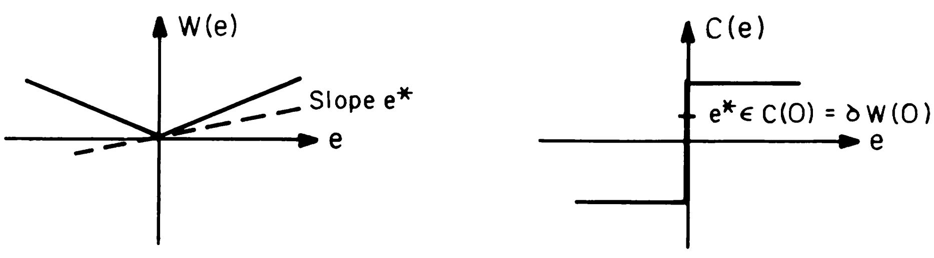

FIG. 2. Corner in W and multivalued C: rigid-plastic materials.

Another possibility is for $W(e)$ to become infinite. At such an e there is no tangent plane, and the function $C(e)$ is not defined. The "subdifferential" $\partial W(e)$ is the empty set. This takes place in the case of a locking material (Fig. 3), and we emphasize that W is still a convex function—the region above its graph is still a convex set. Furthermore W is lower semicontinuous, which means it takes the lowest available value at a jump—or more generally that $W(x) \le \liminf W(x_n)$ if $x_n \to x$.

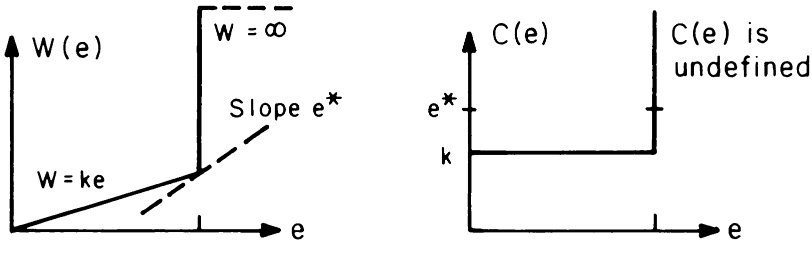

FIG. 3. Infinite values for W and no value for C: locking material.

We assume throughout that W, which maps E to the extended real numbers $\overline{R} = R \cup \{+\infty\}$, is convex and lower semicontinuous. Then its subdifferential $C = \partial W$ is maximal monotone; it is defined as widely as possible, admitting the slopes of all tangent planes below the graph W. These are the essential properties for the duality described below.

When e is a scalar, the stored energy is $W = \int C(e)\,de$. *It is the area under the graph of* C (compare Fig. 4). When e is a vector this figure moves into finite dimensions. In continuum mechanics W is the integral over the physical region of the stored energy density—which we denote by w, and which may vary from point to point if the material is not homogeneous. In this case $W(e) = \int w(e(x), x)\,dx$. A material associated with such a function is called *hyperelastic*.

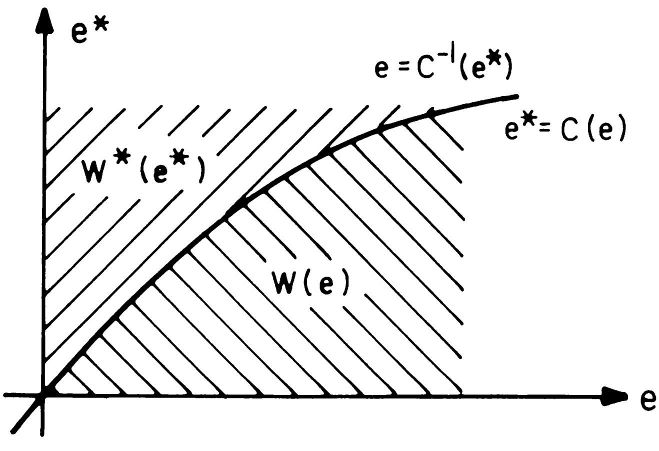

FIG. 4. The graphs of $C(e)$ and $C^{-1}(e^*)$, and the areas $W(e)$ and $W^*(e^*)$.

We should recognize a limitation on C in the middle of all this generality. Any function which arises as a gradient has an inescapable symmetry. In the linear finite-dimensional case, when W is a quadratic $\frac{1}{2}e^T Ce$ and its gradient is Ce and its second gradient (or Hessian) is C, that matrix is automatically symmetric. If we attempt to set $C_{12} = 3$ and $C_{21} = 5$, the quadratic terms $\frac{1}{2}(3e_2e_1 + 5e_1e_2)$ combine into $4e_1e_2$ and the second derivatives $C_{12} = C_{21} = 4$ override the asymmetry. In the nonlinear case this is a consequence of $\partial^2 W/\partial e_i \partial e_j = \partial^2 W/\partial e_j \partial e_i$. In the applications it follows that the tangent stiffness matrix—the discrete form of the tangent stiffness map $A^* C'(e)A$—is again symmetric. It is in this context, controlled by the presence of a superpotential W, that we turn to the dual problem.

Figure 4 suggests the fundamental construction which underlies duality. On the left we see not only the constitutive law $e^* = C(e)$, but also its inverse $e = C^{-1}(e^*)$. Looking at the graph sideways, C^{-1} is also a monotone function. The area under its graph is therefore convex, and it is $W^*(e^*)$—the *complementary superpotential.* At any point on the curve the two areas add to the area of the rectangle:

$$W(e) + W^*(e^*) = ee^*. \tag{6}$$

At points not on the curve, as in the second figure, this becomes an inequality: always

$$W(e) + W^*(e^*) \geq ee^*. \tag{7}$$

The areas now include an extra piece above the rectangle, producing the inequality. If we fix e^*, and look at the difference $ee^* - W(e)$, Eqs. (6) and (7) say that this difference is largest *at the point on the curve*—the point where $e^* = C(e)$. At that point the difference equals $W^*(e^*)$, according to (6). Therefore, with the help of the figure, we obtain the fundamental property

$$W^*(e^*) = \max_e \{ee^* - W(e)\}. \tag{8}$$

In differentiating to find the maximum, we will certainly recover $e^* - W'(e) = 0$, or $e^* = C(e)$. The role of convexity is to assure that this stationary point is actually a maximum.

