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THE PRODUCT OF THREE INVERSIONS*

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1. Introduction. In a series of related papers Synge [8], Tuckerman [9], and Coxeter [4] discuss the product of three reflections in plane mirrors in Euclidean 3-space. Two cases arise depending on whether the mirrors form a prism or have a point in common. In the first case the reflections act like reflections in the lines of a Euclidean plane, and in the second case like reflections in the lines (great circles) of a sphere. This prompts us to complete the analysis by discussing the product of reflections in three lines of the hyperbolic plane.

The points and lines of spherical, Euclidean, or hyperbolic geometry can be modelled by certain of the points and circles of the Riemann sphere. In each case reflection in a line is represented by inversion in the corresponding circle. Thus it is not unexpected that strong parallels exist among the theories appropriate to the three classical geometries but it is surprising that the intricacy increases as markedly as it does from spherical and Euclidean to hyperbolic geometry. Certain invariants of inversive geometry help us to manage the considerable complexity of the hyperbolic case. On the other hand, any three circles of the Riemann sphere can be interpreted simultaneously as lines in a suitable model of one of the classical geometries and so our analysis leads to a complete understanding of products of three inversions and of related matters concerning the pencils which any three circles of inversion determine in pairs.

2. The classical geometries. In spherical, Euclidean, and hyperbolic plane geometry, any isometry is the product of at most three reflections in lines; the isometry is sense-preserving (direct) or sense-reversing (opposite) as the number of reflections is even or odd. The different types of direct isometry that can occur in a given geometry are governed by the different types of pairs of lines that can occur: in spherical geometry every pair of lines intersects and the product of two reflections must be a rotation; in Euclidean geometry, a pair of lines may either intersect or be parallel and the corresponding product of two reflections is a rotation or a translation; in hyperbolic geometry, a pair of lines may intersect, or be parallel, or admit a common perpendicular and accordingly the product of two reflections is a rotation, parallel displacement, or translation along the common perpendicular.

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The situation with respect to opposite isometries is much simpler. In all three geometries, the product of three reflections is either a single reflection or a glide where a glide is defined to be the commuting product of a single reflection in a line called the axis of the glide and the product of two reflections in a pair of lines perpendicular to this axis. The length of the glide is equal to the magnitude of the displacement of points on the axis and it follows that a glide is determined by its axis, its length, and its sense.

Traditionally, spherical glides have been called rotatory reflections and Euclidean and hyperbolic glides, glide reflections. The apparent distinction between rotatory reflections and glide reflections is scarcely warranted, however, and probably arose because the Euclidean and hyperbolic transformations each involve a "translation." But in point of fact this double use of the word translation, which is far too firmly fixed to be changed, is somewhat misleading. From the point of view of the underlying inversive geometry, it is the hyperbolic parallel displacement that is the same type of transformation as the Euclidean translation; the hyperbolic translation is a quite different type of transformation. Of course when it comes to inversive geometry, and it will do before our discussion is over, we shall be forced to admit that the three glides are also mutually distinct types of transformation. However, it remains true, and this is the essential point for now, that there are strong similarities among the glides when each is considered in its respective geometry.

Suppose now that we are given three lines a, b, c in any one of the geometries. The product of reflections in these lines abc is either a reflection in a line β or a glide whose axis is a line that we again denote by β. In the same way the related transformations bca and cab determine lines that we denote by γ and α. Because reflections are involutions, the three transformations are mutually conjugate:

$$a(abc)a = bca, \qquad b(bca)b = cab, \qquad c(cab)c = abc$$

and therefore either they all represent reflections or they all represent glides of the same length. The first case occurs if the lines a, b, c belong to a pencil, and the second case if they do not. It follows from the conjugacy relations that the corresponding mirrors or axes satisfy

$$\beta^a = \gamma, \qquad \gamma^b = \alpha, \qquad \alpha^c = \beta.$$

We shall be concerned with the way in which the Latin lines a, b, c determine the Greek lines α, β, γ and vice versa. There will be a premium on formulating results in a way that preserves symmetry and is independent of whether the geometry in question is spherical, Euclidean, or hyperbolic. For example we have already unified the notion of a glide in such a way that a glide of length 0 is a reflection. Complete information on the transformations is therefore available if we can determine the length $l = l(a, b, c)$ as well as the directed axes α, β, γ. In this connection we also note that transformations corresponding to permutations of abc other than the cyclic ones already mentioned merely represent the inverses of these transformations. For example $bac = (cab)^{-1}$ so if cab is a glide of length l along α, bac is a glide of length l along α but in the opposite sense; if $l = 0$, bac and cab are both equal to reflection in α.

3. Inversive geometry. The group generated by inversions in the circles of the Riemann sphere is the Möbius group. Any Möbius transformation is the product of at most four inversions and presently we shall list canonical forms for the conjugacy classes and describe their canonical factorizations. But to understand the way in which they are named and the ways in which all of them except the product of four inversions arise as isometries in the classical geometries, we must first say a word about pencils and bundles of circles and this is done most easily in terms of the planes of the projective 3-space which extends the Euclidean 3-space that contains the Riemann sphere.

The planes of an axial pencil which meet the Riemann sphere in more than one point cut it in the circles of a *pencil.* Thus if a and b are any two circles of the Riemann sphere they belong to a unique pencil $[a, b]$ which is called elliptic, parabolic, or hyperbolic depending on whether the axis of the corresponding pencil of planes is a secant, tangent, or nonsecant of the Riemann sphere. The planes through a given point which meet the Riemann sphere in more than one point cut it in the circles of a *bundle.* Thus if a, b, c are any three circles of the Riemann sphere which do not belong to a pencil, they determine a unique bundle $\langle a, b, c\rangle$ which is called spherical, Euclidean, or hyperbolic as the common point P lies inside, on, or outside the Riemann sphere. The circles of a spherical bundle can play the role of great circles in an inversive model of spherical geometry. The circles of a Euclidean bundle can play the role of lines in an inversive model of Euclidean geometry whose points are the points of the Riemann sphere other than the common point P. The circles of a hyperbolic bundle are exactly the circles perpendicular to the circle ω where the cone of tangents from the common point P touches the Riemann sphere; they can play the role of lines in a Poincaré model of hyperbolic geometry whose points are all the points in either the open disks determined by the absolute circle ω.

The various types of Möbius transformation together with the roles they play in spherical (S), Euclidean (E), or hyperbolic geometry (H) are as listed in Table 1. In all cases of E the point P is ∞ and in the various cases of H at least one possibility is indicated for the position of the absolute circle ω.

Note that the length d of the Euclidean translation $z \to z + d$ and the Euclidean glide $z \to \bar{z} + d$ is not inversively significant. This is the one instance in which the inversive geometry fails to capture an important feature of the classical geometry.

