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A UNIQUENESS THEOREM FOR INCOMPRESSIBLE MICROPOLAR FLOWS

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Abstract. It is proved that subject to certain regularity assumptions on velocity and microrotation and a certain convergence condition on pressure, the flow of an incompressible micropolar fluid around a finite solid body is uniquely determined by the motion of the body.

1. Introduction. The theories of polar fluids are models for fluids whose microstructure is significant. Such theories have been applied in recent years for modeling rheologically complex liquids such as blood and dilute polymeric suspensions.

The linear constitutive equations for fluids with microstructure have appeared under a variety of names over the past few years (see [1]). A mathematically amenable model for such fluids, called micropolar fluids, was introduced by Eringen [2] in 1966 and has been the subject of extensive study in recent years. This model introduces a new kinematic variable called microrotation describing the individual rotation of particles within the continuum, independent of the velocity field. The resulting equations of motion include an angular momentum equation because of the asymmetry of the stress tensor and the presence of couple stresses.

In this note we prove a uniqueness theorem for an incompressible micropolar fluid in the presence of a solid body. Specifically, given a finite solid body within a mass of incompressible micropolar fluid that extends to infinity, we seek to determine if the motion of the fluid is uniquely determined by the motion of the solid body. An affirmative result is proved subject to certain smoothness and boundedness conditions on the velocity, microrotation, and their derivatives together with a certain convergence condition on pressure at large distances from the solid body. The method used is essentially an extension of that of Dyer and Edmunds [3], who use it to prove similar theorems in MHD flows.

2. Basic formulation. The constitutive relations for micropolar fluids with isotropic microstructure are given by [2]

$$\begin{aligned} t_{ij} &= (-p + \lambda v_{k,k})\delta_{ij} + (2\mu + \kappa)d_{ij} + \varepsilon_{ijk}(\omega_k - \nu_k), \\ m_{ij} &= \alpha\nu_{k,k}\delta_{ij} + \beta\nu_{i,j} + \gamma\nu_{j,i}, \end{aligned}$$

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where $\mathbf{v}$ is the velocity, $\boldsymbol{\nu}$ the microrotation, p the pressure, $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$, $\omega_i = \varepsilon_{ijk} v_{j,k}$, and the constants α, β, γ, λ, μ, κ are viscosity coefficients characterizing the fluid. These viscosities satisfy the inequalities

$$\mu \geq 0, \qquad \kappa \geq 0, \qquad 3\lambda + 2\mu + k \geq 0, \qquad 3\alpha + \beta + \gamma \geq 0.$$

The equations of motion of an incompressible micropolar fluid assume the form [2]

$$v_{i,i} = 0, \tag{1}$$

$$(\mu + \kappa) v_{i,jj} + \kappa \varepsilon_{ijk} \nu_{j,k} - p_{,i} + \rho f_i = \rho[\partial v_i / \partial t + v_k v_{i,k}], \tag{2}$$

$$\gamma \nu_{i,jj} + (\alpha + \beta) \nu_{j,ij} - 2\kappa \nu_i + \kappa \varepsilon_{ijk} v_{j,k} + \rho l_i = \rho j[\partial \nu_i / \partial t + v_k \nu_{i,k}]. \tag{3}$$

Here ρ is the density (> 0), j is the local microinertia moment (> 0), and f_i and l_i are applied body force and couple, respectively.

Consider a solid body S with sufficiently smooth boundary ∂S immersed in a micropolar fluid of infinite extent. Let E denote the set of points of space exterior to S and let G be a time interval $(0, t_0)$ where t_0 is arbitrary but fixed. To completely specify the problem we assume

(i) the flow variables $\mathbf{v}$, $\boldsymbol{\nu}$ are prescribed throughout $E \cup S$ at $t = 0$;

(ii) the v_i and ν_i are prescribed on ∂S at all times $t \geq 0$;

(iii) the f_i and l_i are prescribed at all times and all points of space.

In addition, we require the following boundedness and continuity conditions.

(iv) The velocity components v_i and their first partial derivatives with respect to space and time are continuous bounded functions of these variables in $E \times T$ and the second-order spatial derivatives are continuous in $E \times T$.

(v) The microrotation components ν_i and their first partial derivatives with respect to space and time are bounded continuous functions and the second-order space derivatives of ν_i are continuous in $(E \cup \partial S) \times T$.

(vi) The pressure p is continuous and has continuous first-order spatial derivatives in $E \times T$.

We further assume the following convergence condition at infinity.