Moving beyond the scalar case in Fig. 4, this construction still applies. At each point x it produces $w^*(e^*(x), x)$, the complementary energy density, from $w(e(x), x)$. Integrating over the region, this is the link between the superpotential W and its "convex conjugate" W^*. The maximization will involve C and C^{-1}, when we differentiate, but the definition of W^* does not. The step from W to W^* is the key to duality, and it is the *Legendre–Fenchel transformation:*

$$W^*(e^*) = \sup_e \{\langle e, e^* \rangle - W(e)\}. \tag{9}$$

The transform of a quadratic energy $W = \frac{1}{2}e^T Ce$ is its complementary energy $W^* = \frac{1}{2}(e^*)^T C^{-1} e^*$. The transform of $W = |e|^p/p$, corresponding to the power law $C(e) = e^{p-1}$, is the energy $W^* = |e^*|^q/q$ associated with the inverse power law. (Here $q = \frac{1}{p-1} + 1$, or $p^{-1} + q^{-1} = 1$.) The transform of a Lagrangian is a Hamiltonian. And a repetition of the Legendre transform brings back W:

$$(W^*)^*(e) = \sup_{e^*} \{\langle e, e^* \rangle - W^*(e^*)\} = W(e). \tag{10}$$

We refer to [2] and [3] for a deeper presentation of convex analysis, and to [4] for a particularly neat statement of variational principles in nonlinear elasticity. The following equations, or more properly inclusions, summarize the *equivalent statements* of the monotone and possibly multiple-valued constitutive law:

$$\begin{aligned} e^* &\in \partial W(e) \\ e &\in \partial W^*(e^*) \\ \langle e, e^* \rangle &= W(e) + W^*(e^*). \end{aligned} \tag{11}$$

Those describe the internal properties of the system.

A similar pattern can be established for the external properties. Formally, it has the advantage of specifying constraints on u in a systematic way: the external superpotential can be $F(u) = 0$ when the constraints are satisfied, and $F(u) = +\infty$ when u is not admissible. The source term u^* can come from the convex conjugate F^*—which in many applications is simply linear. The external relations, with the sign convention of convex analysis that leads to a minimum principle, become

$$\begin{aligned} -u^* &\in \partial F(u) \\ u &\in \partial F^*(-u^*) \\ (u, -u^*) &= F(u) + F^*(-u^*). \end{aligned} \tag{12}$$

For us the main advantage is to give consistent expressions for the *total potential energy*

$$\Pi(u, e) = W(e) + F(u) \tag{13}$$

and the *total complementary energy*

$$\Pi^*(-u^*, e^*) = W^*(e^*) + F^*(-u^*). \tag{14}$$

Those are still conjugate. The Legendre–Fenchel transform takes one to the other, and repeating the transform returns to the first. The framework is settled, and *the minimum of* Π *equals the maximum of* $-\Pi^*$—this saddle point is at the point of equilibrium. But the framework needs modification when A is nonlinear.

3. Geometric nonlinearity. Suppose the geometric mapping $e = A(u)$ is not linear. Then the construction of the adjoint A^* given by (1) requires change. We want to separate out the *linearization* of A at a point u, by computing the directional derivative (= Gateaux derivative). In each direction $v \in \mathbf{U}$, starting from the point u,

$$\delta e(u; v) = \lim_{t \to 0^+} \frac{A(u + tv) - A(u)}{t} = T(u)v. \tag{15}$$

This determines the *tangent geometric mapping*, defined at each point u and transforming the space $\mathbf{U}$ (containing v) linearly to the space $\mathbf{E}$ (containing δe). In the finite-dimensional case, the matrix $T(u)$ is the Jacobian of $A(u)$ at u.

We can construct the adjoint $T^*(u)$ by Gauss' law. It again maps the stress space $\mathbf{E}^*$ to the source space $\mathbf{U}^*$, but now it depends on the configuration variable u:

$$\langle T(u)v, e^* \rangle = (v, T^*(u)e^*) \quad \forall v \in \mathbf{U} \quad \text{and} \quad \forall e^* \in \mathbf{E}^*. \tag{16}$$

Since the principle of virtual work still applies at the point of equilibrium—and applies in particular to arbitrary small virtual displacements tv—*it is this linear part which enters the equilibrium mapping.* The virtual work equation (4) becomes

$$\langle T(u)v, C(A(u))\rangle = (v, u^*). \tag{17}$$

Gauss' law (16) matches this weak form with the strong form

$$T^*(u)C(A(u)) = u^*. \tag{18}$$

But the symmetry between T and T^* is lost—the nonlinear A has taken the place of T—and we turn to our principal example.

4. Nonlinear elasticity. For a finite displacement $u(x)$, the Green strain tensor is (see [6])

$$e(u) = \frac{1}{2}[D^T D - \mathbf{I}] \quad \text{with} \quad D_{ij} = \delta_{ij} + u_{ij}. \tag{19}$$

In terms of the operator ∇ or the components $u_{ij} = \partial u_i/\partial x_j$, this is

$$\begin{aligned} A(u)u &= \frac{1}{2}[\nabla u + u\nabla + (\nabla u)\cdot(u\nabla)] \\ e_{i,j} &= \frac{1}{2}[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}] \end{aligned} \tag{20}$$

This is typical of the geometric mappings we consider, which have the special form

$$A(u) = L(u)u. \tag{21}$$

The operator L is *linear*, but depends on u. It is not the tangent mapping T, because when we differentiate (21) there are two terms from $L(u)u$. In fact the linearization of (20) is easy to see directly: the tangent mapping in this example is defined by

$$T(u)v = \frac{1}{2}[(\nabla v) + (v\nabla) + (\nabla u)\cdot(v\nabla) + (\nabla v)\cdot(u\nabla)]. \tag{22}$$

The corresponding $L(u)$ is in (28) below, and we will write $N(u)$ as

$$N(u) = L(u) - T(u).$$

Now we identify a further simplifying property of the Green strain tensor (and of other important nonlinear mappings). $A(u)$ is a *quadratic mapping.* By this we mean that the derivative δT is a linear map that *does not depend* on u. Thus L is affine, a constant map plus one that is linear in u. In other words, $\delta^2 L = 0$ and the second derivative of the original $e(u)$ is a constant (and symmetric) linear map. Straightforward calculations give the basic identities which connect A, L, T, and N for quadratic mappings.

