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A TRANSPORT THEOREM FOR MOVING INTERFACES*

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1. The theorem. When studying surface effects within the framework of continuum mechanics one is often confronted with terms of the form

$$\frac{d}{dt}\int_{\mathscr{S}(t)} f(\mathbf{x},t)\,da(\mathbf{x}), \tag{1}$$

where $\mathscr{S}(t)$ is a surface which evolves with time t, $f(\mathbf{x},t)$, defined for all $\mathbf{x}\in\mathscr{S}(t)$ and all t, is the density (per unit area) of a superficial quantity such as energy, and $da(\mathbf{x})$ is the area measure on surfaces in $\mathbb{R}^3$. The evaluation of (1) is nontrivial when $\mathscr{S}(t)$ evolves within a fixed region $\Omega\subset\mathbb{R}^3$ and $\partial\mathscr{S}(t)\subset\partial\Omega$ is nonempty, for then a portion of (1) must balance an outflow of f due to the transport of portions of $\mathscr{S}(t)$ across $\partial\Omega$.

We assume that $\mathscr{S}(t)$ is smooth and oriented by $\mathbf{n}(\mathbf{x},t)$, a particular choice of continuous unit-normal field, and we write $V(\mathbf{x},t)$ and $\kappa(\mathbf{x},t)$ for the **normal velocity** and **total curvature**. (Total curvature is twice the normal curvature.) It is the purpose of this note to prove the **transport theorem**:[1]

$$\begin{gathered}\frac{d}{dt}\int_{\mathscr{S}(t)} f\,da=\int_{\mathscr{S}(t)}(f^{\circ}-f\kappa V)\,da-\operatorname{outflow}(f,\partial\mathscr{S}(t)),\\ \operatorname{outflow}(f,\partial\mathscr{S}(t))=\int_{\partial\mathscr{S}(t)} fVp(1-p^2)^{-1/2}\,ds,\quad p=\mathbf{n}\cdot\boldsymbol{\nu}.\end{gathered} \tag{2}$$

Here f° is the normal time derivative of f as defined below, ds is the measure of length on curves in $\mathbb{R}^3$, and $\boldsymbol{\nu}(\mathbf{x})$ is the outward unit normal on $\partial\Omega$.

2. Assumptions and preliminary definitions. It is convenient to identify $\mathbb{R}^4$ with $\mathbb{R}^3\times\mathbb{R}$.

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[1]An argument in support of (2) is contained in the work Moeckel [1]. Moeckel assumes that the interface can be identified with a "fictitious" (sic) evolving membrane whose boundary coincides with the boundary of the interface at each time, and then appeals to a standard transport theorem for membranes. Unfortunately, Moeckel expresses the outflow in terms of the *membrane* velocity, which is not intrinsic, and which obscures the influence of the confining region Ω. Moreover, the existence of such an evolving membrane is not at all obvious, and, in fact, seems to constitute a mathematical problem more difficult than the original problem of verifying (2). Angenent and Gurtin [2] establish (2) for an evolving curve in a two-dimensional space, but their proof does not extend.

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded, open region with smooth boundary $\partial\Omega$, and write $\boldsymbol{\nu}(\mathbf{x})$ for the outward unit normal on $\partial\Omega$. We assume that $\mathscr{S}(t) \subset \mathbb{R}^3$ is defined for all t in an open interval T and: (S1) $\mathscr{S}(t)$ is the intersection with Ω of a smooth, nonintersecting, oriented surface, and $\partial\mathscr{S}(t) \subset \partial\Omega$; (S2) $\mathbf{n}(\mathbf{x}, t)$, the unit normal to $\mathscr{S}(t)$, satisfies $|\mathbf{n}(\mathbf{x}, t) \cdot \boldsymbol{\nu}(\mathbf{x})| \neq 1$ on $\partial\mathscr{S}(t)$; (S3) the set

$$\mathscr{S}_T = \{(\mathbf{x}, t) \colon \mathbf{x} \in \mathscr{S}(t),\ t \in T\}$$

is a smooth three-dimensional surface in $\mathbb{R}^4$ with normal never parallel to the time direction.

We assume that $f(\mathbf{x}, t)$ is a smooth scalar field on $\mathscr{S}_T$.