Let $r^2 = x_i x_i$ (using summation convention). At infinity p converges to a constant p_0 such that for all t in T

$$p = p_0 + O(r^{-1/2-\varepsilon}) \quad \text{as } r \to \infty, \ \varepsilon \text{ being an arbitrary small positive constant.}$$

The main result is

THEOREM. There can be at most one solution of Eqs. (1)–(3) satisfying the conditions (i)–(vi) and the convergence condition on pressure at infinity.

Proof. Assume that (i)–(vi) and the convergence condition on pressure hold. Let $\{v_i, \nu_i, p\}$ and $\{v_i + v_i', \nu_i + \nu_i', p + p'\}$ be two possible solutions of the problem.

Since both solutions satisfy the basic equations (1)–(3) we have, by subtraction

$$v'_{i,i} = 0, \tag{4}$$

$$(\mu+\kappa)v'_{i,jj} + \kappa\varepsilon_{ijk}\nu'_{j,k} - p'_{,i} = \rho\left[\frac{\partial v'_i}{\partial t} + v'_k(v'_i + v_i)_{,k} + v_k v'_{i,k}\right], \tag{5}$$

$$\gamma\nu'_{i,jj} + (\alpha+\beta)\nu'_{j,ij} - 2\kappa\nu'_i + \kappa\varepsilon_{ijk}v'_{j,k} = \rho j\left[\frac{\partial \nu'_i}{\partial t} + v'_k(\nu'_i + \nu_i)_{,k} + v_k \nu'_{i,k}\right]. \tag{6}$$

Let B_r be a closed ball centered at the origin, of radius r and having surface C_r. Let us choose r large enough so that the solid body S remains within B_r for $t \in T$. Let $B'_r = B_r \cap E$.

On multiplying (5) by v'_i and integrating over B'_r, using Green's theorem and the boundary conditions $v'_i = \nu'_i = 0$ on ∂S, we get

$$\begin{aligned}\int_{B'_r} -(\mu+\kappa)v'_{i,j}v'_{i,j}\,dB_r + (\mu+\kappa)\int_{C_r} v'_i v'_{i,j} n_j\,dC_r + \int_{B'_r} \kappa\varepsilon_{ijk}v'_i\nu'_{j,k}\,dB_R - \int_{C_r} v'_i p' n_i\,dC_R \\ = \int_{B'_r}\frac{\rho}{2}\frac{\partial}{\partial t}(v'_i v'_i)\,dB_R + \int_{B'_r}\rho v'_i v'_k (v_i + v'_i)_{,k}\,dB_R + \int_{C_r}\frac{\rho}{2}v'_i v'_i v_k n_k\,dC_r.\end{aligned} \tag{7}$$

Here n_i is the outward-drawn unit normal on C_r. Similarly, multiplying (6) by ν'_i and integrating over B'_r, one gets, using Green's theorem and the boundary conditions on S,

$$\begin{aligned}&-\gamma\int_{B'_r}\nu'_{i,j}\nu'_{i,j}\,dB_r + \gamma\int_{C_r}\nu'_{i,j}\nu'_i n_j\,dC_r - (\alpha+\beta)\int_{B_r}\nu'_{i,j}\nu'_{j,i}\,dB_r \\ &\quad + (\alpha+\beta)\int_{C_r}\nu'_i\nu'_{j,i}n_i\,dC_r - 2\kappa\int_{B'_r}\nu'_i\nu'_i\,dB_r - \int_{B'_r}\kappa\varepsilon_{ijk}\nu'_{i,k}v'_j\,dB_r \\ &\quad + \int_{C_r}\kappa\varepsilon_{ijk}\nu'_i v'_j n_k\,dC_r + \gamma\int_{\partial S}\nu'_{i,j}\nu'_i n_j\,dS + (\alpha+\beta)\int_{\partial S}\nu'_i\nu'_{j,i}n_i\,dS \\ &= \int_{B'_r}\frac{\rho j}{2}\frac{\partial}{\partial t}(\nu'_i\nu'_i)\,dB_r + \int_{B'_r}\rho j\nu'_i v'_k(v_i + v'_i)_{,k}\,dB_r + \int_{C_r}\frac{\rho j}{2}\nu'_i\nu'_i v_k n_k\,dC_r \\ &\quad + \frac{\rho j}{2}\int_{\partial S}\nu'_i\nu'_i v_k n_k\,dS,\end{aligned} \tag{8}$$

where dS denotes a surface element on ∂S, and $\mathbf{n}$ is an outward normal on it.