LEMMA 1. A quadratic mapping $e = A(u) = L(u)u$ satisfies

$$N(u) = -[\delta L(u)]u \tag{23}$$

$$\delta[N(u)u] = -[\delta T(u)]u \tag{24}$$

$$\delta^2 e(u; v, w) = -2N(v)w = -2N(w)v. \tag{25}$$

Proof. The derivative of $e(u) = L(u)u$ in the direction v is

$$\begin{aligned} T(u)v &= \lim_{t\to 0^+} \frac{L(u+tv)(u+tv) - L(u)u}{t} \\ &= \lim_{t\to 0^+} \frac{L(u+tv)(u+tv) - L(u)(u+tv) + L(u)(u+tv) - L(u)u}{t} \\ &= \lim_{t\to 0^+} \frac{L(u+tv) - L(u)}{t}(u+tv) + L(u)v \\ &= \delta L(u;v)u + L(u)v. \end{aligned}$$

Since $T - L = -N$, and $\delta L(u;v)$ is $[\delta L(u)]v$ for quadratic mappings, this is (23). The identity (24) comes directly from the splitting $N = L - T$:

$$\delta[N(u)u] = \delta[L(u)u - T(u)u] = T(u) - [\delta T(u)]u - T(u). \tag{26}$$

For the second derivative in (25) we need another limit as $t \to 0^+$ (and also the key property that $\delta^2 L = 0$ for quadratic maps). Briefly

$$\delta e(u;v) = [\delta L(u)v]u + L(u)v \tag{27}$$

and the derivative of this in the direction of w in $\mathbf{U}$ is

$$\begin{aligned} \delta^2 e(u;v,w) &= \lim_{t\to 0^+} \frac{\delta L(u+tw)(u+tw)v + L(u+tw)v - \delta L(u)uv - L(u)v}{t} \\ &= \lim_{t\to 0^+} \frac{\delta L(u+tw)uv - \delta L(u)uv}{t} + \lim_{t\to 0^+} \delta L(u+tw)wv \\ &\quad + \lim_{t\to 0^+} \frac{L(u+tw)v - L(u)v}{t} \\ &= 0 + \delta L(u;w)v + [\delta L(u)w]v. \end{aligned}$$

That turns into the first part of (25), and reversing v and w gives the second part—the symmetry of the second derivative N. An example is needed.

Example. For the Green strain tensor $A(u) = L(u)u$, the affine map $L(u) = [(\mathbf{I} + (u\nabla))\nabla]_{\text{sym}}$ applied to v is

$$\frac{1}{2}\left[(\nabla v) + (v\nabla) + \frac{1}{2}(\nabla u)\cdot(v\nabla) + \frac{1}{2}(\nabla v)\cdot(u\nabla)\right]. \tag{28}$$

That should be compared with the tangent map T in (22). The extra factor $\frac{1}{2}$ in (28) leaves a residual operator, which is N. In other words T was $[(\mathbf{I} + 2(u\nabla))\nabla]_{\text{sym}}$.

Since the Gauss law will be used repeatedly in what follows, we recall that $L = T + N$ and therefore $L^* = T^* + N^*$. In parallel with the law (16) for T and T^*, we have

$$\langle N(u)v, e^*\rangle = (v, N^*(u)e^*) \quad \forall v \in \mathbf{U} \quad \text{and} \quad \forall e^* \in \mathbf{E}^*. \tag{29}$$

5. Symmetry broken: The local form of the governing equations. We consider the bilinear functional $\ell = \langle e, e^*\rangle + (v, -v^*)$. Substituting the geometrical equation $e = A(v) = Lv$ or the conjugate geometrical equation $v^* = L^*\varepsilon^*$ yields the primal functional

$$\ell_p(v) = \langle L(v)v, e^*\rangle + (v, -u^*) \tag{30}$$

and the dual functional

$$\ell_d(\varepsilon^*) = \langle e, \varepsilon^* \rangle + (u, -L(u)^*\varepsilon^*). \tag{31}$$

Those functionals ℓ_p and ℓ_d, defined on $\mathbf{U}$ and $\mathbf{E}^*$, lead to the complementary virtual work principles for nonlinear systems:

PROPOSITION 1. If A is Gateaux-differentiable, then $\forall v \in \mathbf{U}$ and $\forall \varepsilon^* \in E^*$

$$\delta\ell_p(u; v) = 0 \quad \forall v \in \mathbf{U} \leftrightarrow T^*e^* - u^* = 0 \tag{32}$$

$$\delta\ell_d(e^*; \varepsilon^*) = 0 \quad \forall \varepsilon^* \in \mathbf{E}^* \leftrightarrow A(u) - e = 0. \tag{33}$$

This proposition shows that the primal-dual symmetry was broken by the nonlinearity of A.

Tonti [5] calls (33) the definition equation and (32) the balance equation—in which T^* has replaced A^*. Together they describe the "geometric" properties of the system. Combining these geometrical relations with the "physical" relations (11) and (12), the abstract governing equations may be written as

$$\begin{cases} A(u) - e = 0 \\ T^*e^* - u^* = 0 \\ e^* \in \partial W(e) \quad \text{or} \quad e \in \partial W^*(e^*) \\ -u^* \in \partial F(u) \quad \text{or} \quad u \in \partial F^*(-u^*). \end{cases} \tag{34}$$

The scheme of Fig. 5 illustrates this abstract system of governing equations.

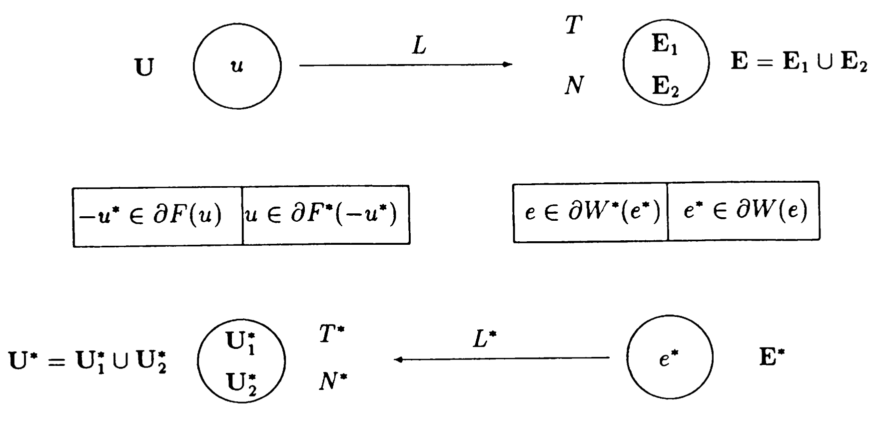

FIG. 5. Interrelations in the nonlinear systems of mathematical physics.