4. Latin triangles. The simplest case of our problem, and one that occurs in all three geometries, concerns mirrors a, b, c that form a triangle. This means that $[a, b]$, $[b, c]$, $[c, a]$ are distinct elliptic pencils and the analysis can proceed in a manner that is independent of the nature of $\langle a, b, c\rangle$.

Following the usual procedure for analyzing the isometry abc we move b and c in their pencil until $a(bc) = a(b'c')$ with $b' \perp a$. Then $abc = (ab')c'$ is equal to the product of a halfturn about $a \cap b'$ and a reflection in c' and hence can be recognized as a glide with axis β passing through $a \cap b'$. We refer to $a \cap b'$ as the foot of $[b, c]$ on a and note that if we repeat the preceding analysis with the alternative bracketing

$$abc = (ab)c = (a''b'')c = a''(b''c)$$

TABLE 1. Representatives of the Conjugacy Classes of the Möbius Group, cf. [11]

Name of Class	Formula for Representative	Canonical Factorization	Action as an Isometry
$(0,1)$-elliptic	$z \to -\bar{z}$	inversion in $\operatorname{Re} z = 0$	S, E, H reflection ω: $\operatorname{Im} z = 0$ or $\lvert z\rvert = 1$
$(1,0)$-elliptic with angle θ	$z \to e^{2i\theta} z$	product of inversions in two circles of the elliptic pencil through 0 and ∞ that are inclined at the angle θ	S, E, H rotation through 2θ ω: $\lvert z\rvert = 1$
$(0,0)$-parabolic	$z \to z + 1$	product of inversions in two circles of the parabolic pencil tangent to $\operatorname{Re} z = 0$ at ∞	E translation H parallel displacement ω: $\operatorname{Im} z = 0$
$(0,0)$-hyperbolic with parameter δ	$z \to e^{2\delta} z$	product of inversions in two circles of the hyperbolic pencil with limiting points 0 and ∞ that are separated by the inversive distance δ	H translation through 2δ along $\operatorname{Re} z = 0$ ω: $\operatorname{Im} z = 0$
$(1,1)$-elliptic with angle θ	$z \to \dfrac{e^{2i\theta}}{\bar{z}}$	product of three inversions: one in $\lvert z\rvert = 1$ commuting with two in the elliptic pencil above	S glide with axis $\lvert z\rvert = 1$ and length 2θ
$(0,1)$-parabolic	$z \to \bar{z} + 1$	product of three inversions: one in $\operatorname{Im} z = 0$ commuting with two in the parabolic pencil above	E glide with axis $\operatorname{Im} z = 0$
$(0,1)$-hyperbolic with parameter δ	$z \to -e^{2\delta} z$	product of three inversions: one in $\operatorname{Re} z = 0$ commuting with two in the hyperbolic pencil above	H glide with axis $\operatorname{Re} z = 0$ and length 2δ ω: $\operatorname{Im} z = 0$
$(1,0)$-hyperbolic with parameters θ and δ	$z \to e^{2(\delta + i\theta)} z$	product of four inversions: two in the elliptic pencil above commuting with two in the hyperbolic pencil above	does not occur as an isometry

we can identify the axis fully as the line joining the foot of $[b, c]$ on a and the foot of $[a, b]$ on c. This construction of β applies whenever these two feet exist; it is not restricted to the case in which a, b, c form a triangle but we shall see that it does not apply in all cases.

The simplicity of the present "triangle case" arises from the fact that both of the feet required to identify each of the axes are guaranteed to exist. If the mirrors a, b, c form a triangle then the axes α, β, γ form its pedal triangle [4]. The general fact noted previously that mirror a must reflect axis β into axis γ, etc., means that the sides and altitudes of the original Latin triangle must be the angle-bisectors (internal or external) of the derived Greek triangle and this observation has two remarkable consequences.

First, it yields an instantaneous proof, valid in all three geometries, that the altitudes of a triangle belong to a pencil, cf. [1] p. 57. Figure 1 shows that the altitudes of an acute-angled Latin triangle are the internal angle-bisectors of the corresponding Greek triangle and therefore they meet at its incentre. Figure 3 shows that the

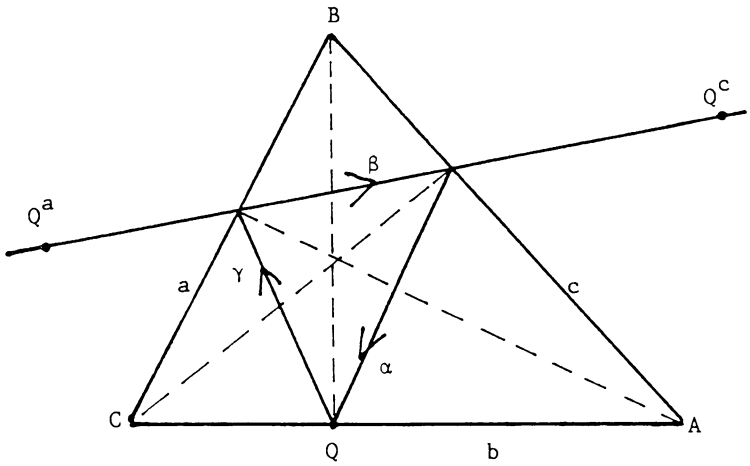

FIG. 1.

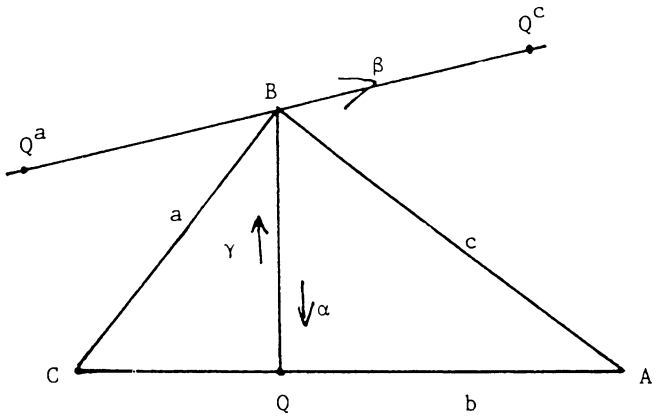

FIG. 2.

altitudes of an obtuse-angled Latin triangle are two external and one internal angle-bisector of the corresponding Greek triangle. In spherical and Euclidean geometry this means that they meet at its excentre opposite the internal angle-bisector; in hyperbolic geometry it means that they meet, are parallel, or share a common perpendicular depending on whether the relevant excycle is a circle, horocycle, or hypercycle.

