ON THE STABILITY OF VARLEY-DAY SOLUTIONS FOR HARMONIC MATERIALS

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1. Introduction. Varley and Day have considered the problem of determining deformations that can occur in a homogeneous, isotropic elastic body at constant pressure on the boundary and null Cauchy deviatoric stress and have found certain solutions to this problem [1] which possess constant principal stretches. Ericksen [2-3] has conjectured that the solutions obtained in [1] should be at best neutrally stable. This conjecture was proved for plane deformations by Adeleke [4].

In a recent paper [5] nonconformal plane deformations characterized by the requirement that the ratio of the principal stretches of the deformation is constant were shown to be Varley-Day type solutions (in the absence of body forces) for certain unconstrained isotropic elastic solids. By employing the stability criterion due to Beatty [6] the solutions in this class were also shown in [5] to be at best neutrally stable. The solutions found in [5] however are not the only plane nonconformal Varley-Day solutions with nonconstant principal stretches. Such solutions may occur whenever the Baker-Ericksen (B-E) inequality [7, Sec. 51] is violated but the weakened B-E inequality is not [5].

One of the implications of the requirement that a Harmonic material [9] admits a regular state of uniaxial tension in plane strain is that the B-E inequality cannot be satisfied at all states of deformation [10]. Specifically, at deformations characterized by the condition that the sum of their principal stretches is a certain material-dependent constant the B-E inequality fails but the weakened B-E inequality does not [9, 10]. As it will become clear in the following this condition characterizes the class of plane Varley-Day solutions for Harmonic materials.² ³

In this paper we show by means of examples that among the Varley-Day solutions for Harmonic materials which admit a regular state of uniaxial tension there are solutions with nonconstant principal stretches. Then, by using Beatty's stability...
criterion for the infinitesimal stability of equilibrium configurations of unconstrained elastic solids subject to uniform hydrostatic loading everywhere on the boundary [6], we show that the Varley-Day solutions can be at best neutrally stable and, in particular, that the Varley-Day solutions with constant principal stretches are unstable. An illustrative example is considered in the last section of the paper.

2. Preliminaries. Consider the plane deformations described by a suitable smooth and invertible transformation

\[ x = x(X) \]  

(2.1)

where \( X \) and \( x \) are points that belong, respectively, to the domains \( D \) and \( D \) of the two-dimensional Euclidean Space \( \mathbb{R}^2 \). Associated with such deformations is the deformation gradient, given by

\[ F = \frac{\partial x}{\partial X} \]  

(2.2)

for which

\[ \det F > 0, \]  

(2.3)

where \( \det(\cdot) \) stands for determinant. The deformation gradient can be uniquely represented in the form

\[ F = VQ, \]  

(2.4)

where \( Q \) is a proper orthogonal tensor and \( V \) is a symmetric, positive-definite tensor often referred to as the left-stretch tensor. By the spectral theorem there is an orthogonal basis \( \{ e^1, e^2 \} \) such that

\[ V = \lambda_1 e^1 \otimes e^1 + \lambda_2 e^2 \otimes e^2, \]  

(2.5)

where \( \otimes \) denotes the tensor product of two vectors. The scalars \( \lambda_1, \lambda_2 \) are called principal stretches while the vectors \( e^1, e^2 \) are the principal axes of strain. One set of invariants of \( V \) is related to the principal stretches by the relations

\[ \text{tr } V^2 \equiv I = \lambda_1^2 + \lambda_2^2, \quad \det V \equiv J = \lambda_1 \lambda_2, \]  

(2.6)

where \( \text{tr} \) denotes the trace operator.

The plane part of the stress-deformation relation for a plane deformation of an isotropic homogeneous elastic solid can be written in the form

\[ T = t(V) = \varphi_0 I + \varphi_1 V^2. \]  

(2.7)

In Eq. (2.7) \( I \) is the identity tensor, \( \varphi_0 \) and \( \varphi_1 \) are scalar-valued functions of the invariants of \( V \), and \( T \) is the Cauchy stress tensor. When a strain-energy function \( W \) exists, then

\[ W = W(I, J) \equiv \tilde{W}(\lambda_1, \lambda_2), \]  

(2.8)

and

\[ \varphi_0 = \frac{\partial W}{\partial J}, \quad \varphi_1 = \frac{2}{J} \frac{\partial W}{\partial I}. \]  

(2.9)

If we combine (2.5), (2.7), and (2.9) we find that

\[ T = t_1 e^1 \otimes e^1 + t_2 e^2 \otimes e^2. \]  

(2.10)
where the principal stresses \( t_1, t_2 \) are given by
\[
t_i = \frac{\partial W}{\partial J} + \frac{2t_i^2}{J} \frac{\partial W}{\partial I}, \quad i = 1, 2.
\]

The equilibrium condition, in the absence of body forces, is
\[
\text{div}\, \mathbf{T} = 0, \quad \text{on } \overline{D},
\]
where \( \text{div}(\cdot) \) denotes the divergence operator with respect to \( \mathbf{x} \).