We write $\mathbf{N}(\mathbf{x}, t)$ and $\mathbf{U}(\mathbf{x})$, respectively, for $\mathbf{n}(\mathbf{x}, t)$ and $\boldsymbol{\nu}(\mathbf{x})$ considered as unit vectors in $\mathbb{R}^4$, and $\mathbf{E}$ for the unit vector in $\mathbb{R}^4$ in the time direction:

$$\mathbf{N} = (\mathbf{n}, 0), \qquad \mathbf{U} = (\boldsymbol{\nu}, 0), \qquad \mathbf{E} = (\mathbf{0}, 1). \tag{3}$$

By (S3) there is a scalar field V such that $\mathbf{N} - V\mathbf{E}$ is normal to $\mathscr{S}_T$; the field V represents the **normal velocity** of the surface in the direction $\mathbf{n}$. We write $\mathbf{M}$ for the unit vector in the direction of $\mathbf{N} - V\mathbf{E}$:

$$\mathbf{M} = q(\mathbf{N} - V\mathbf{E}), \qquad q = (1 + V^2)^{-1/2}. \tag{4}$$

Then $\mathbf{M}(\mathbf{x}, t)^\perp$ is the tangent plane to $\mathscr{S}_T$ at $(\mathbf{x}, t)$. We write $\mathbf{E}^*$ for the normalized projection of $\mathbf{E}$ onto $\mathbf{M}^\perp$:

$$\mathbf{E}^* = q(V\mathbf{N} + \mathbf{E}). \tag{5}$$

Given any field $\boldsymbol{\Phi}$ on $\mathscr{S}_T$, we write $\nabla\boldsymbol{\Phi}$ for the surface gradient[2] of $\boldsymbol{\Phi}$ in $\mathscr{S}_T$: $\nabla\boldsymbol{\Phi}(\mathbf{x}, t)$ is a vector in $\mathbf{M}(\mathbf{x}, t)^\perp$ if $\boldsymbol{\Phi}$ is scalar-valued; it is a linear transformation from $\mathbf{M}(\mathbf{x}, t)^\perp$ into $\mathbb{R}^4$ if $\boldsymbol{\Phi}$ is vector-valued. For $\boldsymbol{\Phi}$ a scalar field, we define the **normal time derivative** $\overset{\circ}{\boldsymbol{\Phi}}$ through

$$\overset{\circ}{\boldsymbol{\Phi}} = \nabla\boldsymbol{\Phi} \cdot (V\mathbf{N} + \mathbf{E}). \tag{6}$$

We write div for the surface divergence on $\mathscr{S}_T$: if $\boldsymbol{\Phi}$ is a vector field on $\mathscr{S}_T$, $\operatorname{div}\boldsymbol{\Phi} = \operatorname{trace}[\mathbf{P}\nabla\boldsymbol{\Phi}]$, where $\mathbf{P}(\mathbf{x}, t)$ is the projection of $\mathbb{R}^4$ onto $\mathbf{M}(\mathbf{x}, t)^\perp$. It is not difficult to verify that

$$\kappa = -\operatorname{div}\mathbf{N} \tag{7}$$

is the **total curvature** of $\mathscr{S}(t)$.

The identity

$$\operatorname{div}\mathbf{E}^* = -q\kappa V \tag{8}$$

is useful. Its verification is not difficult: since $\nabla q = -q^3 V \nabla V$ and $q - q^3 V^2 = q^3$, (5) and (7) yield

$$\operatorname{div}\mathbf{E}^* = qV\operatorname{div}\mathbf{N} + q^3\nabla V \cdot \mathbf{N} - q^3 V\nabla V \cdot \mathbf{E} = -qV\kappa + q^3\nabla V \cdot (\mathbf{N} - V\mathbf{E})$$

which implies (8), since $\mathbf{N} - V\mathbf{E}$ is normal to $\mathscr{S}_T$ (cf. (4)).

[2] Many of the definitions and identities that we use concerning surfaces can be found in [3, 4].

3. Proof of the transport theorem. Given a time interval $R = [t_0, t_1] \subset T$, the surface divergence theorem applied to the vector field $f\mathbf{E}^*$ on

$$\mathscr{S}_R = \{(\mathbf{x}, t) \colon \mathbf{x} \in \mathscr{S}(t),\ t \in R\}$$

has the form

$$\int_{\partial\mathscr{S}_R} f\mathbf{E}^* \cdot \mathbf{W}\, dA_2 = \int_{\mathscr{S}_R} \operatorname{div}(f\mathbf{E}^*)\, dA_3. \tag{9}$$