6. Symmetry restored: The complementary gap function. Using only $u \in \mathbf{U}$ as primal variable, the governing equations (34) may be represented by the inclusion

$$0 \in T^*\partial W(A(u)) + \partial F(u). \tag{35}$$

In the dual system, the governing equations are equivalent to the dual inclusion

$$0 \in \partial W^*(e^*) - T\partial F^*(-T^*e^*) - Nu. \tag{36}$$

To repeat the symmetry in the linear case (with $T = A$ and $N = 0$),

$$0 \in A^* \partial W(Au) + \partial F(u), \tag{37}$$

$$0 \in A \partial F^*(-A^* e^*) - \partial W^*(e^*). \tag{38}$$

To recover symmetry in the nonlinear case, we introduce a *complementary gap function* $G: \mathbf{U} \times \mathbf{E}^* \to \overline{R}$:

$$G(v, \varepsilon^*) = \langle -N(v)v, \varepsilon^* \rangle. \tag{39}$$

Using Gauss' law, the gap function may also be written as

$$G^*(v, \varepsilon^*) = (v, -N^*(v)\varepsilon^*). \tag{40}$$

Clearly $G = G^*$. This is a neutral functional, and for given $(u, e^*) \in \mathbf{U} \times \mathbf{E}^*$, it may be represented in either form

$$G_p(v) = (v, -N^*(u)e^*), \quad \text{or} \quad G_d^*(-N^*(u)\varepsilon^*) = (u, -N^*(u)\varepsilon^*). \tag{41}$$

Obviously $G_p: \mathbf{U} \to \overline{R}$ and $G_d^*: \mathbf{E}^* \to \overline{R}$ are linear. Let

$$F_c(v) := F(v) + G_p(v) \tag{42}$$

$$F_c^*(-L^*\varepsilon^*) := F^*(-T^*\varepsilon^*) + G_d^*(-N^*\varepsilon^*). \tag{43}$$

Then $F_c: \mathbf{U} \to \overline{R}$ and $F_c^*: \mathbf{E} \to \overline{R}$ are convex and lower semicontinuous. Including this gap function, the primal-dual governing inclusions become

$$0 \in L^* \partial W(Lu) + \partial F_c(u) \tag{44}$$

$$0 \in L \partial F_c^*(-L^* e^*) - \partial W^*(e^*). \tag{45}$$

With the aid of the complementary gap function G, the symmetry is recovered for nonlinear systems. We will find that this complementary gap function plays an important role in the analysis of variational problems.

Examples. In finite deformation theory, let a deformed body be contained in an open domain Ω with boundary $\Gamma = \partial\Omega$. For the Green strain tensor $\varepsilon = A(u) = L(u)u$ in (20) and the corresponding Kirchhoff stress tensor $S = \varepsilon^*$, the gap function should be

$$G(v, S) = \int_\Omega \frac{1}{2} S: [\varepsilon^T(v) \cdot \varepsilon(v)]\, d\Omega = \int_\Omega \frac{1}{2} S_{ij} v_{k,i} v_{k,j}\, d\Omega.$$

The gap function with plasticity is studied in [6]. In the one-dimensional beam bending problem, the gap function [7] involves the deflection w and axial force N_x:

$$G(w, N_x) = \int_0^l \frac{1}{2} N_x \left(\frac{\partial w}{\partial x}\right)^2 dx.$$

Closely related ideas are introduced in an important series of papers by Telega and Bielski (see [8–9]). They also extend key elements of the theory to plates and shells. We mention the books by Hanyga [10] and Panagiotopoulos [3] as valuable references to the mathematical analysis of nonlinear continuum mechanics. This subject has a long and highly developed history, and our goal is to work on one specific point of difficulty.

7. Complementary-dual variational principles. Let $\Pi: \mathbf{U} \times \mathbf{E} \to \overline{R}$ be the *total superpotential*

$$\Pi(v, \varepsilon) := W(\varepsilon) + F(v). \tag{46}$$

THEOREM 1. If $\Pi(v, A(v))$ is Gateaux-differentiable, its critical points are the solutions of the governing equations (34).

Proof. Taking the differential of $\Pi(v, A(v))$ at $u \in \mathbf{U}$ yields

$$\begin{aligned}\delta\Pi(u, A(u); \delta u) &= \langle T(u)\delta u, \delta W(A(u))\rangle + (\delta u, \delta F(u)) \\ &= (\delta u, T^*\delta W(A(u)) + \delta F(u)) \\ &= (\delta u, \delta\Pi(u, A(u))).\end{aligned}$$

Let $\mathbf{U}_s$ be the set of critical points, where $\delta\Pi(u, A(u)) = 0$ or equivalently $0 \in \partial\Pi(u, A(u))$. This is the Euler–Lagrange inclusion:

$$0 \in T^*\partial W(A(u)) + \partial F(u). \tag{47}$$

So $u \in \mathbf{U}_s$ solves the abstract problem (34).

We write P for the superpotential

$$P(v) = \Pi(v, A(v)). \tag{48}$$

THEOREM 2. Suppose A is quadratic and u is a critical point of P. If the gap function is nonnegative,

$$G(v, e^*(u)) \geq 0 \quad \forall v \in \mathbf{U}, \tag{49}$$

then the abstract problem (34) is equivalent to the optimization problem

$$(\mathrm{P}) \qquad P(u) = \inf_{v \in \mathbf{U}} P(v). \tag{50}$$

Moreover, if $\mathbf{U}_a$ is a bounded subset of a reflexive Banach space $\mathbf{U}$, then the problem (50) has at least one solution over $\mathbf{U}_a$. The solution is unique if G is strictly positive.

Proof. According to the definition of the subdifferential, for any critical point u and associated $e = A(u)$, the physical relations $-u^* \in \partial F(u)$ and $e^* \in \partial W(e)$ lead to the variational inequality

$$W(\varepsilon) + F(v) - W(e(u)) - F(u) \geq \langle \varepsilon - e(u), e^*(u)\rangle + (v - u, -u^*(u)) \quad \forall (v, \varepsilon) \in \mathbf{U} \times \mathbf{E}. \tag{51}$$

If $\varepsilon = A(v)$, we have

$$\Pi(v, A(v)) - \Pi(u, A(u)) \geq \ell_p(v) - \ell_p(u) \quad \forall v \in \mathbf{U}. \tag{52}$$

Let $v = u + w$, and apply Taylor's formula and the lemma for quadratic maps:

$$\varepsilon(u + w) = \varepsilon(u) + \delta\varepsilon(u)w + \frac{1}{2}\delta^2\varepsilon(u)w^2 = A(u)u + T(u)w - N(w)w. \tag{53}$$

Then

$$\ell_p(v) - \ell_p(u) = (w, T^*e^*(u) - u^*(u)) + \langle -N(w)w, e^*(u)\rangle = G(w, e^*(u)). \tag{54}$$

From the nonnegativity of the gap function—the critical assumption for stability—it follows that $P(v) - P(u) \geq 0$ for all v. Thus u is minimizing, and $P: \mathbf{U} \to \overline{R}$ is convex and lower semicontinuous. Since $\mathbf{U}_a \subset \mathbf{U}$ is bounded, the theory of convex analysis (cf. [2]) assures that the minimization problem (50) has at least one solution over $\mathbf{U}_a$. If $G(v, e^*(u)) > 0\ \forall v \in \mathbf{U}$, then $P(v)$ is strictly convex over $\mathbf{U}$ and the problem has a unique solution.