A second consequence of the observation that the Latin lines are angle-bisectors of the Greek triangle is a formula for the common length of the glides $l(a, b, c)$ in terms of the lengths of the sides of the Greek triangle. To compute this length via abc we follow the point $P = \alpha \cap \beta$ to P^a on γ, P^{ab} on α, and finally P^{abc} back on β. The sides of the Greek triangle unfold onto this line so that $l(a, b, c) = PP^{abc}$ is equal to a sum of their signed lengths. If triangle abc is acute-angled (as in Fig. 4) each Latin line is an external angle-bisector of the Greek triangle and $l(a, b, c) = \alpha + \beta + \gamma$: the

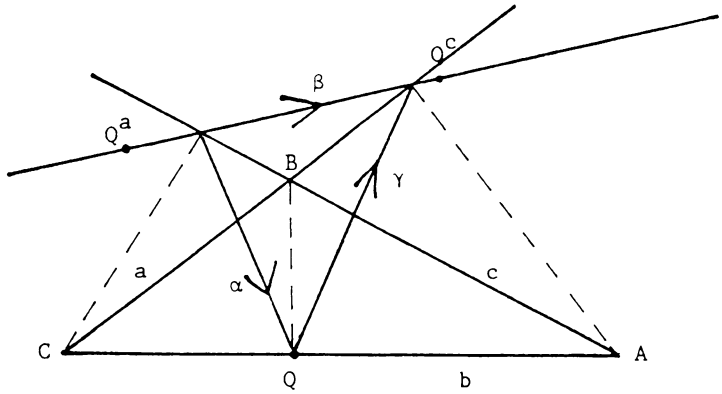

FIG. 3.

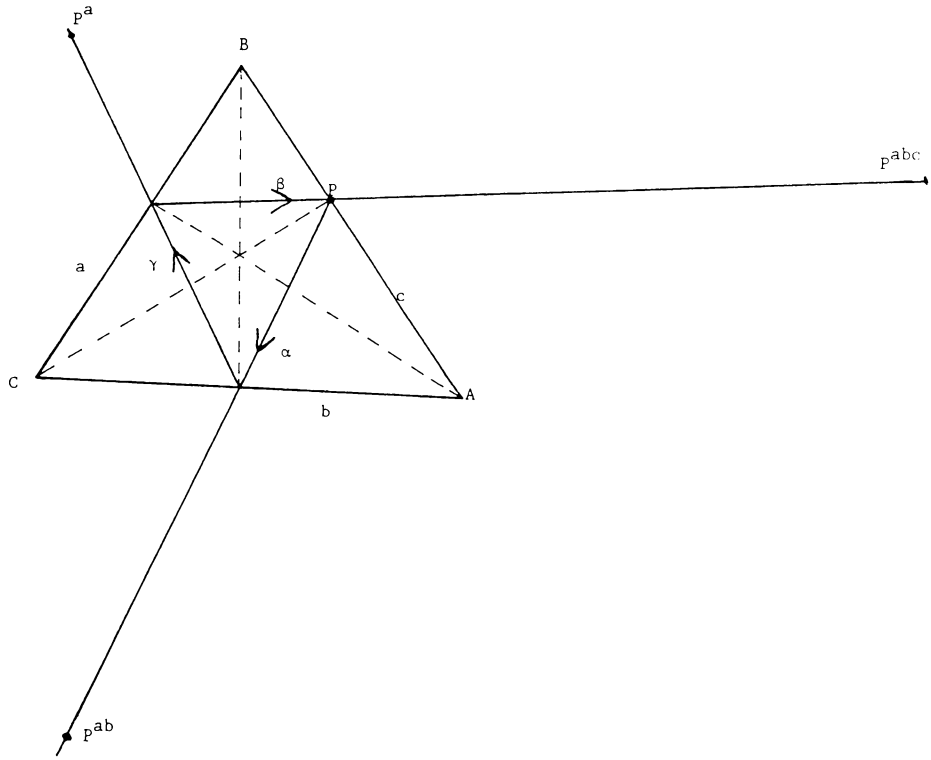

FIG. 4.

length of the glide is equal to the perimeter of the pedal triangle. If triangle abc has an obtuse angle opposite b then a and c become internal angle-bisectors of the Greek triangle and $l(a, b, c) = \alpha - \beta + \gamma$ [4].

Note that these formulae for $l(a, b, c)$ apply whenever the axes form a triangle and, at least in the hyperbolic plane, this goes beyond the case in which the mirrors form

a triangle. When a, b, c do form a triangle we can obtain formulae for $l(a, b, c)$ in terms of the parameters of this triangle. The point $Q = \alpha \cap \gamma$ reflects to points Q^a and Q^b which lie on β and satisfy $Q^{a(abc)} = Q^{bc} = Q^c$ so that $l(a, b, c)$ which we obtained earlier as $l(a, b, c) = PP^{abc}$ is also given by $l(a, b, c) = Q^a Q^c$. On the other hand (cf. Fig. 1 for the acute case and Fig. 3 for the obtuse case) Q is the foot of the altitude h from B and so $Q^a B Q^c$ is an isosceles triangle whose base, equal sides, and vertical angle are respectively l, h, and $2B$ or $2(\pi - B)$. This leads to an assortment of interesting expressions for l. In Euclidean geometry we have, with standard usage for undefined symbols [4],

$$\begin{aligned}\frac{l}{2} = h \sin B &= a \sin B \sin C \\ &= 2R \sin A \sin B \sin C \\ &= \frac{abc}{(2R)^2} \\ &= \frac{\Delta}{R} \\ &= \frac{4\Delta^2}{abc} \\ &= \frac{4s(s-a)(s-b)(s-c)}{abc}.\end{aligned}$$

In spherical and hyperbolic geometry we have exact analogues of this last formula

$$\sin \tfrac{l}{2} = \frac{4 \sin s \sin(s-a) \sin(s-b) \sin(s-c)}{\sin a \sin b \sin c}$$

and

$$\sinh \tfrac{l}{2} = \frac{4 \sinh s \sinh(s-a) \sinh(s-b) \sinh(s-c)}{\sinh a \sinh b \sinh c}.$$

These formulae remain valid in the intermediate case (Fig. 2) when the angle opposite b becomes $B = \frac{\pi}{2}$ and the Greek triangle collapses. When this occurs the axes α and γ coalesce with the altitude h, the axis $\beta = \alpha^c = \gamma^a$ remains well defined and the length of the glide becomes $l = 2h = \alpha + \gamma$ consistent with the fact that the third side length of the degenerate Greek triangle is $\beta = 0$.

The analysis above is essentially complete. While there do exist spherical triangles with more than one obtuse angle the same side lines determine a triangle with at most one obtuse angle. This observation together with our freedom to permute a, b, c cyclically shows that it remains to consider the possibility, which exists only in spherical geometry, that the triangle abc might possess two or even three right angles. If a and c are both perpendicular to b, the mirrors are in canonical form for a glide and $\alpha = \beta = \gamma = b$. If a, b, c are mutually perpendicular abc is the antipodal map and it can be written as a glide of length π using any axis [4].