Here dA_n ($n = 1, 2, 3$) is the "area" measure on n-dimensional surfaces in $\mathbb{R}^4$, while $\mathbf{W}$ is the outward unit normal to $\partial\mathscr{S}_R$. $\partial\mathscr{S}_R$ is the union of the sets

$$\operatorname{top}(\mathscr{S}_R) = \{(\mathbf{x}, t_1) \colon \mathbf{x} \in \mathscr{S}(t_1)\},$$
$$\operatorname{bot}(\mathscr{S}_R) = \{(\mathbf{x}, t_0) \colon \mathbf{x} \in \mathscr{S}(t_0)\},$$
$$\operatorname{side}(\mathscr{S}_R) = \{(\mathbf{x}, t) \colon \mathbf{x} \in \partial\mathscr{S}(t),\ t \in T\},$$

whose intersection has zero A_1-measure, and, trivially,

$$\mathbf{E}^* \cdot \mathbf{W} = 1 \quad \text{on } \operatorname{top}(\mathscr{S}_R), \qquad \mathbf{E}^* \cdot \mathbf{W} = -1 \quad \text{on } \operatorname{bot}(\mathscr{S}_R). \tag{10}$$

The computation of $\mathbf{E}^* \cdot \mathbf{W}$ on $\operatorname{side}(\mathscr{S}_R)$ is not so simple. Since

$$p = \mathbf{n} \cdot \boldsymbol{\nu} = \mathbf{N} \cdot \mathbf{U}, \tag{11}$$

(4) and (5) yield

$$\mathbf{U} \cdot \mathbf{M} = qp, \qquad \mathbf{U} \cdot \mathbf{E}^* = qpV. \tag{12}$$

If $\mathbf{A} = \mathbf{U} - (\mathbf{U} \cdot \mathbf{M})\mathbf{M}$, the projection of $\mathbf{U}$ onto $\mathbf{M}^{\perp}$, then $\mathbf{W} = \mathbf{A}/|\mathbf{A}|$ on $\operatorname{side}(\mathscr{S}_R)$. Thus, using (12),

$$\mathbf{W} = (1 - q^2p^2)^{-1/2}(\mathbf{U} - qp\mathbf{M}) \quad \text{on } \operatorname{side}(\mathscr{S}_R), \tag{13}$$

and, since $\mathbf{M} \cdot \mathbf{E}^* = 0$ and

$$(1 - q^2p^2) = (1 - p^2 + V^2)/(1 + V^2), \tag{14}$$

a simple calculation using (12) leads to

$$\mathbf{E}^* \cdot \mathbf{W} = Vp(1 - p^2 + V^2)^{-1/2} \quad \text{on } \operatorname{side}(\mathscr{S}_R). \tag{15}$$

By (5), (6), and (8), $\operatorname{div}(f\mathbf{E}^*) = q(-fV\kappa + \overset{\circ}{f})$; thus (9) yields

$$\begin{aligned} &\int_{\operatorname{top}(\mathscr{S}_R)} f\, dA_2 - \int_{\operatorname{bot}(\mathscr{S}_R)} f\, dA_2 + \int_{\operatorname{side}(\mathscr{S}_R)} fVp(1 - p^2 + V^2)^{-1/2}\, dA_2 \\ &\quad = \int_{\mathscr{S}_R} q(\overset{\circ}{f} - f\kappa V)\, dA_3. \end{aligned} \tag{16}$$

Further,

$$\int_{\operatorname{top}(\mathscr{S}_R)} f\, dA_2 = \int_{\mathscr{S}(t_1)} f\, da, \qquad \int_{\operatorname{bot}(\mathscr{S}_R)} f\, dA_2 = \int_{\mathscr{S}(t_0)} f\, da. \tag{17}$$

The final step is to rewrite the remaining terms in (16) as iterated integrals. For any function g on $\mathscr{S}_R$,

$$\int_{\mathscr{S}_R} g\, dA_3 = \int_{t_0}^{t_1} \left\{ \int_{\mathscr{S}(t)} g(\mathbf{E}^* \cdot \mathbf{E})^{-1}\, da \right\} dt = \int_{t_0}^{t_1} \left\{ \int_{\mathscr{S}(t)} gq^{-1}\, da \right\} dt, \tag{18}$$

where we have used (5). On the other hand,

$$\int_{\text{side}(\partial\mathscr{S}_R)} g\,dA_2 = \int_{t_0}^{t_1}\left\{\int_{\partial\mathscr{S}(t)} g(\mathbf{B}\cdot\mathbf{E})^{-1}\,ds\right\}dt, \tag{19}$$

where $\mathbf{B}(\mathbf{x},t)$ with $\mathbf{B}\cdot\mathbf{E} > 0$ is that unit vector in the tangent plane to side$(\partial\mathscr{S}_R)$ which is normal to $\partial\mathscr{S}(t)$. In fact, $\mathbf{B} = \mathbf{C}/|\mathbf{C}|$, where $\mathbf{C}$ is the projection of $\mathbf{E}^*$ onto $\mathbf{W}^\perp$:

$$\mathbf{C} = \mathbf{E}^* - (\mathbf{E}^*\cdot\mathbf{W})\mathbf{W}.$$

By $(4)_2$ and (15),

$$|\mathbf{C}|^2 = q^{-2}(1-p^2)/(1-p^2+V^2).$$

Further, since $\mathbf{E}^*\cdot\mathbf{M} = \mathbf{U}\cdot\mathbf{E} = 0$, (4), (5), (12), and (13) yield

$$\mathbf{E}^*\cdot\mathbf{E} = q, \qquad \mathbf{E}^*\cdot\mathbf{W} = qpV(1-q^2p^2)^{-1/2}, \qquad \mathbf{E}\cdot\mathbf{W} = q^2pV(1-q^2p^2)^{-1/2},$$

and hence, using (14),

$$\mathbf{B}\cdot\mathbf{E} = (1-p^2)^{1/2}(1-p^2+V^2)^{-1/2}.$$

Thus (19) yields

$$\int_{\text{side}(\partial\mathscr{S}_R)} g\,dA_2 = \int_{t_0}^{t_1}\left\{\int_{\partial\mathscr{S}(t)} g\{(1-p^2+V^2)/(1-p^2)\}^{1/2}\,ds\right\}dt. \tag{20}$$

Finally, in view of (17), (18), and (20), (16) reduces to

$$\int_{\mathscr{S}(t_1)} f\,da - \int_{\mathscr{S}(t_0)} f\,da + \int_{t_0}^{t_1}\left\{\int_{\partial\mathscr{S}(t)} fVp/(1-p^2)^{1/2}\,ds\right\}dt$$
$$= \int_{t_0}^{t_1}\left\{\int_{\mathscr{S}(t)} (f^\circ - f\kappa V)\,da\right\}dt;$$

and differentiation with respect to t_1 yields (2).

REMARK 1. $\mathscr{S}(t)$ is the intersection with Ω of an oriented surface $\mathscr{M}(t)$; let $\boldsymbol{\mu}(\mathbf{x},t)$, a *tangent* vector to $\mathscr{M}(t)$ at $\mathbf{x}\in\mathscr{M}(t)$, denote the outward unit normal to $\partial\mathscr{S}(t)$ as a curve in $\mathscr{M}(t)$. The calculation of the outflow term in (2) is essentially the calculation of the velocity $\sigma(\mathbf{x},t)$ of $\partial\mathscr{S}(t)$ in the direction $\boldsymbol{\mu}(\mathbf{x},t)$. In fact, if we consider an arbitrary (smoothly-evolving) patch $\mathscr{S}(t)$ of an evolving surface $\mathscr{M}(t)$, then

$$\frac{d}{dt}\int_{\mathscr{S}(t)} f\,da = \int_{\mathscr{S}(t)} (f^\circ - f\kappa V)\,da + \int_{\partial\mathscr{S}(t)} f\sigma\,ds. \tag{21}$$

REMARK 2. It is important to identify the term outflow$(f,\partial\mathscr{S}(t))$ in (2) as a term representing an outflow of $f(\mathbf{x},t)$ due to the transport of portions of $\mathscr{S}(t)$ across $\partial\Omega$. If one writes, for example, balance of energy for a continuous body Ω consisting of two phases separated by an interface $\mathscr{S}(t)$ with interfacial energy f, then a term of the form outflow$(f,\partial\mathscr{S}(t))$ should appear (cf. Gurtin [4]). Moeckel [1] fails to include such an outflow in his balance laws. Fernandez-Diaz and Williams [5] point this out, but unfortunately the outflow term they propose is incorrect, as it does not include the scale factor $(1-p^2)^{-1/2}$.

REMARK 3. It is possible to write the transport identity (2) in terms of a nonnormal velocity. Indeed, for $\mathbf{v} = V\mathbf{n} + \mathbf{u}$ with $\mathbf{u} \cdot \mathbf{n} = 0$,

$$\frac{d}{dt}\int_{\mathscr{S}(t)} f\,da = \int_{\mathscr{S}(t)} (f^{\circ} + f \operatorname{div}\mathbf{u})\,da - \operatorname{outflow}(f, \partial\mathscr{S}(t)) \tag{22}$$

where $f^{\circ} = \nabla f \cdot (\mathbf{v} + \mathbf{E})$ is the derivative following $\mathbf{v}$, div is the surface divergence on $\mathscr{S}(t)$, and

$$\operatorname{outflow}(f, \partial\mathscr{S}(t)) = \int_{\partial\mathscr{S}(t)} f[Vp(1-p^2)^{-1/2} + \mathbf{u}\cdot\boldsymbol{\nu}(1+p^2)^{-1/2}]\,ds, \qquad p = \mathbf{n}\cdot\boldsymbol{\nu}. \tag{23}$$

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