QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVIII, NUMBER 2
JUNE 1990, PAGES 361-370

ON THE EXTREME VARIATIONAL PRINCIPLES FOR NONLINEAR ELASTIC PLATES

By

YANG GAO *

Massachusetts Institute of Technology, Cambridge, Massachusetts
Hefei University of Technology, Anhui, P.R. China

Abstract. The min-maximum variational principles for Von Karman plates are formulated by using the theory of convex analysis. It is shown that the global extremum criteria for both the total potential and complementary variational functional is directly related to a so-called dual gap function. The existence and uniqueness of the variational solutions are proved. And the saddle point condition of the generalized variational principle is also discussed.

1. Introduction. Although there lists substantial literature on the variational problems for geometrical nonlinear elastic plates (cf., e.g., [1–3]), the basic features in this field still remain somewhat obscure. These include the convexity of the total potential energy and the total complementary energy; the criteria for the existence and uniqueness of the variational solutions; and the saddle point condition for the generalized variational principles, etc. These properties are very important in both theoretical analysis and engineering applications.

Recently, a systematic contribution has been given in [4,5] for nonlinear variational boundary value problems. By introducing a so-called dual gap function, a remarkable symmetry, which yields a series of important results in nonlinear mechanics [5–7], can be observed. In this paper, we will present two dual-complementary extreme variational principles for Von Karman plates. It is shown that in the geometrical nonlinear plate theory, this dual gap function gives the criteria not only for the convexity of the total potential and complementary energy, but also the existence and uniqueness of the variational solutions. Moreover, the saddle point condition of the generalized variational principle is also proved to be related to this gap function.

2. Preliminary Relations. Let $S \subset \mathbb{R}^2$ be the midsurface of the undeformed plates with regular boundary $\partial S = \Gamma$, such that $\Gamma = \Gamma_u \cup \Gamma_t$, $\Gamma_u \cap \Gamma_t = \varnothing$. $\mathbf{a}_\alpha (\alpha = 1, 2)$ are the base vectors on the surface and $\mathbf{n}$ is the normal to S. By $\mathbf{u} = \{u_\alpha, w\}$ we

Received July 3, 1989.
*Current address: Department of Mathematics, M.I.T., Cambridge, MA 02139.
This research was in part supported by the Young Scientists' Foundation of China.

denote the displacement vector of the midsurface of the plate

$$\mathbf{u} = u_\alpha \mathbf{a}_\alpha + w\mathbf{n}. \tag{1}$$

Then the governing equations for Von Karman plates can be described as follows:
i) Geometric relations:

$$\gamma_{\alpha\beta}(u_\alpha, w) = \vartheta_{\alpha\beta}(u_\alpha, w) + \frac{1}{2} w_{,\alpha} w_{,\beta} \tag{2a}$$

$$\kappa_{\alpha\beta}(u_\alpha, w) = -w_{,\alpha\beta} \tag{2b}$$

$$u_\alpha = 0, \qquad w = 0, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_u \tag{2c}$$

where

$$\vartheta_{\alpha\beta}(u_\alpha, w) = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right). \tag{2d}$$

ii) Equilibrium relations:

$$N_{\alpha\beta,\beta} = 0 \quad \text{in } S, \tag{3a}$$

$$M_{\alpha\beta,\alpha\beta} + (N_{\alpha\beta} w_{,\beta})_{,\alpha} + \overline{p} = 0 \quad \text{in } S, \tag{3b}$$

$$N_{\alpha\beta} n_\beta - \overline{P}_\alpha = 0 \quad \text{on } \Gamma_t, \tag{3c}$$

$$(N_{\alpha\beta} w_{,\beta} + M_{\alpha\beta,\beta}) n_\alpha + \frac{\partial}{\partial s}(M_{\alpha\beta} t_\alpha n_\beta) - \overline{P} = 0 \quad \text{on } \Gamma_t, \tag{3d}$$

$$M_{\alpha\beta} n_\alpha n_\beta - \overline{M}_n = 0 \quad \text{on } \Gamma_t, \tag{3e}$$

where t, s, n denote the unit vectors on Γ.
iii) Constitutive relations:

$$N_{\alpha\beta} - H_{\alpha\beta\lambda\mu} \gamma_{\lambda\mu} = 0 \quad \text{in } S, \tag{4a}$$

$$M_{\alpha\beta} - h_{\alpha\beta\lambda\mu} \kappa_{\lambda\mu} = 0 \quad \text{in } S, \tag{4b}$$

where

$$H_{\alpha\beta\lambda\mu} = H_{\beta\alpha\lambda\mu} = H_{\lambda\mu\alpha\beta}, \qquad h_{\alpha\beta\lambda\mu} = h_{\beta\alpha\lambda\mu} = h_{\lambda\mu\alpha\beta}.$$

For isotropic plates, we have

$$H_{\alpha\beta\lambda\mu} = \frac{Eh}{2(1+\nu)}\left(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda} + \frac{2\nu}{1-\nu}\delta_{\alpha\beta}\delta_{\lambda\mu}\right),$$

$$h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}.$$

Here h is the plate thickness, E and ν denote the Young's modulus and Poisson's ratio, respectively.