The formulae for l can be used to show that when the Latin triangle has a given semi-perimeter s, the greatest value of l belongs to the equilateral triangle with $a = b = c = \frac{2}{3}s$. In the case of hyperbolic geometry this reveals a surprise: the longest glide which can arise from mirrors that form a triangle has length l_0 given by

$$\sinh\frac{l_0}{2} = \lim_{s\to\infty}\frac{4\sinh s\sinh^3\frac{s}{3}}{\sinh^3\frac{2s}{3}} = 2.$$

The formula for l_0 leads to $l_0 = 6\log\tau$ where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Glides of this length can arise as the product of reflections in the sides of a trebly asymptotic triangle, a fact that can be verified independently by showing that the perimeter of the pedal triangle of a trebly asymptotic triangle is equal to l_0. Longer glides in the hyperbolic plane must arise from nontriangular configurations of mirrors and we shall see later that such configurations also occur in the Euclidean plane.

Finally we mention a case that is even simpler than the triangle case and can be treated uniformly in the three geometries. If a, b, c belong to a pencil (and this includes all cases in which the lines are not distinct) then the transformations abc, etc., are reflections and the mirrors can be located by shifting mirrors in the pencil: $abc = (ab)c = (\beta c)c = \beta$.

5. The Greek problem. In the Greek problem we are given lines α, β, γ and asked to find mirrors a, b, c such that abc is a glide along β (or reflection in β); bca, a glide along γ; and cab, a glide along α. We have seen in the last section that if $\langle\alpha, \beta, \gamma\rangle$ is a spherical bundle then α, β, γ can each be regarded as the axis of the antipodal map $cab = abc = bca$ which can be written using any three mutually perpendicular mirrors a, b, c. Barring this possibility, our earlier considerations show that candidates for a, b, c must satisfy $\beta^a = \gamma$, $\gamma^b = \alpha$, $\alpha^c = \beta$, and hence if α, β, γ are distinct, $a \in [\beta, \gamma]$, $b \in [\gamma, \alpha]$, $c \in [\alpha, \beta]$. If the pencil $[x, y]$ is parabolic or hyperbolic there is a unique mirror m such that $x^m = y$; if it is elliptic, there are two such mirrors and they are mutually perpendicular [7]. Thus the Greek problem for distinct lines α, β, γ has at most eight candidate solution triples a, b, c and they all satisfy

$$\beta^{abc} = \gamma^{bc} = \alpha^c = \beta, \quad \text{etc.}$$

Now if g is a proper glide (not a reflection and not, in the spherical case, the antipodal map) then the axis of g is characterized as the unique line λ that satisfies $\lambda^g = \lambda$. On the other hand if g is a reflection then the equation $\lambda^g = \lambda$ means that λ is either the mirror of the reflection or a line perpendicular to this mirror. These remarks allow us to assess our trial solutions to the Greek problem.

If α, β, γ are distinct lines in a pencil Γ then a, b, c must belong to Γ, the transformations abc, etc., are reflections in lines of Γ, and the equations $\beta^{abc} = \beta$, etc., imply that abc is a reflection in β as desired or in a perpendicular line $\beta^\perp$. Identical results hold for bca and cab by conjugation. If Γ is a parabolic or hyperbolic pencil, the trial solution a, b, c is unique and must solve the problem because Γ does not contain a line perpendicular to β. However if Γ is elliptic there are eight trial solutions and four of them work while the other four produce reflections in lines perpendicular to the desired mirrors. To see that this is so we note, by way of example, that the two

mirrors which reflect β into γ are a perpendicular pair, say a and $a^{\perp}$. The effect of replacing a by $a^{\perp}$ in the product abc is to multiply the product by the halfturn $a^{\perp}a$ and thereby change the product reflection to one in a perpendicular mirror. Since two such changes restore the original mirror, the result follows.

If α, β, γ are distinct lines not in a pencil then the lines of a trial solution may lie in a pencil or not. If a, b, c do not lie in a pencil the transformations abc, etc., are glides and the equations $\beta^{abc} = \beta$, etc., show that we have produced glides with the required axes. But if a, b, c do lie in a pencil, the transformations abc, etc., are reflections in lines of this pencil and therefore cannot represent reflections in β, etc.; the appropriate interpretation of the equation $\beta^{abc} = \beta$ is that abc is a reflection in a mirror perpendicular to β.

These considerations give a beautiful technique for proving that certain lines belong to a pencil. Suppose α, β, γ are three lines not in a pencil and a, b, c are chosen so that $\beta^a = \gamma$, $\gamma^b = \alpha$, $\alpha^c = \beta$. Suppose further that orientations cannot be assigned to α, β, γ in a manner invariant under these reflections. Then a, b, c must belong to a pencil, for otherwise the transformations abc, etc., would be glides whose senses would orient α, β, γ in an invariant way. The simplest application of this technique is to prove, by an unorthodox method, the standard result that the three internal angle-bisectors of a triangle belong to a pencil and likewise two external and the third internal bisector. In contrast the three external angle-bisectors or two internal and the third external bisector correspond to consistent orientations of the sides of the triangle which we have already seen realized by glides. Before we are done we shall find triples of lines in the hyperbolic plane which admit no consistent orientation and for which the Greek problem has no solution. As a byproduct of this result, we shall obtain the existence of certain pencils derived earlier [6] in connection with the problem of Apollonius.

Our discussion of three lines α, β, γ not in a pencil shows that the Greek problem for oriented axes on at least two lines has at most one solution. It follows that if two sets of three mirrors not in a pencil and no two perpendicular to a third satisfy

$$abc = a'b'c', \qquad bca = b'c'a', \qquad cab = c'a'b'$$

we can conclude that $a = a'$, $b = b'$, $c = c'$. This is a remarkable result considering the flexibility in factoring a single glide; it stands in marked contrast to our 4-fold solution of the Greek problem for lines in an elliptic pencil.

Finally we treat the case of the Greek problem in which α, β, γ are not distinct. This means that the geometry is not necessarily determined and from the inversive point of view α, β, γ are circles in a pencil Γ. If $\alpha = \gamma$ and β does not cross this circle then Γ is not elliptic, $c = a$, $b = \alpha = \gamma$ and the transformations abc, etc., represent reflections in hyperbolic geometry if Γ is hyperbolic or in Euclidean or hyperbolic geometry (both interpretations are possible) if Γ is parabolic.

If $\alpha = \gamma$ and β does cross this circle then Γ is elliptic and any one of the three geometries is possible. There are two positions available in Γ for a and c. If we choose $a = c$ we are committed to reflections; the further choice $b = \alpha = \gamma$ gives reflections in the required lines while $b = \alpha^{\perp} = \gamma^{\perp}$ gives reflections in lines of Γ

perpendicular to the desired mirrors. If we choose $a \neq c$ there are two quite different things to do about b. We can keep b in Γ in which case $b = \alpha^{\perp} = \gamma^{\perp}$ gives the required reflections while $b = \alpha = \gamma$ gives perpendicular reflections. Otherwise we can take b perpendicular to $\alpha = \gamma$ but not in Γ. Then the transformations abc, etc., represent glides, the type of geometry is determined by the position of b, and the situation is similar to that shown in Fig. 2 except that abc need not form a triangle if the geometry is hyperbolic.