Harmonic materials are unconstrained elastic solids with a strain-energy density function in plane strain given by [9]
\[
W(I, J) = 2\mu[H(Q) - J], \quad Q \equiv (I + 2J)^{1/2} = \lambda_1 + \lambda_2, \quad (2.13)
\]
where \( \mu \) is a constant and \( H(\cdot) \in C^2(0, \infty) \). Combining (2.7), (2.9), and (2.13) we find
\[
\mathbf{T} = 2\mu \left\{ \left[ \frac{H'(Q)}{Q} - 1 \right] 1 + \frac{H'(Q)}{QJ} \mathbf{v}^2 \right\}.
\]
The requirement that both the strain-energy and the stress vanish in the undeformed configuration (i.e., for \( \lambda_1 = \lambda_2 = 1 \)) yields
\[
H(2) = H'(2) = 1. \quad (2.15)
\]

A material is strongly-elliptic at \( \mathbf{F} \) if [7, Sec. 44] \(^4\)
\[
A_{\alpha\beta\gamma\sigma} a_\alpha a_\beta b_\gamma b_\sigma > 0, \quad \alpha, \beta, \gamma, \sigma = 1, 2, \quad (2.16)
\]
for every pair of nonzero vectors \( \mathbf{a}(a_\alpha), \mathbf{b}(b_\beta) \). Here \( A_{\alpha\beta\gamma\sigma} \) denote the components of the elasticity tensor \( \mathbf{A} \) which is the Fréchet derivative of the Piola stress \(^5\)
\[
\mathbf{S}(\mathbf{F}) = J(\varphi_0 \mathbf{F}^{-T} + \varphi_1 \mathbf{F}). \quad (2.17)
\]
If Cartesian coordinates are assigned to \( D \) and \( \overline{D} \) the components of \( \mathbf{A} \) are given by [11]
\[
A_{\alpha\beta\gamma\sigma} = 4 \frac{\partial^2 W}{\partial I^2} F_{\alpha\beta} F_{\gamma\sigma} + 2J \frac{\partial^2 W}{\partial I \partial J} [F_{\beta\gamma}^{-1} F_{\alpha\sigma} + F_{\alpha\beta} F_{\gamma\sigma}^{-1}] + J^2 F_{\alpha\beta}^{-1} F_{\gamma\sigma}^{-1} \frac{\partial^2 W}{\partial J^2}
+ 2 \frac{\partial W}{\partial I} \delta_{\alpha\gamma} \delta_{\beta\sigma} + J \frac{\partial W}{\partial J} [F_{\beta\gamma}^{-1} F_{\alpha\sigma}^{-1} - F_{\gamma\sigma}^{-1} F_{\alpha\beta}^{-1}], \quad (2.18)
\]
where \( F_{\alpha\beta} \) denote the components of \( \mathbf{F} \), \( F_{\alpha\beta}^{-1} \) the components of \( \mathbf{F}^{-1} \), and \( \delta_{\alpha\beta} \) the two-dimensional Kronecker symbol.

We shall suppose that strong ellipticity holds for infinitesimal deformations of Harmonic materials. As shown in [11], this is the case if and only if
\[
\mu > 0, \quad H''(2) > 0. \quad (2.19)
\]

\(^4\) The summation convention over repeated indices is employed throughout the text.

\(^5\) \( \mathbf{F}^T \) is the transpose of \( \mathbf{F} \) and \( \mathbf{F}^{-T} \) is the inverse of the transpose of \( \mathbf{F} \).
It then follows [10] that the strain-energy density (2.13) is positive, except in the undeformed state, if and only if

\[ H(Q) > \frac{Q^2}{4}, \quad Q \in (0, \infty) - \{2\}. \]  

(2.20)

From (2.6), (2.7), and (2.10) we find that at conformal deformations (with \( \lambda_1 = \lambda_2 \equiv \lambda \))

\[ t_1 = t_2 = \varphi_0(2J, J) + J \varphi_1(2J, J) \equiv t = \bar{t}(\lambda). \]  

(2.21)