In this paper the abstract notations [7, 8] will be used for convenience. Let $\mathscr{U}$ and $\mathscr{E}$ denote the admissible displacement space and strain space, respectively:

$$\mathscr{U} = \{\mathbf{u} \in L^2(S) \mid \mathbf{u} = \{u_\alpha, w\}\}, \tag{5}$$

$$\mathscr{E} = \{\mathbf{E} \in L^2(S) \mid \mathbf{E} = \{\gamma_{\alpha\beta}, \kappa_{\alpha\beta}\}\}. \tag{6}$$

Their conjugate spaces

$$\mathscr{T} = \{\mathbf{t} \in L^2(S) \mid \mathbf{t} = \{p\} \text{ in } S,\ \mathbf{t} = \{P_\alpha, P, M_n\} \text{ on } \Gamma\}, \tag{7}$$

$$\mathscr{S} = \{\mathbf{T} \in L^2(S) \mid \mathbf{T} = \{\mathbf{N}, \mathbf{M}\}, \mathbf{N} = \{N_{\alpha\beta}\}, \mathbf{M} = \{M_{\alpha\beta}\} \text{ in } S\}, \tag{8}$$

are the admissible force space and stress space, respectively. The bilinear form $\langle \mathbf{E}, \mathbf{T}\rangle = \gamma_{\alpha\beta} N_{\alpha\beta} + \kappa_{\alpha\beta} M_{\alpha\beta}$ puts $\mathscr{E}$ and $\mathscr{S}$ in duality, and we write

$$\langle \mathbf{E}, \mathbf{T}\rangle_S = \iint_S \langle \mathbf{E}, \mathbf{T}\rangle dS.$$

In the same way, the bilinear form $(\mathbf{u}, \mathbf{t})$ puts $\mathscr{U}$ and $\mathscr{T}$ in duality, and we define

$$(\mathbf{u}, \mathbf{t})_{\overline{S}} = \iint_S pw\ dS + \int_\Gamma (P^\alpha u_\alpha + Pw - M_n \frac{\partial w}{\partial n}) d\Gamma.$$

Introducing the geometrical deformation operator $\Lambda:\ \mathscr{U} \to \mathscr{E}$:

$$\Lambda = \begin{Bmatrix} \Lambda_\gamma \\ \Lambda_\kappa \end{Bmatrix} = \begin{Bmatrix} \Lambda_\vartheta + \frac{1}{2}(\Lambda_w \mathbf{u}) \overline{\otimes} \Lambda_w \\ \Lambda_\kappa \end{Bmatrix}, \tag{9}$$

in which,

$$\Lambda_\gamma \mathbf{u} = \left\{\gamma_{\alpha\beta}(u_\alpha, w)\right\},$$
$$\Lambda_\kappa \mathbf{u} = \left\{\kappa_{\alpha\beta}(w)\right\},$$
$$\Lambda_\vartheta \mathbf{u} = \left\{\vartheta_{\alpha\beta}(u_\alpha)\right\},$$
$$\Lambda_w \mathbf{u} = \left\{-w_{,\alpha}\right\}.$$

Then the strain-displacement relations (2) can be written in the following form

$$\Lambda(\mathbf{u})\mathbf{u} - \mathbf{E} = 0 \quad \text{in } S. \tag{10}$$

According to [4,7], the nonlinear operator Λ_γ can be decomposed into two parts,

$$\Lambda(\mathbf{u}) = \Lambda_T(\mathbf{u}) + \Lambda_N(\mathbf{u}) = \begin{Bmatrix} \Lambda_t(\mathbf{u}) \\ \Lambda_\kappa \end{Bmatrix} + \begin{Bmatrix} \Lambda_n(\mathbf{u}) \\ 0 \end{Bmatrix}, \tag{11}$$

$$\Lambda_t(\mathbf{u}) = \Lambda_\vartheta + (\Lambda_w \mathbf{u}) \otimes \Lambda_w, \tag{12}$$

$$\Lambda_n(\mathbf{u}) = -\frac{1}{2}(\Lambda_w \mathbf{u}) \otimes \Lambda_w. \tag{13}$$

We can see here that Λ_n is a symmetric and quadratic operator,

$$\Lambda_n(\mathbf{u})\mathbf{u} = \left\{-\frac{1}{2} w_{,\alpha} w_{,\beta}\right\}. \tag{14}$$

It will play an important part in the analysis of nonlinear plates.