If $\alpha = \beta = \gamma = \lambda$ we can have $a = b = c = \lambda$ giving reflections belonging to our choice of geometries. Otherwise we can have $b = \lambda$ with $a = c$ perpendicular to λ giving reflections in our choice of geometries or $a \neq c$ perpendicular to λ giving glides in spherical, Euclidean, or hyperbolic geometry depending on whether $[a, c]$ is elliptic, parabolic, or hyperbolic.

6. The rest of the Latin problem. We return now to the Latin problem for three distinct lines not in a pencil and not forming a triangle. There are no other possibilities in spherical geometry and so the analysis there is complete.

(i) *Euclidean geometry.* In Euclidean geometry there is one other possibility: a and c parallel and b a transversal. If a and c are perpendicular to b this is the canonical form for a glide and $\alpha = \beta = \gamma = b$. If a and c are not perpendicular to b, the foot of $[a, b]$ on c and the foot of $[b, c]$ on a exist and determine β. Then $\alpha = \beta^c$ and $\gamma = \beta^a$ as in Fig. 5. If a and c are separated by a distance d and inclined to b at an angle θ, $0 < \theta \leq \frac{\pi}{2}$, then $l(a, b, c) = 2d \sin\theta$.

(ii) *Hyperbolic geometry.* The situation here is fairly complicated so we begin by enumerating (Fig. 6) the sixteen basic arrangements of three distinct lines in the

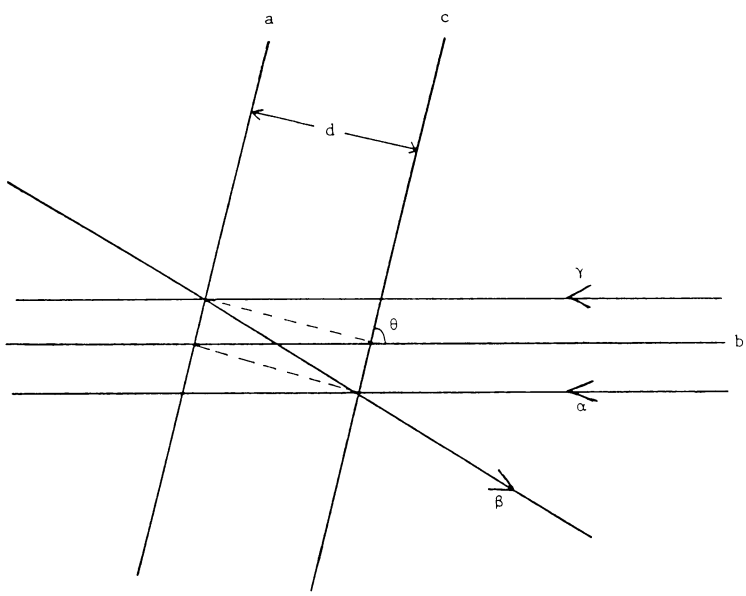

FIG. 5.

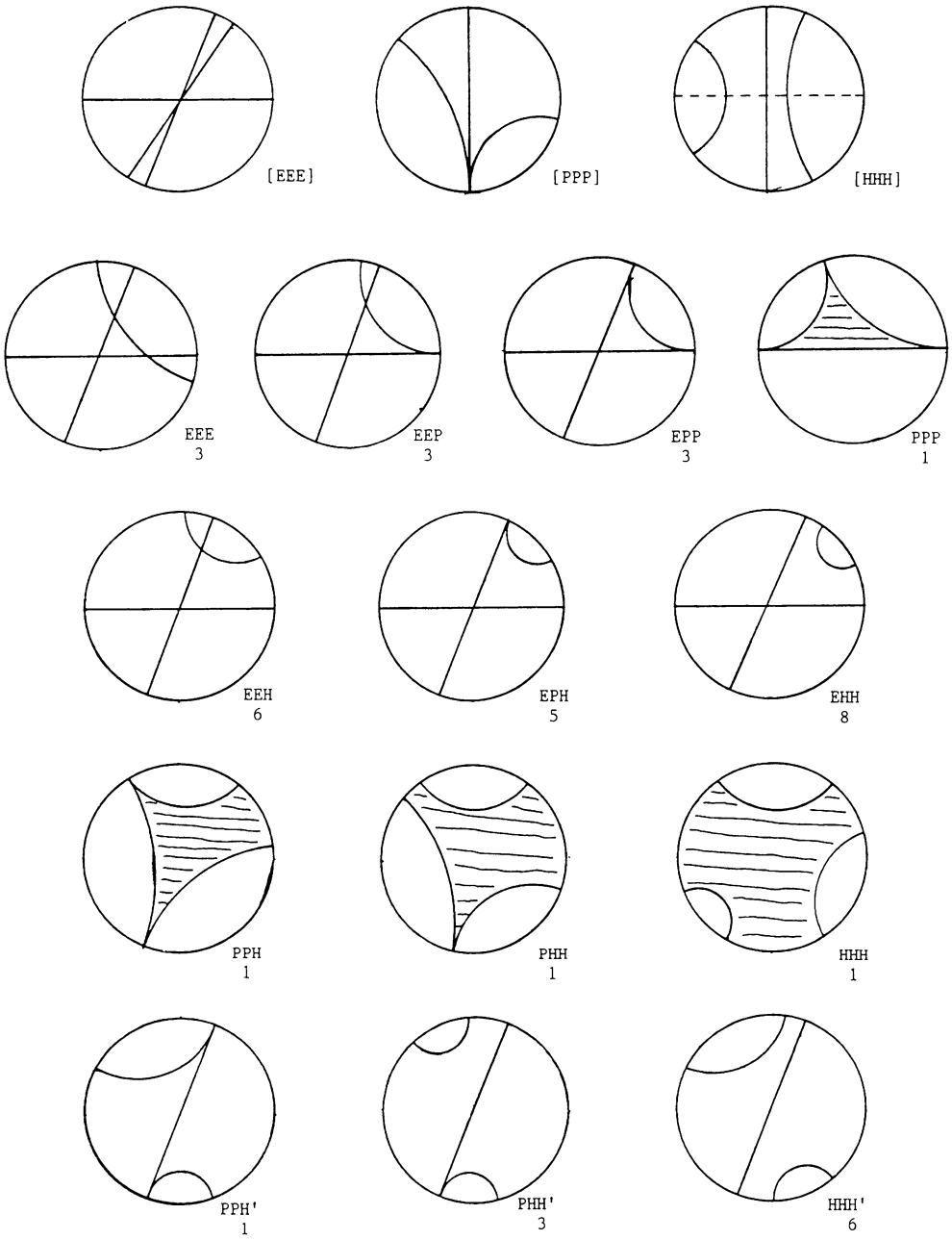

FIG. 6.

hyperbolic plane. We denote these arrangements by listing the types of pencils which their lines determine in pairs. If the types happen to be the same, the simple list means that the pencils are distinct, eg., EEE ≡ "triangle case" and the bracketed list means that they are identical, e.g., [EEE] ≡ three lines in an elliptic pencil. This

simple notation is sufficiently descriptive in all cases but PPH, PHH, HHH, and here we must add a prime if one of the lines separates the other two (cf. [3]).