Theorem 2 shows that for nonlinear systems, although $W\colon \mathbf{E} \to \overline{R}$ and $F\colon \mathbf{U} \to \overline{R}$ are convex, *the superpotential* $P\colon \mathbf{U} \to \overline{R}$ *may be nonconvex* (owing to the nonlinearity of the geometric operator A).

Now let us consider the dual variational problem. From the Fenchel transformation, the conjugate function of Π may be defined as $\Pi^*\colon \mathbf{U}_1^* \times \mathbf{E}^* \to \overline{R}$:

$$\begin{aligned}\Pi^*(-u^*, e^*) &:= \sup_v \sup_\varepsilon \{(v, -u^*) + \langle \varepsilon, e^* \rangle - \Pi(v, \varepsilon)\} \\ &= \sup_\varepsilon \{\langle \varepsilon, e^* \rangle - W(\varepsilon)\} + \sup_v \{(v, -u^*) - F(v)\} \\ &= W^*(e^*) + F^*(-u^*).\end{aligned} \tag{55}$$

Let $P^*\colon \mathbf{E}^* \to \overline{R}$ be the *conjugate superpotential*

$$P^*(\varepsilon^*) := -\Pi^*(-T^*(u)\varepsilon^*, \varepsilon^*) \quad \forall u \in \mathbf{U}_s. \tag{56}$$

Then we have a result that is complementary to Theorem 2.

PROPOSITION 2. Suppose $(u, e^*) \in \mathbf{U} \times \mathbf{E}^*$ are the solutions of (34) and A is quadratic. If G satisfies

$$G(u, \varepsilon^* - e^*) \le 0 \quad \forall \varepsilon^* \in \mathbf{E}^*, \tag{57}$$

then e^* solves the dual problem

$$(\mathrm{D}) \qquad P^*(e^*) = \sup_{\varepsilon^*} P^*(\varepsilon^*). \tag{58}$$

Proof. If $u \in \mathbf{U}$ and the associated $e(u)$ give a critical point of $\Pi(u, A(u))$, then the conjugate physical relations $u \in \partial F^*(-u^*)$ and $e \in \partial W^*(e^*)$ yield

$$W^*(\varepsilon^*) + F^*(-v^*) - W^*(e^*) - F^*(-u^*) \ge \langle e, \varepsilon^* - e^* \rangle + (u, -v^* + u^*) \tag{59}$$

for all ε^* in $\mathbf{E}^*$ and v^* in $\mathbf{U}^*$. If $v^* = T^*(u)\varepsilon^*$, we have

$$P^*(e^*) - P^*(\varepsilon^*) \ge -G(u, \varepsilon^* - e^*) \ge 0 \quad \forall \varepsilon^* \in \mathbf{E}^*. \tag{60}$$

This shows that e^* maximizes P^* under condition (57).

We note again that although $-P^*\colon \mathbf{E}^* \to \overline{R}$ is convex and lower semicontinuous for $u \in \mathbf{U}_s$, the solutions of the abstract problem may not maximize P^*. Hence, for a nonlinear system, *the conjugate of the total superpotential P is not the total complementary superpotential*. Let us consider the functional $\Pi_c^*\colon \mathbf{U} \times \mathbf{E}^* \to \overline{R}$:

$$\Pi_c^*(-L^*(v)\varepsilon^*, \varepsilon^*) := \Pi^*(-T^*(v)\varepsilon^*, \varepsilon^*) + G^*(-N^*(v)\varepsilon^*, v). \tag{61}$$

The corresponding complementary superpotential is

$$P_c^*(\varepsilon^*) := -\Pi_c^*(-L^*(u)\varepsilon^*, \varepsilon^*) = P^*(\varepsilon^*) - G_d^*(-N^*\varepsilon^*) \quad \forall u \in \mathbf{U}_s. \tag{62}$$

For *this* functional we obtain the *dual variational principle.*

THEOREM 3. If A is quadratic, the critical points (u, e^*) of $\Pi_c^*(-L^*(v)\varepsilon^*, \varepsilon^*)$ give the solutions of the abstract problem (34), and e^* is maximizing in the dual problem:

$$(\mathrm{D}_c) \qquad P_c^*(e^*) = \sup_{\varepsilon^* \in \mathbf{E}^*} P_c^*(\varepsilon^*). \tag{63}$$

If $\mathbf{E}^*$ is a bounded set in a reflexive Banach space, then (63) has at least one solution.

Proof. Taking the Gateaux differential of Π_c^* at (u, e^*) yields

$$\begin{aligned}\delta\Pi_c^*(-L^*(u)e^*, e^*; \delta u, \delta e^*) &= \langle \delta W(e^*), \delta e^* \rangle + (\delta_{u^*} F^*(-T^*(u)e^*), \delta(-T^*(u)e^*)) \\ &\quad + \langle \delta_{u^*} G^*(-N^*(u)e^*, u), \delta(-N^*(u)e^*) \rangle \\ &\quad + (\delta u, \delta_u G^*(-N^*(u)e^*, u)) \\ &= \langle \delta W^*(e^*) - T\delta_{u^*} F^*(-T^*(u)e^*) - Nu, \delta e^* \rangle \\ &\quad + \langle -\delta T^*(u)e^*, \delta_{u^*} F^* \rangle + \langle -\delta(N(u)u), e^* \rangle.\end{aligned}$$

From the lemma for quadratic maps, $\delta(N(u)u) = -\delta T(u)u$:

$$\begin{aligned}\delta\Pi^*(\Lambda; \delta\Lambda) &= \langle \delta W^*(e^*) - T\delta_{u^*} F^* - Nu, \delta e^* \rangle + (\delta_{u^*} F^* - u, -\delta T^*(u)e^*) \\ &= \langle \delta\Pi^*(\Lambda), \delta\Lambda \rangle.\end{aligned} \tag{64}$$

Let $\mathbf{E}_s^*$ be a subset of the critical points of P_c^*. The pair $(u, e^*) \in \mathbf{U}_s \times \mathbf{E}_s^*$ is a critical point if and only if it gives the Euler–Lagrange inclusion

$$(0, 0) \in \partial\Pi_c^*(-A^*(u)e^*, e^*). \tag{65}$$

Therefore differentiability gives the two inclusions

$$0 \in \partial W^*(e^*) - T(u)\partial F^*(-T^*(u)e^*) - N(u)u \tag{66}$$

$$0 \in u - \partial F^*(-T^*(u)e^*). \tag{67}$$

So the critical points solve the abstract problem. On the other hand, if (u, e^*) are the critical points of Π^*, then from the variational inequality (60) we have

$$\Pi^*(-T^*(u)\varepsilon^*, \varepsilon^*) + G_d^*(-N^*(u)\varepsilon^*) - \Pi^*(-T^*(u)e^*, e^*) - G_d^*(-N^*(u)e^*) \geq 0 \quad \forall \varepsilon^* \in \mathbf{E}^*. \tag{68}$$

It means that $\forall u \in \mathbf{U}_s, e^* \in \mathbf{E}_s^*$ maximizes P_c^*, and the complementary superpotential $P_c^*: \mathbf{E}^* \to \overline{R}$ is concave. If $\mathbf{E}^*$ is bounded, then the complementary optimization (63) has at least one solution.