Using the Gauss-Green law, it is easy to prove that

$$\begin{aligned} \langle \Lambda(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S &= \langle \Lambda_t(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S + \langle \Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S + \langle \Lambda_\kappa \mathbf{u}, \mathbf{M}\rangle_S \\ &= \langle \mathbf{u}, \Lambda_t^*(\mathbf{u})\mathbf{N}\rangle_{\overline{S}} + \langle \mathbf{u}, \Lambda_\kappa^* M\rangle_{\overline{S}} - \mathbf{G}(\mathbf{u}, \mathbf{N}). \end{aligned} \tag{15}$$

In which, Λ_t^* and Λ_κ^* are the conjugate operators of Λ_t and $\Lambda_{\kappa'}$ respectively,

$$\Lambda_t^*(\mathbf{u})\mathbf{N} = \begin{cases} -\left\{N_{\alpha\beta,\beta}\right\} & \text{in } S, \\ -\left(N_{\alpha\beta}w_{,\beta}\right)_{,\alpha} & \text{in } S, \\ \left\{N_{\alpha\beta}n_\beta\right\} & \text{on } \Gamma_t, \\ N_{\alpha\beta}w_{,\beta}n_\alpha & \text{on } \Gamma_t, \end{cases} \tag{16}$$

$$\Lambda_\kappa^*\mathbf{M} = \begin{cases} -M_{\alpha\beta,\alpha\beta} & \text{in } S, \\ M_{\alpha\beta}n_\alpha + \frac{\partial}{\partial s}(M_{\alpha\beta}n_\alpha t_\beta) & \text{on } \Gamma_t, \\ M_{\alpha\beta}n_\alpha n_\beta & \text{on } \Gamma_t. \end{cases} \tag{17}$$

$G(\mathbf{u}, \mathbf{N})$ is the so-called dual gap function [4, 7]

$$G(\mathbf{u}, \mathbf{N}) = \langle -\Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S = \iint_S \frac{1}{2} w_{,\alpha} w_{,\beta} N_{\alpha\beta} dS. \tag{18}$$

Let

$$\bar{\mathbf{t}} = \begin{cases} p & \text{in } S, \\ \overline{P}_\alpha & \text{on } \Gamma_t, \\ \overline{P} & \text{on } \Gamma_t, \\ \overline{M}_n & \text{on } \Gamma_t, \end{cases} \tag{19}$$

$$\Lambda_T^*(\mathbf{u}) = \begin{Bmatrix} \Lambda_t^*(\mathbf{u}) \\ \Lambda_\kappa^* \end{Bmatrix}, \tag{20}$$

the equilibrium relations (3) can be written in following abstract form:

$$\Lambda_t^*(\mathbf{u})\mathbf{N} + \Lambda_\kappa^*\mathbf{M} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t, \tag{21a}$$

or

$$\Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t. \tag{21b}$$

Let $\mathscr{U}_a \subset \mathscr{U}$ be a kinematically admissible space

$$\mathscr{U}_a = \left\{\mathbf{u} \in \mathscr{U} \mid u_\alpha = w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_u\right\}. \tag{22}$$

The boundary value problem for Von Karman plates is finding displacements $\mathbf{u} \in \mathscr{U}_a$, such that

$$\begin{cases} \Lambda(\mathbf{u})\mathbf{u} - \mathbf{E} = 0 & \text{in } S, \\ \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 & \text{in } S \cup \Gamma_t, \\ \mathbf{T} - \frac{\partial W}{\partial \mathbf{E}} = 0 & \text{in } S, \end{cases} \tag{22}$$

where

$$W(\mathbf{E}) = W(\gamma, \kappa) = \frac{1}{2} H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}g_{\lambda\mu} + \frac{1}{2} h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu},$$

is the elastic stored energy, which is convex and quadratic.

3. Potential extremum principles. The external potential done by the distributed load $\bar{\mathbf{t}}$ is a linear functional of $\mathbf{u}$ over $\mathscr{U}_a$,

$$F(\mathbf{u}) = (\mathbf{u}, -\bar{\mathbf{t}})_{\bar{S}} = -\iint_S \bar{p} w \, dS - \int_{\Gamma_t} (\bar{P}_\alpha u_\alpha + \bar{P} w - \bar{M}_n \frac{\partial w}{\partial n}) d\Gamma. \tag{24}$$

The total potential energy functional $\Pi: \mathscr{E} \times \mathscr{U}_a \to \mathbb{R}$ should be

$$\Pi(\mathbf{E}, \mathbf{u}) = \iint_S W(\gamma, \kappa) dS + F(\mathbf{u}). \tag{25}$$

Substituting the geometrical relation (11) into (25), we write

$$\Xi(\mathbf{u}) = \Pi(\Lambda \mathbf{u}, \mathbf{u}). \tag{26}$$

Then we have the stationary potential principle:

Among all kinematically admissible fields $\mathbf{u} \in \mathscr{U}_a$, *the stationary points* $\bar{\mathbf{u}}$ *of* $\Xi(\mathbf{u})$ *solve the boundary value problem* (23), *i.e.*,

$$\delta \Xi(\bar{\mathbf{u}}, \mathbf{u}) = 0 \quad \forall \mathbf{u} \in \mathscr{U}_a \Leftrightarrow \Lambda_T^*(\bar{\mathbf{u}}) \frac{\partial W(\Lambda \bar{\mathbf{u}})}{\partial (\Lambda \bar{\mathbf{u}})} - \bar{\mathbf{t}} = 0. \tag{27}$$

We should emphasize that although the stored energy function $W(\mathbf{E}): \mathscr{E} \to \mathbb{R}$ is convex in strain tensor $\mathbf{E}$, $W(\Lambda \mathbf{u}): \mathscr{U} \to \mathbb{R}$ is not definitely convex in displacement $\mathbf{u}$ due to the nonlinearity of geometrical operator Λ. The following theorem shows that the convexity of the total potential depends on the property of the gap function G.