Adding (7) and (8) and rearranging, we get

$$
\begin{aligned}
\int_{B'_r} &\left[\frac{1}{2}\rho\frac{\partial}{\partial t}(v'_i v'_i) + \frac{1}{2}\rho j\frac{\partial}{\partial t}(\nu'_i\nu'_i) + (\mu+\kappa)v'_{i,j}v'_{i,j} + \gamma\nu'_{i,j}\nu'_{i,j}\right] dB_r \\
&= -\int_{B'_r}[\rho v'_i v'_k(v_i + v'_i)_{,k} + \rho j\nu'_i v'_k(\nu'_i + \nu_i)_{,k} + 2\kappa\nu'_i\nu'_i + (\alpha+\beta)\nu'_{i,j}\nu'_{j,i} \\
&\qquad + 2\kappa\varepsilon_{ijk}\nu'_{i,k}v'_j]\, dB_r \\
&\quad + \int_{C_r}\left[(\mu+\kappa)v'_i v'_{i,j}n_j - \kappa\varepsilon_{ijk}v'_i\nu'_j n_k - \frac{\rho}{2}v'_i v'_i v_k n_k - v'_i p' n_i + (\alpha+\beta)\nu'_i\nu'_{j,i}n_j \right. \\
&\qquad \left. + \gamma\nu'_i\nu'_{i,j}n_j - \frac{\rho j}{2}\nu'_i\nu'_i v_k n_k\right] dC_r \\
&\quad + \int_{\partial S}\left[\gamma\nu'_{i,j}\nu'_i + (\alpha+\beta)\nu'_i\nu'_{j,i} - \frac{\rho j}{2}\nu'_i\nu'_i v_j\right] n_j\, dS.
\end{aligned} \tag{9}
$$

Integrating (9) with respect to t from 0 to t_1 and then again with respect to t_1 from 0 to a, where a $(0 < a \le t_0)$ is to be prescribed later, it follows that

$$
\begin{aligned}
\int_0^a dt_1 &\left\{\int_{B'_r}\left[\frac{1}{2}\rho v'_i v'_i + \frac{1}{2}\rho j\nu'_i\nu'_i\right] dB_r\right\} \\
&\qquad + \int_0^a dt_1\left\{\int_0^{t_1} dt\int_{B'_r}\left[(\mu+\kappa)v'_{i,j}v'_{i,j} + \gamma\nu'_{i,j}\nu'_{i,j}\right] dB_r\right\} \\
&= -\int_0^a dt_1\left\{\int_0^{t_1} dt\int_{B'_r}[\rho v'_i v'_k(v_i + v'_i)_{,k} + \rho j\nu'_i v'_k(\nu_i + \nu'_i)_{,k} + 2\kappa\nu'_i\nu'_i \right. \\
&\quad \left. + (\alpha+\beta)\nu'_{i,j}\nu'_{j,i} + 2\kappa\varepsilon_{ijk}\nu'_{i,k}v'_j]\, dB_r\right\} \\
&\quad + \int_0^a dt_1\left\{\int_0^{t_1} dt\int_{C_r}\left[(\mu+\kappa)v'_i v'_{i,j}n_j - \kappa\varepsilon_{ijk}v'_i\nu'_j n_k - \frac{\rho}{2}v'_i v'_i v_k n_k - v'_i p' n_i \right.\right. \\
&\quad \left.\left. + (\alpha+\beta)\nu'_i\nu'_{j,i}n_j + \gamma\nu'_i\nu'_{i,j}n_j - \frac{\rho j}{2}\nu'_i\nu'_i v_k n_k\right] dC_r\right\} \\
&\quad + \int_0^a dt_1\left\{\int_0^{t_1}\int_{\partial S}\left[\gamma\nu'_{i,j}\nu'_j + (\alpha+\beta)\nu'_i\nu'_{j,i} - \frac{\rho j}{2}\nu'_i\nu'_i v_j\right] n_j\, dS\right\}.
\end{aligned} \tag{10}
$$

Let us call the left side of (10), $P(r)$. Then because μ, κ, γ, ρ, j are all nonnegative it is easy to see that $P(r) \ge 0$.