THEOREM 4. If $G(v, \varepsilon^*) \geq 0$ for any $(v, \varepsilon^*) \in \mathbf{U} \times \mathbf{E}^*$, then the abstract problem (34) is equivalent to the following optimization problem:

$$\Pi_c^*(-L^*(u)e^*, e^*) = \inf_{v} \inf_{\varepsilon^*} \Pi_c^*(-L^*(v)\varepsilon^*, \varepsilon^*). \tag{69}$$

Proof. With $w = u - v$,

$$e(u) = e(v) + T(v)w - N(w)w = T(v)u + N(v)v - N(w)w. \tag{70}$$

Substituting into (59), we have

$$\begin{aligned}&\Pi^*(-T^*(v)\varepsilon^*, \varepsilon^*) - \Pi^*(-T^*(u)e^*, e^*) \\ &\quad \geq \langle T(v)u, \varepsilon^* \rangle + \langle N(v)v, \varepsilon^* \rangle - \langle N(w)w, \varepsilon^* \rangle \\ &\qquad - \langle T(u)u, e^* \rangle - \langle N(u)u, e^* \rangle + (u, -T^*(v)\varepsilon^* + u^*) \\ &\quad = -G^*(-N^*(v)\varepsilon^*, v) + G^*(N^*(u)e^*, u) + G(w, \varepsilon^*) \quad \forall (v, \varepsilon^*) \in \mathbf{U} \times \mathbf{E}^*.\end{aligned} \tag{71}$$

By assumption $G(w,\varepsilon^*) \geq 0$ for *any* $(w,\varepsilon^*) \in \mathbf{U} \times \mathbf{E}^*$, so $\Pi_c^*(-L^*(v)\varepsilon^*,\varepsilon^*) - \Pi_c^*(-L^*(u)e^*,e^*) \geq 0$ as we wished to prove.

Thus for gap functions of the right sign (the stable case), and for the reconstructed superpotential Π_c^*, the key results of the linear case extend to problems with geometric nonlinearity. Theorem 4 shows that Π_c^* is the true complementary superpotential of a nonlinear system. Then we have duality:

THEOREM 5. If A is quadratic and $G(v,e^*) \geq 0 \ \forall (v,e^*) \in \mathbf{U} \times \mathbf{E}_s^*$, then

$$\inf_v P(v) = \sup_{\varepsilon^*} P_c^*(\varepsilon^*) \quad v \in \mathbf{U} \quad \text{and} \quad \varepsilon^* \in \mathbf{E}^*. \tag{72}$$

Proof. According to Theorem 2, if $u \in \mathbf{U}_s$ solves (P) then

$$\begin{aligned} \inf P(v) &= P(u) = W(L(u)u) + F(u) \\ &= \langle e^*, L(u)u\rangle - W^*(e^*) + (-u^*,u) - F^*(-T^*(u)e^*) \\ &= \langle e^*, Nu\rangle - W^*(e^*) - F^*(-T^*(u)e^*) \\ &= -\langle -N^*(u)e^*, u\rangle + P^*(e^*) \\ &= P_c^*(e^*) = \sup P_c^*(\varepsilon^*). \end{aligned}$$

Theorem 5 shows when there exists a duality gap $G \neq 0$ between (P) and (P^*):

$$\inf P(v) \begin{cases} \geq \sup P^*(\varepsilon^*) & \text{if } G_d^*(-N^*\varepsilon^*) \geq 0 \\ \leq \sup P^*(\varepsilon^*) & \text{if } G_d^*(-N^*\varepsilon^*) \leq 0. \end{cases}$$

For a geometrically linear system, $G = 0$, we always have $\inf P(v) = \sup P^*(\varepsilon^*)$.

8. Hamiltonian and Lagrangian. According to the general theory of convex analysis, the Hamiltonian of problem (P) is a functional from $\mathbf{U} \times \mathbf{E}^*$ into $\overline{R}$:

$$H(v,\varepsilon^*) := \sup_{\varepsilon}\{\langle \varepsilon^*, \varepsilon\rangle - \Pi(v,\varepsilon)\} \tag{73}$$

$$H(v,*)\colon \mathbf{E}^* \to \overline{R} \quad \text{is convex and lower semicontinuous} \tag{74}$$

$$H(*,\varepsilon^*)\colon \mathbf{U} \to \overline{R} \quad \text{is concave and upper semicontinuous} \tag{75}$$

and we have the canonical Hamiltonian inclusions:

$$A(u) \in \partial_{e^*} H(u,e^*), \qquad T^*e^* \in \overline{\partial}_u H(u,e^*). \tag{76}$$

Those are equivalent to the governing equations (34), where $\overline{\partial}$ should be understood as $\overline{\partial}_u H(u,e^*) = -\partial_u(-H(u,e^*))$. The Lagrangian associated with the Hamiltonian H is a functional $L\colon \mathbf{U} \times \mathbf{E}^* \to \overline{R}$ defined as

$$L(v,\varepsilon^*) := \langle A(v), \varepsilon^*\rangle - H(v,\varepsilon^*). \tag{77}$$

By property (74)

$$L(v,*)\colon \mathbf{E}^* \to \overline{R} \quad \text{is concave and upper semicontinuous.}$$

However, the convexity of $L(*,\varepsilon^*)\colon \mathbf{U} \to \overline{R}$ will depend on the gap function G. It is easy to prove that the critical points of L are the complete solutions of (34):

$$(0,0) \in \overline{\partial} L(u,e^*) \leftrightarrow \begin{cases} 0 \in T^*(u)e^* - \overline{\partial}_u H(u,e^*) \\ 0 \in A(u) - \partial_{e^*} H(u,e^*). \end{cases} \tag{78}$$

PROPOSITION 3. The saddle point of L on $\mathbf{U} \times \mathbf{E}^*$, if A is quadratic and $G(v, e^*) \geq 0$ on $\mathbf{U} \times \mathbf{E}_s^*$, is

$$\inf_v \sup_{\varepsilon^*} L(v, \varepsilon^*) = \inf_v P(v) = \sup_{\varepsilon^*} P_c^*(\varepsilon^*). \tag{79}$$

Proof. From the definition of the Hamiltonian (73),

$$\begin{aligned}\sup_{\varepsilon^*} L(v, \varepsilon^*) &= \sup_{\varepsilon^*}\{\langle \varepsilon^*, \varepsilon(v)\rangle - \sup_{\varepsilon}\{\langle \varepsilon^*, \varepsilon\rangle - W(\varepsilon)\} + F(v)\} \\ &= W^{**}(\varepsilon(v)) + F(v).\end{aligned}$$

Since $W: \mathbf{E} \to \overline{R}$ is convex and lower semicontinuous,

$$\sup_{\varepsilon^*} L(v, \varepsilon^*) = W(\varepsilon(v)) + F(v) = P(v).$$

Taking account of Theorem 5, the proposition is proved.