THEOREM 1. Let $\bar{\mathbf{u}}$ and associated $\bar{\mathbf{N}}$ be one stationary point of $\Xi(\mathbf{u})$. If the gap function satisfies inequality

$$G(\mathbf{u}, \bar{\mathbf{N}}(\bar{\mathbf{u}})) = \iint_S \frac{1}{2} \frac{\partial W(\Lambda \bar{\mathbf{u}})}{\partial (\Lambda_\gamma \bar{\mathbf{u}})_{\alpha\beta}} w_{,\alpha} w_{,\beta} dS \geq 0 \quad \forall \mathbf{u} \in \mathscr{U}_a \tag{28}$$

then $\bar{\mathbf{u}}$ minimizes $\Xi(\mathbf{u})$, i.e.,

$$\Xi(\bar{\mathbf{u}}) = \inf_{\mathbf{u} \in \mathscr{U}_a} \Xi(\mathbf{u}). \tag{29}$$

If $\mathscr{U}_a$ is a bounded subset of a relaxive Banach space, then problem (29) has at least one solution. The solution is unique if $G(\mathbf{u} - \bar{\mathbf{u}}, \bar{\mathbf{N}}(\bar{\mathbf{u}})) > 0 \ \forall \mathbf{u} \neq \bar{\mathbf{u}}$.

Proof. From the convexity of the strain energy $W(\mathbf{E})$, the constitutive relation $\bar{\mathbf{T}} = \partial W(\bar{\mathbf{E}}) / \partial \bar{\mathbf{E}}$ yields the following inequality:

$$W(\mathbf{E}) - W(\bar{\mathbf{E}}) \geq \langle \bar{\mathbf{T}}, \mathbf{E} - \bar{\mathbf{E}} \rangle = \langle \bar{\mathbf{N}}, \gamma - \bar{\gamma} \rangle + \langle \bar{\mathbf{M}}, \kappa - \bar{\kappa} \rangle \quad \forall \mathbf{E} \in \mathscr{E}. \tag{30}$$

Suppose $\bar{\mathbf{u}}$ and associated $\bar{\mathbf{E}}, \bar{\mathbf{T}}$ are the stationary points of Ξ. For any given $\mathbf{u} \in \mathscr{U}_a$, we put $\delta \mathbf{u} = \mathbf{u} - \bar{\mathbf{u}}$, then the Taylor formulation gives

$$\begin{aligned} \mathbf{E}(\bar{\mathbf{u}} + \delta \mathbf{u}) &= \mathbf{E}(\bar{\mathbf{u}}) + \delta \mathbf{E}(\bar{\mathbf{u}}; \delta \mathbf{u}) + \frac{1}{2} \delta^2 \mathbf{E}(\bar{\mathbf{u}}; \delta \mathbf{u}, \delta \mathbf{u}) \\ &= \Lambda(\bar{\mathbf{u}}) + \Lambda_T(\bar{\mathbf{u}}) \delta \mathbf{u} - \Lambda_N(\delta \mathbf{u}) \delta \mathbf{u}. \end{aligned}$$

Substituting into inequality (30) gives

$$W(\Lambda(\mathbf{u})) - W(\Lambda(\bar{\mathbf{u}})) \geq \langle \Lambda_T(\bar{\mathbf{u}})\delta\mathbf{u}, \overline{\mathbf{T}}\rangle + \langle -\Lambda_N(\delta\mathbf{u})\delta\mathbf{u}, \overline{\mathbf{T}}\rangle \quad \forall \mathbf{u} \in \mathscr{U}_a.$$

Integrating (30) and using the Gauss-Green transformation (15), we obtain

$$\iint_S W(\Lambda(\mathbf{u}))dS - \iint_S W(\Lambda(\bar{\mathbf{u}}))dS \geq (\delta\mathbf{u}, \Lambda_T^*(\bar{\mathbf{u}}\overline{\mathbf{T}})_{\overline{S}} + \langle -\Lambda_n(\delta\mathbf{u})\delta\mathbf{u}, \overline{\mathbf{N}}\rangle_S$$
$$= (\mathbf{u} - \bar{\mathbf{u}}, \bar{\mathbf{t}})_{\overline{S}} + G(\delta\mathbf{u}, \overline{\mathbf{T}}) \qquad \forall \mathbf{u} \in \mathscr{U}_a$$

which means

$$\Xi(\mathbf{u}) - \Xi(\bar{\mathbf{u}}) \geq G(\mathbf{u} - \bar{\mathbf{u}}, \overline{\mathbf{N}}) \quad \forall \mathbf{u} \in \mathscr{U}_a.$$