Now we need estimates for the various quantities on the right of (10). Using Cauchy's inequality $2ab \le a^2 + b^2$ and conditions (iii) and (iv), we find that

$$
\begin{aligned}
|v_i' v_k' (v_i + v_i')_{,k}| &\le N_1 v_i' v_i', \\
|\nu_i' v_k' (\nu_i' + \nu_i)_{,k}| &\le N_2 (v_i' v_i' + \nu_i' \nu_i'), \\
|v_i' v_{i,j}' n_j| &\le \tfrac{3}{2} v_i' v_i' + \tfrac{1}{2} v_{i,j}' v_{i,j}', \\
|v_i' v_i' v_k n_k| &\le N_3 v_i' v_i', \\
|\nu_i' \nu_{j,i}' n_j| &\le \tfrac{3}{2} \nu_i' \nu_i' + \tfrac{1}{2} \nu_{i,j}' \nu_{i,j}', \\
|\nu_i' \nu_{i,j}' n_j| &\le \tfrac{3}{2} \nu_i' \nu_i' + \tfrac{1}{2} \nu_{i,j}' \nu_{i,j}', \\
|\nu_i' \nu_i' v_k n_k| &\le N_4 \nu_i' \nu_i', \\
|\nu_{j,i}' \nu_{i,j}'| &\le N_5 \nu_{i,j}' \nu_{i,j}', \\
|\varepsilon_{ijk} v_j' \nu_{i,k}'| &\le N_6 (v_i' v_i' + \nu_{i,j}' \nu_{i,j}'),
\end{aligned}
$$

where $N_1, \ldots, N_6$ are positive constants. For example, $N_3 = \frac{1}{3} \sup |v_k|$, the supremum taken in $E \times T$, over all indices k.

Using Schwarz's inequality and the convergence condition on p, it follows that there exists a constant N_7 such that

$$
\left| \int_{C_r} p' v_i' n_i \, dC_r \right| \le N_7 r^{1/2-\varepsilon} \left| \int_{C_r} v_i' v_i' \, dC_r \right|^{1/2}.
$$

Using these inequalities in (10) we have

$$
\begin{aligned}
0 \le P(r) \le{}& (N_1 + \rho N_2 + 2\kappa N_6) a \int_0^a dt_1 \int_{B_r'} v_i' v_i' \, dB_r + (2\kappa + \rho N_2) a \int_0^a dt_1 \int_{B_r'} \nu_i' \nu_i' \, dB_r \\
&+ [(\alpha + \beta) N_5 + 2\kappa N_6] \int_0^a dt \left\{ \int_0^{t_1} dt_1 \int_{B_r'} \nu_{i,j}' \nu_{i,j}' \, dB_r \right] \right\} \\
&+ \int_0^a dt_1 \left\{ \int_{C_r} \left[\tfrac{3}{2}(\mu + \kappa) + \frac{\rho}{2} N_3 + \kappa \right] v_i' v_i' \, dC_r \right\} \\
&+ \int_0^a dt_1 \left\{ \int_{C_r} \left[\frac{3}{2}(|\alpha + \beta| + \gamma) + \frac{\rho j}{2} N_4 + \kappa \right] \nu_i' \nu_i' \, dC_r \right\} \\
&+ \int_0^a dt_1 \left\{ \int_0^{t_1} dt \int_{C_r} \frac{\mu + \kappa}{2} v_{i,j}' v_{i,j}' \, dC_r \right\} \\
&+ \int_0^a dt_1 \left\{ \int_0^{t_1} dt \int_{C_r} \tfrac{1}{2}(|\alpha + \beta| + \gamma) \nu_{i,j}' \nu_{i,j}' \, dC_r \right\} \\
&+ N_7 r^{1/2-\varepsilon} \int_0^a dt_1 \left\{ \int_0^{t_1} dt \left[\int_{C_r} v_i' v_i' \, dC_r \right]^{1/2} \right\} \\
&+ \int_0^a dt_1 \left\{ \int_0^{t_1} dt \int_{\partial S} \left[\gamma \nu_{i,j}' \nu_i' + (\alpha + \beta) \nu_i' \nu_{j,i}' - \frac{\rho j}{2} \nu_i' \nu_i' v_j \right] n_j \, dS \right\}.
\end{aligned} \tag{11}
$$

Put

$$m = \max\left\{\frac{(N_1 + \rho N_2 + 2\kappa N_6)}{\rho}, \frac{(2\kappa + \rho N_2)^2}{\rho j}\right\}$$

and choose

$$a = \max\left\{\frac{t_0}{n} : \frac{t_0}{n} \le \frac{1}{2m}, \quad n \text{ integer}\right\}.$$