We have fully analyzed the pencil cases [EEE], [PPP], [HHH] and the triangle case EEE with its three subcases depending on whether the triangle is acute-angled, right-angled or obtuse-angled. The closely related cases EEP with three subcases EPP with three subcases and PPP without subcases (in which the triangle becomes, respectively, singly, doubly, or trebly asymptotic) are essentially like EEE: there is a well-defined pedal triangle which collapses to an altitude in the right-angled subcases. In point of fact this type of analysis, which is based on the property that each of the three pencils should have a well-defined foot on the remaining line, can be used to locate the three axes in many of the other cases. Moreover if just one of the pencils fails to have a foot (as in the Euclidean variation of EEP) then we can use the axis that is determined, say β, to locate the others as $\alpha = \beta^c$, $\gamma = \beta^a$.

Now elliptic and parabolic pencils have a foot on any line outside the pencil and hyperbolic pencils have a foot on any line that does not cross their common perpendicular. Thus if $[a, c]$ is a hyperbolic pencil with common perpendicular b' the qualitative nature of the glide axes will depend, among other factors, on the position of the third mirror b relative to the line b'. A detailed study of the remaining nine cases shows that they include the following numbers of subcases: EEH, six; EPH, five; EHH, eight; PPH, one; PPH′, one; PHH, one; PHH′, three; HHH, one; and HHH′, six. Thus the five subcases in spherical geometry and five in Euclidean geometry blossom to forty-two in hyperbolic geometry. There is too much material here to present full details but the reader may be interested in a careful discussion of EEH.

Figure 7 shows a fixed hyperbolic pencil $[a, c]$ with common perpendicular b'. The third mirror b is always a transversal to a and c but the resulting configuration of oriented glide axes α, β, γ depends on the angles which b makes with a and c and on the position of b relative to b'. At first $abcb'$ forms a convex quadrilateral so the sum of the angles of interest is less than π. We obtain an evolution of subcases similar to that of the triangle case as the angle b makes with a changes from acute to a right angle to obtuse while the angle it makes with c stays acute. Then we obtain three more subcases as b moves on to be parallel to b', to cross b' and to coincide with b'.

The case EPH evolves like EEH except that the last subcase is impossible.

In EHH it is natural to assume that $[a, c]$ is elliptic. Then if a misses a' and c misses c' we obtain three subcases depending on whether the angle between a and c is acute, a right angle, or obtuse. Beyond this we get four more subcases in which a and a' are parallel or cross in the presence of c and c' missing or being parallel. Finally there is the subcase in which a crosses a' and c crosses c'; feet no longer suffice to locate the axes belonging to this subcase and one must proceed by moving mirrors instead.

The cases PPH, PPH′, PHH do not divide into multiple subcases and are simple in the additional sense that in each case the axes can be located using only feet. For PHH′, if we take b between a and c with $[b, c]$ parabolic then the subcases depend on whether c misses, is parallel to, or crosses c' and the last subcase alone requires moving mirrors.

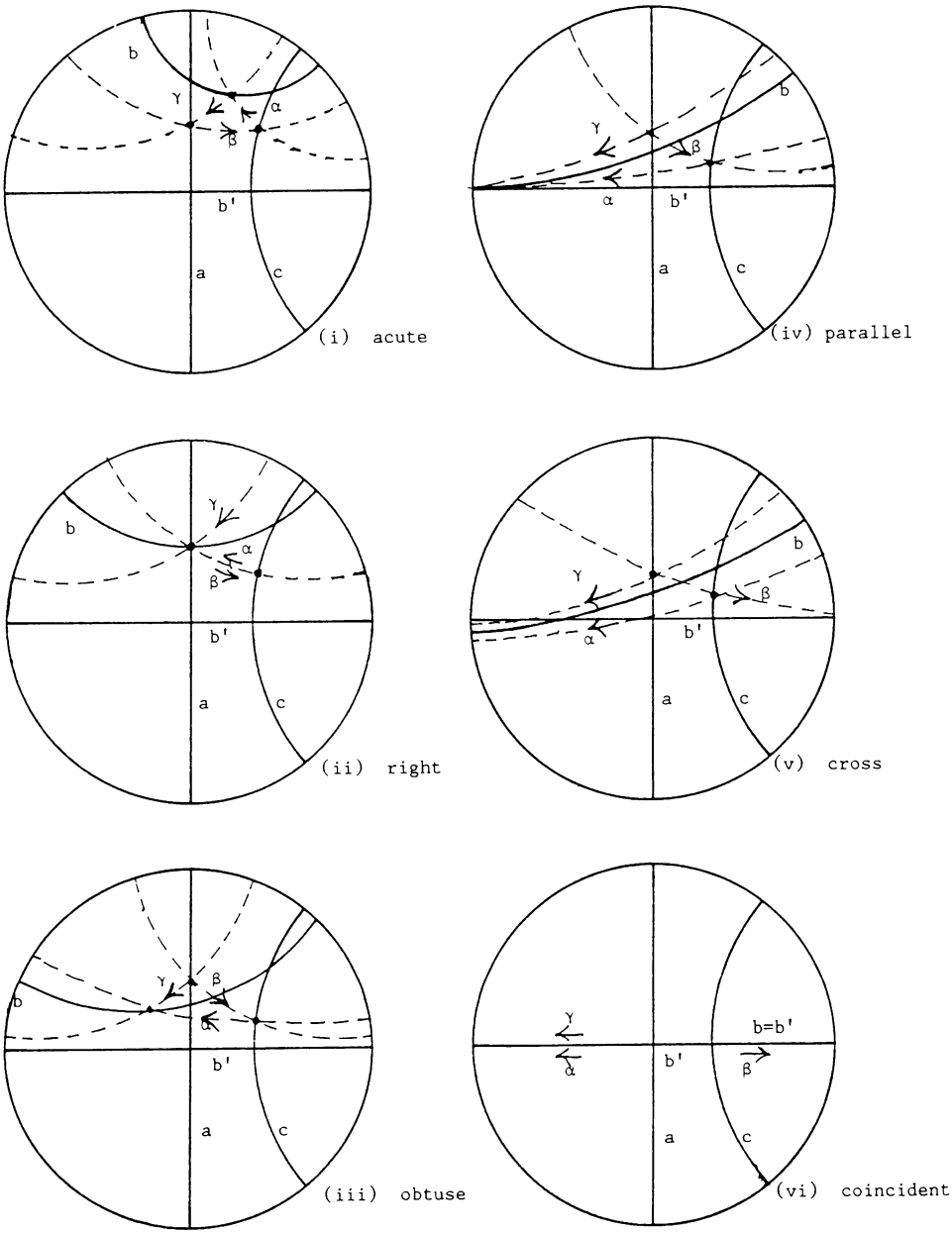

FIG. 7.