Let us consider the so-called pseudo-Lagrangian $L_p: \mathbf{U} \times \mathbf{E}^* \times \mathbf{E} \to \overline{R}$:

$$L_p(v, \varepsilon^*, \varepsilon) := \langle \varepsilon^*, A(v)v - \varepsilon\rangle + \Pi(v, \varepsilon). \tag{80}$$

From the definition (73), we have

$$L(v, \varepsilon^*) = \inf_{\varepsilon} L_p(v, \varepsilon^*, \varepsilon) \quad \forall (v, \varepsilon^*) \in \mathbf{U} \times \mathbf{E}^*. \tag{81}$$

If $(u, e^*, e) \in \mathbf{U}_s \times \mathbf{E}_s^* \times \mathbf{E}_s$ are the complete solutions of (34), then they satisfy

$$L_p(u, e^*, e) = \inf_v \sup_{\varepsilon^*} \inf_{\varepsilon} L_p(v, \varepsilon^*, \varepsilon) = \inf_v \inf_{\varepsilon} \sup_{\varepsilon^*} L_p(v, \varepsilon^*, \varepsilon). \tag{82}$$

9. Applications in continuum mechanics. In finite deformation theory, the spaces $\mathbf{U}$, $\mathbf{E}$ and their conjugate spaces $\mathbf{U}^* = \mathbf{L}$, $\mathbf{E}^* = \Sigma$ denote the displacement, strain, force, and stress space, respectively. Let Ω be an open bounded connected subset of R^3, with a Lipschitz boundary $\Gamma = \Gamma_u \cup \Gamma_t$. First of all, when the nonlinear operator A was

$$A(u)u = \frac{1}{2}[\nabla u + u\nabla + (\nabla u) \cdot (u\nabla)], \tag{20}$$

then $e = A(u) \in \mathbf{E}$ was the Green strain tensor. Its conjugate variable $e^* \in \Sigma$ is the Kirchhoff stress tensor. In ordinary notation, we put $S = e^*$. The geometrical splitting of A is given by

$$T(u)u = \frac{1}{2}(\nabla u + u\nabla) + (\nabla u) \cdot (u\nabla), \qquad N(u)u = -\frac{1}{2}(\nabla u) \cdot (u\nabla). \tag{83}$$

The equilibrium equations, in this case, should be

$$T^*(u)S = \left\{ \begin{array}{ll} -((\mathbf{I} + u\nabla) \cdot S) \cdot \nabla & \text{in } \Omega \\ (\mathbf{I} + u\nabla) \cdot S \cdot n & \text{on } \Gamma \end{array} \right\} = \bar{l}, \tag{84}$$

where $\bar{l} = \bar{u}^* \in \mathbf{L}$ is the external force vector, determined by $-\bar{l} \in \partial F(u)$. Here the superpotential F is given by

$$F(u) = -\int_{\Omega} \bar{b} \cdot u d\Omega - \int_{\Gamma_t} \bar{t} \cdot u d\Gamma + \Psi_{\mathbf{U}_a}(u), \tag{85}$$

where $\bar{b}, \bar{t}$ are body force and surface traction, respectively, and $\Psi_{\mathbf{U}_a}$ is the indicator of the admissible configuration subset $\mathbf{U}_a \subset \mathbf{U}$:

$$\Psi_{\mathbf{U}_a}(u) := \left\{ \begin{array}{ll} 0 & \text{if } u \in \mathbf{U}_a; \\ +\infty & \text{otherwise.} \end{array} \right. \tag{86}$$

The admissible displacement space $\mathbf{U}_a$ is given by

$$\mathbf{U}_a = \{v \in \mathbf{U} | v = 0 \text{ on } \Gamma_u\}. \tag{87}$$

Let w be the stored energy per unit volume. The governing equations (34) for nonlinear elasticity may be written as:

$$\begin{cases} \frac{1}{2}[\nabla u + u\nabla + (\nabla u)\cdot(u\nabla)] - e = 0 \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \Gamma_u \\ ((\mathbf{I} + u\nabla)\cdot S)\cdot\nabla + \bar{b} = 0 \quad \text{in } \Omega \qquad (\mathbf{I} + u\nabla)\cdot S\cdot n - \bar{t} = 0 \quad \text{on } \Gamma_t \\ S \in \partial w(e) \quad \text{or} \quad e \in \partial w^*(S) \quad \text{in } \overline{\Omega}. \end{cases} \tag{88}$$

In this case, the total superpotential functional (48) should be written as

$$P(v) = \int_\Omega w(e(v))\, d\Omega - \int_\Omega \bar{b}\cdot v d\Omega - \int_{\Gamma_t} \bar{t}\cdot u d\Gamma + \Psi_{\mathbf{U}_a}(v). \tag{89}$$

According to Theorem 2, if the gap function is

$$G(v, S(u)) = \int_\Omega \frac{1}{2}[(\nabla v)\cdot(v\nabla)] : S(u) d\Omega \geq 0 \quad \forall v \in \mathbf{U}, \tag{90}$$

(note $\xi : \zeta = \operatorname{tr}(\xi \cdot \zeta)$), then we have the minimum potential energy principle for nonlinear elasticity:

$$P(u) = \inf_{v \in \mathbf{U}} P(v). \tag{91}$$

The total complementary superpotential (61) for nonlinear elasticity is:

$$\Pi_c^*(-L^*(v)S, S) = \int_\Omega w^*(S) d\Omega + \Psi_{\Sigma_a}(v, S) + \int_\Omega \frac{1}{2}[(\nabla v)\cdot(v\nabla)] : S d\Omega. \tag{92}$$

Here Ψ_{Σ_a} is the indicator of Σ_a, the statically admissible stress space Σ_a given by

$$\Sigma_a = \{(v, S) \in \mathbf{U} \times \mathbf{E}^* | - T^*(v)S \in \partial F(v)\}. \tag{93}$$