The classical Pressure-Compression inequality is the requirement that \( t \) be a strictly increasing function of \( \lambda \) [7, Sec. 51]. Slightly stronger than this is the requirement that

\[ \frac{d\bar{t}}{d\lambda} > 0, \quad \lambda \in (0, \infty). \]  

(2.22)

For Harmonic materials this condition becomes

\[ QH''(Q) - H'(Q) > 0, \quad Q \in (0, \infty). \]  

(2.23)

At a deformation with \( \lambda_1 \neq \lambda_2 \) the stress-deformation relation (2.7) is said to satisfy the B-E inequality if [7, Sec. 51]

\[ \varphi_1 > 0 \]  

(2.24)

and the weakened B-E inequality if

\[ \varphi_1 \geq 0. \]  

(2.25)

At conformal deformations B-E inequality reduces to (2.25) [7, Sec. 51].

The inequality (2.24) is satisfied by the constitutive equation of Harmonic materials (2.14) if and only if

\[ H'(Q) > 0, \quad Q \in (0, \infty). \]  

(2.26)

However, we shall suppose that the Harmonic materials admit a regular state of uniaxial tension in plane strain [10]. As shown in [10], for this condition to be satisfied by the Harmonic materials which obey (2.15), (2.19), and (2.23) it is necessary and sufficient that there exists a number \( Q_0 \in (1, 2) \) such that

\[ H'(Q_0) = 0, \quad H'(Q)/Q \to 1 \quad \text{as} \quad Q \to \infty \quad \text{and} \quad H''(Q) > 1 \quad \text{for} \quad Q \in (Q_0, \infty). \]  

(2.27)

A simple continuity argument shows that (2.27) implies

\[ H''(Q_0) \geq 1. \]  

(2.28)

An inspection of (2.14) and (2.12) now shows that the Varley-Day solutions for the class of Harmonic materials under consideration are characterized by the condition

\[ \lambda_1 + \lambda_2 = Q_0. \]  

(2.29)

\[ ^6 \text{Note that at deformations with} \lambda_1 + \lambda_2 = Q_0 \text{ the Coleman-Noll (C-N) convexity condition [7, Sec. 87] merely requires} H''(Q_0) > \frac{1}{2} [10]. \]
3. Nonconformal Varley-Day solutions with nonconstant principal stretches. We start with the deformation
\[ r = Rf(\Xi), \quad \theta = g(R, \Xi), \] (3.1)
where \((r, \theta)\) and \((R, \Xi)\) are spatial and referential polar coordinates, respectively. The functions \(f(\cdot)\) and \(g(\cdot, \cdot)\) are to be determined such that the conditions (2.3) and (2.29) are satisfied.

The physical components of the deformation gradient corresponding to (3.1) are given by
\[ F = \begin{bmatrix} f & f_r \xi' \\ Rf g, R & f g, R \end{bmatrix} \] (3.2)
where \((\cdot)\) stands for differentiation with respect to \((\cdot)\) (whether it is partial or total).

From (2.4), (2.6), (3.2), and (2.29) we find the condition
\[ I + 2J = f^2 + f_r^2, R + 2(f_r g, R - Rf f, \Xi) = Q_0^2. \] (3.3)

To obtain our first example we assume that \(g, R = 0\) so that (3.3) reduces to
\[ f^2(1 + g, R^2) + f_r^2, R = Q_0^2. \] (3.4)

Taking \(f, R = k, k = \text{const.}\), \(k \in (0, Q_0)\), we obtain from (3.4)
\[ f = k\Xi + k_1, \quad g = k^{-1}\sqrt{Q_0^2 - k^2}\ln(k\Xi + k_1) - \Xi + k_2, \] (3.5)
where \(k_1 > 0\) and \(k_2\) are constants of integration. Using (2.4), (2.6), (3.2), and (3.5) it is easy to show that
\[ I - 2J = 4(k\Xi + k_1)^2 - 4\sqrt{Q_0^2 - k^2}(k\Xi + k_1) + Q_0^2 \neq \text{const.} \] (3.6)

Since there exists a number \(\hat{k}\), independent on \(Q_0\), such that, for \(k_1 \in (0, \hat{k})\), the condition (2.3) is satisfied it follows that (3.1) and (3.5) define a deformation which meets our requirements.