The full figure for HHH is a convex hexagon $ab'ca'bc'$ with a right angle at every vertex. The foot of $[a,c]$ on b is determined by the common perpendicular of b and b'. It follows that these common perpendiculars are the internal angle bisectors of the Greek triangle $\alpha\beta\gamma$ and hence they are concurrent. For other information

about the common perpendiculars of opposite sides of a convex right hexagon see [2] pp. 160–161.

The full figure for HHH′ is a nonconvex right hexagon which can be labelled $ab'ca'bc'$ in such a way that b separates a from c. The line b must then cross b' but six subcases arise depending on whether a and a' miss, are parallel, or cross while c and c' miss, are parallel, or cross but are never more intimately related than a and a'. All but the first two cases require moving mirrors to locate the axes.

7. Concluding remarks on the hyperbolic Greek problem. We mentioned earlier that for certain triples of lines α, β, γ in the hyperbolic plane the Greek problem has no solution. Instances of this are given by PPP, PPH, PHH, and HHH. For by regarding these lines as the boundaries of the shaded "triangles" of Fig. 6 we can see that there is no way of orienting their sides which is compatible with the unique trial solution a, b, c consisting of the "internal bisectors of the triangle." It follows that the lines of this failed solution a, b, c belong to a pencil which is elliptic, parabolic, or hyperbolic as the cycle tangent to α, β, γ (the incycle of the "triangle," as it were) is a circle, horocycle, or hypercycle. Only the first possibility obtains in the case PPP but all three possibilities are realized in the remaining cases and in particular in HHH. This observation regarding HHH was made quite differently in [6] and used to give an interesting corollary. Here we remark that the equation $\beta^{abc} = \beta$ indicates that the transformation $abc = b^*$ must be a reflection in the radius of the cycle perpendicular to β at its point of contact, and similarly for $bca = c^*$ and $cab = a^*$.

To see that there are no further instances of a Greek problem without a solution, we return to the details of our work in §6 and observe that every other triple of lines arises and with every apparently consistent orientation. These sets of oriented axes are drawn in Fig. 8 and beside each of the twelve possibilities are listed the various types of arrangements of mirrors which can be responsible for it.

8. The length of the glide. We have satisfactory formulae for the length $l(a, b, c)$ of the glide abc in spherical and Euclidean geometry and in hyperbolic geometry as well in the case in which the mirrors a, b, c form a triangle. But as we have just explained in some detail, the triangle case is only one of many cases in the hyperbolic plane and accordingly the hyperbolic version of the formula for $l(a, b, c)$ needs to be elaborated. We turn for help to the underlying inversive geometry and produce a formula which applies in spherical as well as hyperbolic geometry, but not in Euclidean geometry. (All Euclidean glides are conjugate within the group of similarities and this group is the natural Möbius subgroup to associate with a given inversive model of Euclidean geometry.)

In inversive geometry each circle a on the Riemann sphere determines a pair of complementary caps which have inversive coordinates A and $-A$ (see [10] for a detailed discussion of these coordinates). Associated with these coordinates, which are real 4-vectors, is the Lorentz bilinear form $A * B$. If circles a and b intersect at an angle θ, are tangent, or are disjoint and separated by an inversive distance δ (see [5] for a discussion of inversive distance) then the coordinates of the caps which they bound satisfy, respectively, $A * B = \pm\cos\theta$, ± 1, or $\pm\cosh\delta$. The sign, which