Obviously, for any given $\mathbf{u} \in \mathscr{U}_a$, if the gap function G satisfies (28), we have

$$\Xi(\mathbf{u}) - \Xi(\bar{\mathbf{u}}) \geq 0 \quad \forall \mathbf{u} \in \mathscr{U}_a,$$

i.e., $\Xi(\mathbf{u})$ is convex over $\mathscr{U}_a$. Hence the stationary points $\mathbf{u}$ of Ξ minimize Ξ. If $\mathscr{U}_a$ is a bounded subset of a reflexive Banach space, the theory of convex analysis (see e.g. [8]) assures that the optimal problem (29) has at least one solution. If the gap function G satisfies $G(\mathbf{u} - \bar{\mathbf{u}}, \overline{\mathbf{N}}(\bar{\mathbf{u}})) > 0 \; \forall \mathbf{u} \neq \bar{\mathbf{u}}$, then $\Xi(\mathbf{u})$ is strictly convex over $\mathscr{U}_a$. So the solution is unique.

4. Conjugate transformation and complementary energy. According to the theory of convex analysis, the conjugate function of the stored energy W can be given by using the Legendre-Fenchel transformation:

$$W^*(\mathbf{T}) = \sup_{\mathbf{E}\in\mathscr{E}}\{\langle \mathbf{E}, \mathbf{T}\rangle - W(\mathbf{E})\}$$
$$= \frac{1}{2}H^{-1}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu} + \frac{1}{2}h^{-1}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}. \tag{31}$$

Obviously $W^*: \mathscr{S} \to \mathbb{R}$ is convex and quadratic. The conjugate function of $F(\mathbf{u})$ is given by

$$F^*(-\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}) = \sup_{\mathbf{u}\in\mathscr{U}_a}\{(\mathbf{u}, -\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T})_{\overline{S}} - F(\mathbf{u})\}$$
$$= \sup_{\mathbf{u}\in\mathscr{U}_a}\{(\mathbf{u}, -\Lambda_T^*(\mathbf{u})\mathbf{T} + \bar{t})_{\overline{S}}\}$$
$$= \begin{cases} 0 & \text{if } \Lambda_T^*(\bar{\mathbf{u}})\mathbf{T} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t \\ +\infty & \text{otherwise.} \end{cases} \tag{32}$$

So the conjugate functional of the total potential energy Π should be

$$\Pi^*(\mathbf{T}, -\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}) = \iint_S W^*(\mathbf{T})dS + F^*(-\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}). \tag{33}$$

Unfortunately, for geometrical nonlinear problems, the conjugate functional Π^* of the total potential energy is not the total complementary energy functional [4]. The difference is just the gap function G. Hence, the total complementary energy functional for geometrical nonlinear thin elastic plates is

$$\Pi_c^*(T, -\Lambda^*(\mathbf{u})\mathbf{T}) = \Pi^*(\mathbf{T}, -\Lambda_T^*(\mathbf{u})\mathbf{T}) + G(\mathbf{u}, \mathbf{N})$$
$$= \iint_S W^*(\mathbf{T})dS + \iint_S \frac{1}{2}w_{,\alpha}w_{,\beta}N_{\alpha\beta}dS + F^*(-\Lambda_T^*(\mathbf{u})\mathbf{T}). \tag{34}$$

We note that the variational arguments in Π_c^* is not only the stress tensor $\mathbf{T} = \left\{ \begin{smallmatrix} \mathbf{N} \\ \mathbf{M} \end{smallmatrix} \right\}$, but also the displacement $\mathbf{u} = \left\{ \begin{smallmatrix} u_\alpha \\ w \end{smallmatrix} \right\}$ due to the nonlinearity of the geometric operator Λ.

Let $\mathscr{S}_a \subset \mathscr{U} \times \mathscr{S}$ be a statically admissible space

$$\mathscr{S}_a = \{(u, T) \in \mathscr{U} \times \mathscr{S} \mid \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 \ \text{ in } S \cup \Gamma_t\}, \tag{35}$$

then over $\mathscr{S}_a$, Π_c^* is degenerated to

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}) = \iint_S W^*(\mathbf{M}, \mathbf{N})dS + \iint_S \frac{1}{2} w_{,\alpha} w_{,\beta} \mathbf{N}_{\alpha\beta} dS.$$

We have the stationary complementary energy principle of the geometrical nonlinear elastic plates:

Among all the $(\mathbf{u}, \mathbf{t}) \in \mathscr{S}_a$ *the stationary points* $(\bar{\mathbf{u}}, \overline{\mathbf{T}})$ *of* Π_c^* *solve the boundary value problem* (23)*, i.e.*,

$$\delta\Pi_c^*(\overline{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\overline{\mathbf{T}}) = 0 \Leftrightarrow \begin{cases} \Lambda\bar{\mathbf{u}} - \frac{\partial W^*(\overline{\mathbf{T}})}{\partial \overline{\mathbf{T}}} = 0 & \text{in } S, \\ \bar{u}_\alpha = \overline{w} = \frac{\partial \overline{w}}{\partial n} = 0 & \text{on } \Gamma_u. \end{cases} \tag{36}$$

For a given stationary point $\bar{\mathbf{u}}$ of Π_c^*, over $\mathscr{S}_a$, let

$$\begin{aligned} \Xi^*(T) &= -\Pi_c^*(\mathbf{T}, -\Lambda^*(\bar{\mathbf{u}})\mathbf{T}) \\ &= -\iint_S W^*(\mathbf{T})dS - \iint_S \frac{1}{2}\overline{w}_{,\alpha}\overline{w}_{,\beta}N_{\alpha\beta}dS. \end{aligned} \tag{37}$$

Then the dual variational principle for nonlinear shells can be described as the following:

THEOREM 2. Among all statically admissible fields $(\bar{\mathbf{u}}, \mathbf{T}) \in \mathscr{S}_a$, the solutions $\overline{\mathbf{T}}$ of the boundary value problem (23) maximize $\Xi(\mathbf{T})$, i.e.,

$$\Xi^*(\overline{\mathbf{T}}) = \sup_{\mathbf{T} \in \mathscr{S}_a} \Xi^*(\mathbf{T}). \tag{38}$$

Proof. Considering the convexity of the complementary energy W^*, the constitutive equation $\overline{\mathbf{E}} = \frac{\partial W^*(\overline{\mathbf{T}})}{\partial \overline{\mathbf{T}}}$ yields the following inequality:

$$W^*(\mathbf{T}) - W^*(\overline{\mathbf{T}}) \geq \langle \overline{\mathbf{E}}, \mathbf{T} - \overline{\mathbf{T}} \rangle \quad \forall \mathbf{T} \in \mathscr{S}, \tag{39a}$$

i.e.,

$$W^*(\mathbf{N}, \mathbf{M}) - W^*(\overline{\mathbf{N}}, \overline{\mathbf{M}}) \geq \langle \bar{\gamma}, \mathbf{N} - \overline{\mathbf{N}} \rangle + \langle \bar{\kappa}, \mathbf{M} - \overline{\mathbf{M}} \rangle \quad \forall (\mathbf{N}, \mathbf{M}) \in \mathscr{S}. \tag{39b}$$

Substituting the geometric relation $\overline{\mathbf{E}} = \Lambda\bar{\mathbf{u}}$ into (39) gives

$$\begin{aligned} W^*(\mathbf{N}, \mathbf{M}) - W^*(\overline{\mathbf{N}}, \overline{\mathbf{M}}) &\geq \langle \Lambda_t(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N} - \overline{\mathbf{N}} \rangle + \langle \Lambda_\kappa \bar{\mathbf{u}}, \mathbf{M} - \overline{\mathbf{M}} \rangle \\ &\quad + \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N} - \overline{\mathbf{N}} \rangle \quad \forall (\mathbf{N}, \mathbf{M}) \in \mathscr{S}. \end{aligned} \tag{40}$$

Integrating (40) over the midsurface S and using the Gauss-Green transformation, we obtain

$$\iint_S W^*(\mathbf{N}, \mathbf{M})dS - \iint_S W^*(\overline{\mathbf{N}}, \overline{\mathbf{M}})dS \geq (\bar{\mathbf{u}}, \Lambda_t^*(\bar{\mathbf{u}})\mathbf{N} + \Lambda_\kappa^*(\bar{\mathbf{u}})\mathbf{M})_{\overline{S}} + \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N}\rangle_S$$
$$- (\bar{\mathbf{u}}, \Lambda_t^*(\bar{\mathbf{u}})\overline{\mathbf{N}} + \Lambda_\kappa^*(\bar{\mathbf{u}})\overline{\mathbf{M}})_{\overline{S}} - \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \overline{\mathbf{N}}\rangle$$
$$= (\bar{\mathbf{u}}, \Lambda_T^*(\bar{\mathbf{u}})\mathbf{T} - \bar{\mathbf{t}})_{\overline{S}} - G(\bar{\mathbf{u}}, \mathbf{N}) + G(\bar{\mathbf{u}}, \overline{\mathbf{N}})$$
$$\forall \mathbf{T} \in \mathscr{S}. \tag{41}$$

So for any given statically admissible stresses $(\bar{\mathbf{u}}, \mathbf{T}) \in \mathscr{S}_a$, we have

$$\Xi^*(\mathbf{T}) - \Xi^*(\overline{\mathbf{T}}) \leq 0 \quad \forall \mathbf{T} \in \mathscr{S}_a, \tag{42}$$

which means that $\overline{\mathbf{T}}$ maximizes Ξ^* over $\mathscr{S}_a$.