According to Theorem 4, if the gap function $G(v, s) \geq 0$ $\forall (v, S) \in \mathbf{U} \times \Sigma$, then we have the minimum complementary energy principle:

$$\Pi_c^*(-L^*(u)\overline{S}, \overline{S}) = \inf_{v \in \mathbf{U}} \inf_{S \in \Sigma} \Pi_c^*(-L^*(v)S, S). \tag{94}$$

On the other hand, if we take $A = \nabla$, then the definition equation should be

$$A(u)u = \nabla u = e = D - \mathbf{I} \tag{95}$$

where D is the deformation gradient. The dual variable of e is the Piola stress tensor $\tau = e^*$. In this case A is linear, $T = A$, $N = 0$, and the governing equations (34) become:

$$\begin{cases} \nabla u - (D - \mathbf{I}) = 0 \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \Gamma_u \\ \tau\cdot\nabla - \bar{b} = 0 \quad \text{in } \Omega \qquad \tau\cdot n - \bar{t} = 0 \quad \text{on } \Gamma_t \\ \tau \in \partial w(D - \mathbf{I}) \quad \text{or} \quad D - \mathbf{I} \in \partial w^*(\tau) \quad \text{in } \Omega. \end{cases} \tag{96}$$

We stress here that according to Fig. 1, the dual variable of Piola stress τ should be $D - \mathbf{I}$. But in the theory of finite deformation, some authors consider (D, τ) as dual pair, which leads to difficulty in the complementary variational problems for nonlinear elasticity. Using the Fenchel transformation, the conjugate function of $w(D-\mathbf{I})$ is given by

$$\begin{aligned} w^*(\tau) &= \sup_D\{\langle \tau, D-\mathbf{I}\rangle - w(D-\mathbf{I})\} \\ &= \sup_D\{\langle \tau, D\rangle - w(D-\mathbf{I})\} - \operatorname{tr}\tau \\ &= w_1^*(\tau) - \operatorname{tr}\tau. \end{aligned}$$

Here $w_1^*(\tau)$ is a convex, semicontinuous stored energy function, and

$$D \in \partial w_1^*(\tau). \tag{97}$$

Hence, in this case, the total complementary energy for nonlinear elasticity is

$$\Pi_c^*(-A^*\tau, \tau) = \int_\Omega [w_1^*(\tau) - \operatorname{tr}\tau] + \Psi_{\Sigma_a}(\tau). \tag{98}$$

A third application is to the rate problem. The geometric relation between velocity $\dot{u} \in \dot{\mathbf{U}}$ and strain rate tensor $\dot{e} \in \dot{\mathbf{E}}$ can be written in total Lagrangian form:

$$\dot{e} = A(u)\dot{u} = \frac{1}{2}[\nabla\dot{u} + \dot{u}\nabla + (\nabla\dot{u})\cdot(u\nabla) + (\nabla u)\cdot(\dot{u}\nabla)]. \tag{99}$$

The dual variable to $\dot{e}$ is the Kirchhoff stress rate tensor $\dot{S}$. The constitutive relations can be given by introducing the rate potential $\dot{w}$ and its conjugate function $\dot{w}^*$:

$$\dot{S} \in \partial\dot{w}(\dot{e}) \quad \text{or} \quad \dot{e} \in \partial\dot{w}^*(\dot{S}) \quad \text{in } \Omega. \tag{100}$$

In this example, $A(u)\colon \dot{\mathbf{U}} \to \dot{\mathbf{E}}$ is also linear. So its conjugate operator $A^*\colon \dot{\mathbf{E}}^* \to \dot{\mathbf{U}}^*$ is

$$A^*(u)\dot{S} = \begin{cases} -[(\mathbf{I} + u\nabla)\cdot\dot{S}]\cdot\nabla & \text{in } \Omega \\ (\mathbf{I} + u\nabla)\cdot\dot{S}\cdot n & \text{on } \Gamma_t. \end{cases} \tag{101}$$

According to the general structure of Fig. 1, the dual variable to $\dot{u}$ is

$$\dot{l}(\dot{u}, S) = \dot{u}^* = \begin{cases} \rho_0\dot{\bar{b}} + [(\dot{u}\nabla)\cdot S]\cdot\nabla & \text{in } \Omega \\ \dot{\bar{t}} - [(\dot{u}\nabla)\cdot S]\cdot n & \text{on } \Gamma_t. \end{cases} \tag{102}$$

So the superpotential $F\colon \dot{\mathbf{U}} \to \overline{R}$ in this case is

$$F(\dot{u}) = -\int_\Omega \dot{\bar{b}}\cdot\dot{u}\,d\Omega - \int_{\Gamma_t} \dot{\bar{t}}\cdot\dot{u}\,d\Gamma + \Psi_{\dot{\mathbf{U}}_a}(\dot{u}) + \int_\Omega \frac{1}{2}S\colon[(\nabla\dot{u})\cdot(\dot{u}\nabla)]\,d\Omega. \tag{103}$$

We have the total superpotential for the rate problem:

$$\Pi(\dot{u}, \dot{e}(\dot{u})) = \int_\Omega \dot{w}(\dot{e}(\dot{u}))\,d\Omega - \int_\Omega \dot{\bar{b}}\cdot\dot{u}\,d\Omega - \int_{\Gamma_t} \dot{\bar{t}}\cdot\dot{u}\,d\Gamma + \Psi_{\dot{\mathbf{U}}_a}(\dot{u}) + G(\dot{u}, S). \tag{104}$$

Its conjugate functional is

$$\Pi^*(-l(\dot{u}, \dot{S}), \dot{S}) = \int_\Omega \dot{w}^*(\dot{S})\,d\Omega + \Psi_{\dot{\Sigma}_a}(\dot{u}, \dot{S}) + G(\dot{u}, S), \tag{105}$$

in which

$$\dot{\Sigma}_a = \{(\dot{u}, \dot{S}) \in \dot{\mathbf{U}} \times \dot{\mathbf{E}}^* \mid A^*(u)\dot{S} \in \partial F(\dot{u})\}. \tag{106}$$

It is easy to find that in rate problems, extremum principles similar to Theorem 2 and 3 can be obtained when the gap function $G(\dot{u}, S)$ takes the right sign.

The last two examples show that in the analysis of mathematical physics, to establish the dual-complementary variational principles, the key step is to determine correctly the dual pairs in spaces $\mathbf{U} \times \mathbf{U}^*$ and $\mathbf{E} \times \mathbf{E}^*$. For each strain measure we need the appropriate (dual) stress measure—and the gap function requires study. This paper is a first step, by amateurs of continuum mechanics, to identify an approach in which duality survives in the presence of geometric nonlinearity.

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