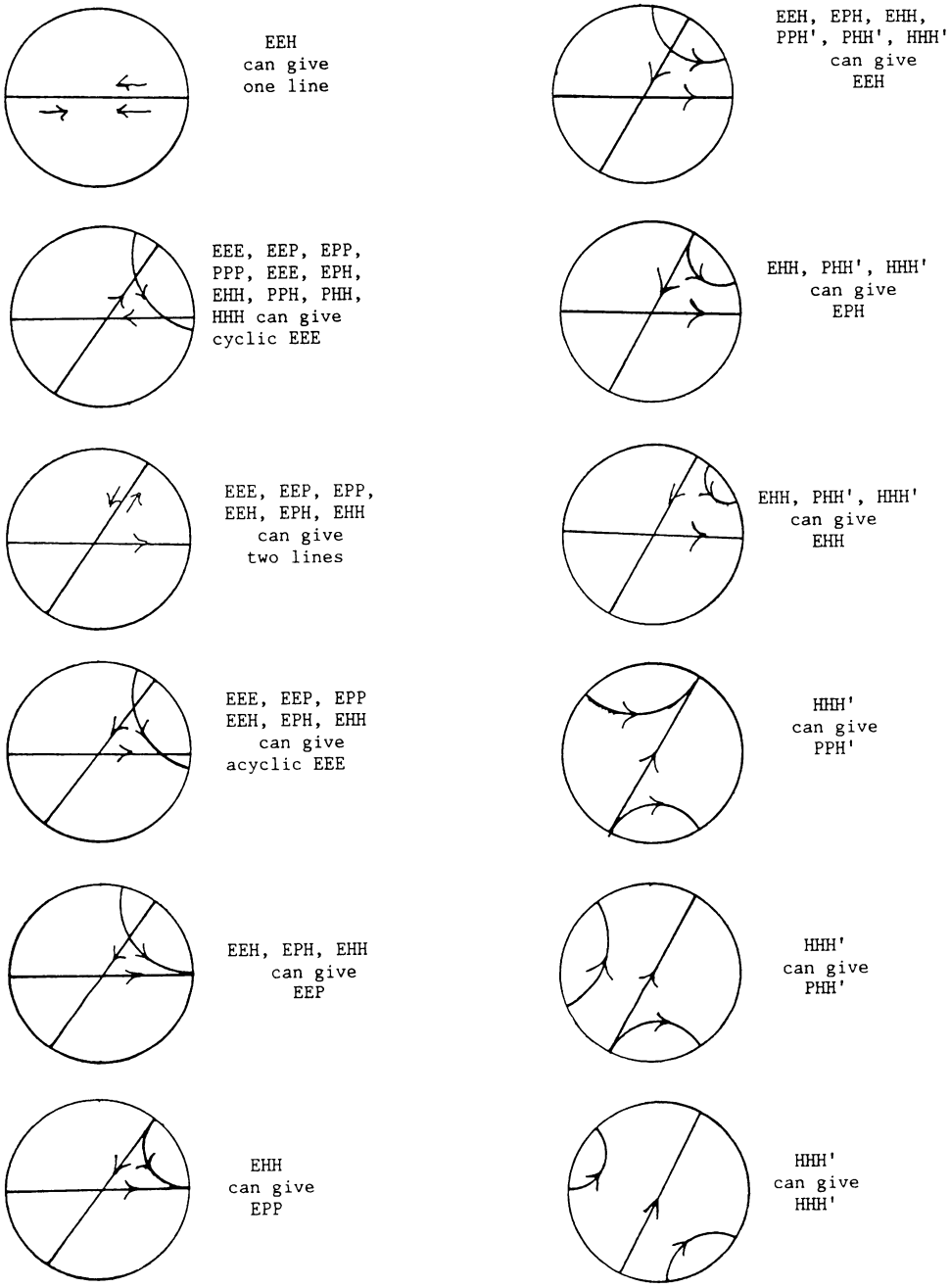

FIG. 8.

is not important in our discussion, depends on whether the caps chosen to represent the circles are nested or not. It follows that if a and b represent lines intersecting at an angle θ in spherical, Euclidean, or hyperbolic geometry, $(A * B)^2 = \cos^2 \theta$; if they represent parallel lines in Euclidean or hyperbolic geometry, $(A * B)^2 = 1$; and if they

represent ultraparallel lines in the hyperbolic plane standing a hyperbolic distance δ apart, $(A * B)^2 = \cosh^2 \delta$.

Now let us assume that a, b, c are three lines in one of the classical geometries with a and c perpendicular to b so that they are mirrors in canonical form for a glide of length $l = l(a, b, c)$ along $\beta = b$. We form and evaluate the Gram determinant of corresponding inversive coordinate vectors A, B, C:

$$\begin{vmatrix} A*A & A*B & A*C \\ B*A & B*B & B*C \\ C*A & C*B & C*C \end{vmatrix} = \begin{vmatrix} 1 & 0 & A*C \\ 0 & 1 & 0 \\ C*A & 0 & 1 \end{vmatrix} = 1 - (A*C)^2.$$

In Euclidean geometry the value is automatically 0 but in spherical geometry it is $1 - \cos^2 l/2 = \sin^2 l/2$ and in hyperbolic geometry, $1 - \cosh^2 l/2 = -\sinh^2 l/2$. We shall see presently that Gram determinants are invariant under substitutions corresponding to the technique of moving mirrors in their pencils to bring them to canonical form. Thus if a, b, c are any three lines in the hyperbolic plane with associated coordinate vectors A, B, C the glide abc has length l given by the general formula

$$\sinh^2 \frac{l}{2} = -\begin{vmatrix} A*A & A*B & A*C \\ B*A & B*B & B*C \\ C*A & C*B & C*C \end{vmatrix}$$

which happily is unaffected by the ambiguity of sign in these coordinates. In particular, if a, b, c are the sides of a trebly asymptotic triangle we can use the external halfplanes A, B, C and write

$$\sinh^2 \frac{l_0}{2} = -\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = 4$$

in agreement with our earlier discussion of this special case. While the hyperbolic formula above is a genuine extension of the triangle formula derived earlier, the companion spherical formula

$$\sin^2 \frac{l}{2} = \begin{vmatrix} A*A & A*B & A*C \\ B*A & B*B & B*C \\ C*A & C*B & C*C \end{vmatrix}$$

is merely a reformulation of our previous triangle formula in terms of angles rather than edge lengths.

Corresponding to the standard technique of moving mirrors in their pencil $abc = a(bc) = a(b'c')$ there is a change of the Gram determinant from

$$G = \begin{vmatrix} A*A & A*B & A*C \\ B*A & B*B & B*C \\ C*A & C*B & C*C \end{vmatrix} \quad \text{to} \quad G' = \begin{vmatrix} A*A & A*B' & A*C' \\ B'*A & B'*B' & B'*C' \\ C'*A & C'*B' & C'*C' \end{vmatrix}$$

and it remains to show that $G' = G$. The fact that $[b', c'] = [b, c]$ means that B' and C' lie in the span of B and C and the fact that $b'c' = bc$ means that $B' * C' = B * C = \lambda$, say. There are two different cases depending on whether $[b, c]$ is parabolic or not,

that is, whether $\lambda = \pm 1$ or $\lambda \neq \pm 1$. In the second case

$$G = \begin{vmatrix} 1 & A*B & A*C \\ B*A & 1 & \lambda \\ C*A & \lambda & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1 - \lambda^2 - A*B(A*B - \lambda A*C) + A*C(\lambda A*B - A*C) \\ &= 1 - \lambda^2 - D*D \end{aligned}$$

where $D = (A*C)B - (A*B)C$ and the fact that $\lambda^2 \neq 1$ facilitates a proof that $D' = \pm D$ hence $G' = G$. For if $B' = rB + sC$ and $C' = tB + uC$ the conditions $B'*B' = C'*C' = 1$ and $B'*C' = \lambda$ give the matrix equation

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} r & t \\ s & u \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

and hence the determinant equation $(ru-st)^2(1-\lambda^2) = 1-\lambda^2$. This and the condition $\lambda^2 \neq 1$ imply that $ru - st = \pm 1$. On the other hand if we substitute our expressions for B' and C' into the formula for D' a short calculation reveals that $D' = (ru-st)D$.

In the parabolic case we can assume, by changing the sign of B if necessary, that $\lambda = 1$. Then the vectors B' and C' can be chosen so that any two of B, C, B', C' satisfy $X*Y = 1$. The fact that $bc = b'c'$ can be used to impose further conditions on B' and C'. Using the standard formula for inversions [10] we compute in two ways the image of the halfplane B under this transformation. On the one hand

$$B^{bc} = (-B)^c = -B - 2C*(-B)C = -B + 2C.$$

On the other hand

$$\begin{aligned} B^{b'c'} &= (B - 2(B'*B)B')^{c'} \\ &= B - 2B' - 2C'*(B - 2B')C' \\ &= B - 2B' + 2C'. \end{aligned}$$

It follows that

$$-B + 2C = B - 2B' + 2C'$$

hence $C' - B' = C - B$. This proves the result because in the parabolic case

$$\begin{aligned} G &= \begin{vmatrix} 1 & A*B & A*C \\ B*A & 1 & 1 \\ C*A & 1 & 1 \end{vmatrix} \\ &= -A*BA*(B-C) + A*CA*(B-C) \\ &= -[A*(C-B)]^2. \end{aligned}$$

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