Furthermore, we can prove the following complementary extremum principle (two-fields complementary variational extremum principle):

THEOREM 3. If the gap function $G(\mathbf{u}, \mathbf{N}) \geq 0$ for any given $(\mathbf{u}, \mathbf{N}) \in \mathscr{S}_a$, the solutions $(\bar{\mathbf{u}}, \overline{\mathbf{T}})$ of the boundary value problem (23) minimize Π_c^*, i.e.,

$$\Pi_c^*(\overline{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\overline{\mathbf{T}}) = \inf_{(\mathbf{u}, \mathbf{t}) \in \mathscr{S}_a} \Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}). \tag{43}$$

Proof. With $\delta\mathbf{u} = \bar{\mathbf{u}} - \mathbf{u}$,

$$\begin{aligned}\mathbf{E}(\bar{\mathbf{u}}) &= \mathbf{E}(\mathbf{u}) + \Lambda_T(\mathbf{u})\delta\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u} \\ &= \Lambda_T(\mathbf{u})\mathbf{u} + \Lambda_N(\mathbf{u})\mathbf{u} + \Lambda_T(\mathbf{u})\delta\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u} \\ &= \Lambda_T(\mathbf{u})\bar{\mathbf{u}} + \Lambda_N(\mathbf{u})\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u}.\end{aligned}$$

It means that

$$\gamma_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) = \vartheta_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) + w_{,\alpha}\bar{w}_{,\beta} - \frac{1}{2}w_{,\alpha}w_{,\beta} + \frac{1}{2}\delta w_{,\alpha}\delta w_{,\beta},$$
$$\kappa_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) = \kappa_{\alpha\beta}(\bar{u}_\alpha, \bar{w}).$$

Substituting into Eq. (39), we obtain

$$W^*(\mathbf{T}) - W^*(\mathbf{T}) \geq \langle \Lambda_T(\mathbf{u})\bar{\mathbf{u}}, \mathbf{T}\rangle + \langle \Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle - \langle \Lambda_N(\delta\mathbf{u})\delta\mathbf{u}, \mathbf{T}\rangle$$
$$- \langle \Lambda_T(\bar{\mathbf{u}})\bar{\mathbf{u}}, \overline{\mathbf{T}}\rangle - \langle \Lambda_N(\bar{\mathbf{u}})\bar{\mathbf{u}}, \overline{\mathbf{T}}\rangle \quad \forall \mathbf{T} \in \mathscr{S}.$$

By integrating and using the Gauss-Green transformation, we have

$$\iint_S W^*(\mathbf{T})dS + \langle -\Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S - \iint_S W^*(\overline{\mathbf{T}})dS - \langle -\Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \overline{\mathbf{N}}\rangle_S$$
$$\geq (\bar{\mathbf{u}}, \Lambda_T^*(\mathbf{u})\mathbf{T})_{\overline{S}} - (\bar{\mathbf{u}}, \Lambda_T^*(\bar{\mathbf{u}})\overline{\mathbf{T}})_{\overline{S}} - \langle \Lambda_n(\delta\mathbf{u})\delta\mathbf{u}, \mathbf{N}\rangle_s \tag{44}$$

For any given $(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a$, inequality (44) yields

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})(\mathbf{T}) - \Pi_c^*(\overline{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\overline{\mathbf{T}}) \geq G(\delta\mathbf{u}, \mathbf{N}) \quad \forall(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a. \tag{45}$$

By assumption $G(\mathbf{u}, \mathbf{N}) \geq 0$ for any $(\mathbf{u}, \mathbf{T}) \in \mathscr{U} \times \mathscr{S}$, we have

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}) - \Pi_c^*(\overline{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\overline{\mathbf{T}}) \geq 0 \quad \forall(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a,$$

which means that $(\bar{\mathbf{u}}, \overline{\mathbf{T}})$ minimize Π_c^* over $\mathscr{S}_a$.

5. Generalized saddle point variational principle. The equilibrium constraint in extremum principle (43) can be relaxed by Lagrangian $L: \mathscr{U} \times \mathscr{S} \to \mathbb{R}$:

$$\begin{aligned} L(\mathbf{u}, \mathbf{T}) &= (\mathbf{u}, \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}})_{\overline{S}} - \iint_S W^*(\mathbf{T})dS - G(\mathbf{u}, \mathbf{N}) \\ &= \langle \Lambda_T(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - \iint_S W^*(\mathbf{T})dS - \langle \Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - (\mathbf{u}, \bar{\mathbf{t}})_{\overline{S}} \\ &= \langle \Lambda(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - \iint_S W^*(\mathbf{T})dS - (\mathbf{u}, \bar{\mathbf{t}})_{\overline{S}}. \end{aligned} \tag{46}$$