Then (11) becomes

$$\begin{aligned} 0 \le P(r) \le \frac{P(r)}{2} &+ \int_0^a dt_1 \int_{C_R} (\tfrac{3}{2}(\mu + \kappa) + N_3 + \kappa) v_i' v_i' \, dC_r \\ &+ \int_0^a dt_1 \int_{C_r} \left[\tfrac{3}{2}(|\alpha + \beta| + \gamma) + \frac{\rho j}{2} N_4 + \kappa\right] \nu_i' \nu_i' \, dC_r \\ &+ \int_0^a dt_1 \left\{\int_0^{t_1} dt \int_{C_r} \frac{\mu + \kappa}{2} v_{i,j}' v_{i,j}' \, dC_r\right\} \\ &+ \int_0^a dt_1 \left\{\int_0^{t_1} dt \int_{C_r} \tfrac{1}{2}(|\alpha + \beta| + \gamma) \nu_{i,j}' \nu_{i,j}' \, dC_r\right\} \\ &+ a^{3/2} N_7 r^{1/2-\varepsilon} \left[\int_0^a dt \int_{C_r} v_i' v_i' \, dC_r\right]^{1/2} \\ &+ \int_0^a dt_1 \left\{\int_0^{t_1} dt \int_{\partial S} \left[\gamma \nu_{i,j}' \nu_i' + (\alpha + \beta) \nu_i' \nu_{j,i}' - \frac{\rho j}{2} \nu_i' \nu_i' v_j\right] n_j \, dS\right\}. \end{aligned}$$

Because of the boundedness conditions on ν_i and its derivatives on the surface of the body ∂S, it follows that the last term on the right of the above equation is bounded. Thus it is clear that we can find a positive number l, by taking the appropriate suprema, such that

$$0 \le P(r) \le l[P'(r) + r^{1/2-\varepsilon}\sqrt{P'(r)} + 1] \tag{12}$$

where $P'(r) \equiv dP(r)/dr$.

We now proceed to show that $P(r) \equiv 0$. First we observe that because of the boundedness conditions (iii) and (iv), $P(r) \sim O(r^3)$ as $r \to \infty$. Suppose $P(r_0) \ne 0$ for some $r_0 > 0$. Then $P(r_0) > 0$. Also $P'(r) > 0$ for $r > r_0$ (because $P(r)$ is the volume integral of nonnegative quantities), hence $P(r)$ is monotonic increasing and

$$0 < P(r_0) \le P(r) \le l[P'(r) + r^{1/2-\varepsilon}\sqrt{P'(r)} + 1], \qquad r > r_0. \tag{13}$$

Thus

$$P'(r) + \sqrt{P'(r)}r^{1/2-\varepsilon} - \left(\frac{P(r_0)}{l} - 1\right) \ge 0 \tag{14}$$

and hence

$$\sqrt{P'(r)} = \frac{1}{2} r^{1/2-\varepsilon}\left[-1 + \sqrt{1 + 4r^{2\varepsilon-1}\left(\frac{P(r_0)}{l} - 1\right)}\right] > 0, \qquad r > r_0. \tag{15}$$

Thus it follows that

$$\sqrt{P'(r)} \geq \Delta r^{1/2-\varepsilon} r^{2\varepsilon-1}, \tag{16}$$

where $\Delta = P(r_0)/l - 1 > 0$.

Thus we have from (13), using (16),

$$P(r) \leq lP'(r)[1 + (r^{1-2\varepsilon}/\Delta)] + l. \tag{17}$$

Assuming that $1 - 2\varepsilon > 0$, we have, for sufficiently large r,

$$P(r) \leq \frac{2l}{\Delta} P'(r) r^{1-2\varepsilon} + l \tag{18}$$

or

$$\frac{P'(r)}{P(r)} \geq \frac{\Delta}{2l} r^{2\varepsilon-1} + \frac{\Delta}{2} \frac{r^{2\varepsilon-1}}{P(r)}. \tag{19}$$

This gives, upon integration,

$$P(r) \geq P(r_0) \exp\left\{ \frac{\Delta}{2l}(r^{2\varepsilon} - r_0^{2\varepsilon}) + f(r) \right\} \tag{20}$$

where $f(r) \sim O(r^{2\varepsilon-3})$ as $r \to \infty$.

But this contradicts the assertion that $P(r) = O(r^3)$ as $r \to \infty$. Hence $P(r) \equiv 0$. It follows that $v_i' = 0$, $\nu_i' = 0$ throughout $E \times (0, a)$. Integrating (9) with respect to t from a to $a + t_1$ and with respect to t_1 from 0 to a and repeating the above arguments, it follows that $v_i' = \nu_i' = 0$ throughout $E \times (a, 2a)$. In this manner we can cover the whole interval T in steps of length a and it follows that $v_i' = \nu_i' = 0$ in $E \times T$. Finally from (5) and the conditions on p it follows that $p' = 0$ in $E \times T$. This proves the theorem, since t_0 was arbitrary.

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