The generalized variational principle for Von Karman plates states that

Among all $\mathbf{u} \in \mathscr{U}_a$ *and* $\mathbf{T} \in \mathscr{S}$, *the stationary points* $(\bar{\mathbf{u}}, \overline{\mathbf{T}})$ *of* $L(\mathbf{u}, \mathbf{T})$ *solve the boundary value problem* (23), *i.e.*,

$$\delta L(\bar{\mathbf{u}}, \overline{\mathbf{T}}) = 0 \Leftrightarrow \begin{cases} \Lambda(\bar{\mathbf{u}})\bar{\mathbf{u}} - \frac{\partial W^*(\overline{\mathbf{T}})}{\partial \overline{\mathbf{T}}} = 0 & \text{in } S, \\ \Lambda_T^*(\bar{\mathbf{u}})\overline{\mathbf{T}} - \bar{\mathbf{t}} = 0 & \text{in } S \cup \Gamma_t. \end{cases} \tag{47}$$

The following theorem states the saddle point property of Lagrangian $L(\mathbf{u}, \mathbf{T})$.

THEOREM 4. Among all $\mathbf{u} \in \mathscr{U}_a$ and $\mathbf{T} \in \mathscr{S}$, if the gap function G satisfies $G(\mathbf{u}, \mathbf{T}) \geq 0$, then the solutions $(\bar{\mathbf{u}}, \overline{\mathbf{T}})$ of the boundary value problem (23) are the saddle points of $L(\mathbf{u}, \mathbf{T})$, i.e.,

$$L(\bar{\mathbf{u}}, \overline{\mathbf{T}}) = \inf_{\mathbf{u} \in \mathscr{U}_a} \sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}). \tag{48}$$

Proof. Since

$$\begin{aligned} \sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}) &= \sup_{\mathbf{T} \in \mathscr{S}} \left\{ \iint_S [\langle \mathbf{E}(\mathbf{u}), \mathbf{T}\rangle - W^*(\mathbf{T})]dS \right\} + (\mathbf{u}, -\bar{\mathbf{t}})_{\overline{S}} \\ &= \iint_S W(\mathbf{E}(\mathbf{u}))dS + (\mathbf{u}, -\bar{\mathbf{t}})_{\overline{S}} = \Xi(\mathbf{u}). \end{aligned} \tag{49}$$

Recalling Theorem 1, we have

$$\inf_{\mathbf{u} \in \mathscr{U}_a} \sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}) = \inf_{\mathbf{u} \in \mathscr{U}_a} L(\mathbf{u}, \overline{\mathbf{T}}) = \inf_{\mathbf{u} \in \mathscr{U}_a} \Xi(\mathbf{u}) = L(\bar{\mathbf{u}}, \overline{\mathbf{T}}).$$

Concluding remarks. The results in the present paper show that in the theory of geometrical nonlinear plates, there exists a dual gap between the total potential energy functional $\Xi(\mathbf{u})$ and the total complementary energy functional $\Xi^*(\mathbf{T})$. This dual gap function provides a global extremum criteria for the primal-dual variational problems. It is obvious that if $\mathbf{N}(x) \geq 0$ a.e. in S, the gap function

$$G(\mathbf{u}, \mathbf{N}) = \iint_S \frac{1}{2} w_{,\alpha} w_{,\beta} N_{\alpha\beta} dS \geq 0 \quad \forall w \in \mathscr{U}_a.$$

In this case, both Ξ and Ξ^* are convex and the problem (23) is stable. Otherwise the plate may be instable. The application in the one dimensional case can be found in [9].

References

[1] S. N. Atluri, *Alternate stress and conjugate strain measures and mixed variational formulations involving rigid rotations, for computational analysis of finitely deformed solids, with applications to plates and shells.* I, Comput. & Structures **18** (1), 93–116 (1983)

[2] W. R. Bielski and J. J. Telega, *A note on duality for Von Kármán plates in the case of the obstacle problem*, Arch. Mech. **37** (1), 135–141 (1985)

[3] P. G. Ciarlet and P. Rabier, *Les Equations de Von Kármán*, Lecture Notes in Mathematics, Vol. 826, Springer–Verlag, Berlin, 1980

[4] Yang Gao and Gilbert Strang, *Geometrical nonlinearity: Potential energy, complementary energy and the gap function*, 17th IUTAM, Grenoble, France, 1988, Quart. Appl. Math. **47** (3), 487–504 (1989)

[5] Yang Gao and Gilbert Strang, *Dual extremum principles in finite deformation elastoplatic analysis*, to appear in Acta Appl. Math., (1990)

[6] Yang Gao, *Opposite principles in nonlinear conservative systems*, Adv. in Appl. Math. **10** (3), 370–377 (1989)

[7] Yang Gao and Y. K. Cheung, *On the extremum complementary energy principles for nonlinear elastic shells*, Internat. J. Solids and Structures **26** (1990)

[8] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976

[9] Yang Gao and T. Wierzbicki, *Bounding theorem in finite plasticity with hardening effect*, Quart. Appl. Math. **47** (3), 395–403 (1989)