Next we suppose that, in addition to (3.3), the functions \(f\) and \(g\) satisfy
\[ f_r, \Xi = -Rf g, R. \] (3.7)
Combining (3.7) and (3.3) and taking
\[ f = \alpha_0 \exp(\alpha_1\Xi), \quad \alpha_0, \alpha_1 = \text{const.}, \] (3.8)
in the resulting equation yields
\[ g, \Xi = \alpha_0^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2} - 1, \] (3.9)
where we assumed
\[ 4\alpha_0^2\alpha_1^2 \leq Q_0^2 \exp(-4\alpha_1\pi). \] (3.10)

From (3.7), (3.8), and (3.9) we deduce
\[ g(R, \Xi) = -\alpha_0^{-1}\alpha_1^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2} - \Xi - \alpha_1 \ln R \]
\[ + 2\arctan\left[2^{-1}\alpha_0^{-1}\alpha_1^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2}\right] + c, \] (3.11)
where \(c = \text{const.}\).
As condition (2.3) is satisfied if and only if \( g_{,2} > 0 \), we choose \( \alpha_0 \), \( \alpha_1 \) so that

\[
0 < \alpha_0 < 1, \quad 4\alpha_0^2 \alpha_1^2 + \alpha_0 \leq \exp(-4\alpha_1 \pi).
\]  

(3.12)

Noticing that

\[
I - 2J = \alpha_0^2 (1 - g_{,2})^2 \exp(2\alpha_1 \Xi) \neq \text{const.},
\]

(3.13)

we conclude that (3.8), (3.11), (3.1), and (3.12) define another deformation of the type sought here.

4. Stability. A criterion for the infinitesimal stability of equilibrium configurations of unconstrained homogeneous elastic solids subject to uniform hydrostatic loading \( p \) everywhere on the boundary has been obtained by Beatty in [6]. The specialization of this criterion to plane deformations of isotropic bodies takes the form

\[
K \equiv \int_D \{ \text{tr}[H(t(V))H^T] + H \cdot C(F)[H] - p[2 \det E + \text{tr}(RR^T)] \} \, d\overline{D} > 0
\]

(4.1)

for all nonzero plane infinitesimal displacements \( u \). Here \( H(H_{\alpha\beta}) \) is the gradient of \( u \), \( E \) and \( R \) the symmetric and skew-symmetric parts of \( H \), respectively, and

\[
H \cdot C(F)[H] \equiv \left( \frac{1}{J} F_{\nu\sigma} F_{\xi\eta} A_{\xi\eta,\nu\sigma} - \delta_{\alpha\sigma} t_{\beta\sigma} \right) H_{\alpha\beta} H_{\gamma\sigma},
\]

(4.2)

where \( t_{\beta\sigma} \) denote the components of \( t(V) \) and \( A_{\xi\eta,\nu\sigma} \) are given by (2.18). If \( K \) becomes negative for some \( u \) then the equilibrium configuration is unstable while if \( K = 0 \) for some \( u \neq 0 \) but nonnegative for all \( u \) the equilibrium configuration is neutrally stable [6].

If \( F = VQ \) is the deformation gradient corresponding to a Varley-Day solution for a Harmonic material it follows from (2.7) and (2.14) that

\[
dt(V) = p \mathbf{1} = -2\mu \mathbf{1}.
\]

(4.3)

A straightforward but lengthy computation based on (2.18), (2.13), (4.2), (4.1), (4.3), and

\[
\frac{\partial^2 \tilde{W}}{\partial \lambda_n \partial \lambda_\beta} = 4\lambda_1 \lambda_2 \frac{\partial^2 W}{\partial J^2} + 2J \left( \frac{\lambda_n}{\lambda_\beta} + \frac{\lambda_\beta}{\lambda_n} \right) \frac{\partial^2 W}{\partial I \partial J}
\]

\[
+ \frac{J^2}{\lambda_n \lambda_\beta} \frac{\partial^2 W}{\partial J^2} + 2\delta_{n\beta} \frac{\partial W}{\partial I} + (1 - \delta_{n\beta}) \frac{\partial W}{\partial J}
\]

(4.4)

leads to

\[
K = 2\mu \int_D \left\{ \left[ \frac{\lambda_2}{\lambda_1} H''(Q_0) - 1 \right] E_{11}^2 + 2H''(Q_0)E_{11}E_{22} + \left[ \frac{\lambda_2}{\lambda_1} H''(Q_0) - 1 \right] E_{22}^2 \right\} \, d\overline{D}.
\]

(4.5)

Since there exists a nonzero displacement vector \( u \) such that

\[
E_{11} = E_{22} = 0
\]

(4.6)

the deformation with deformation gradient \( F \) can be at best neutrally stable.

\( F_{\alpha\beta} \) are the components of \( F \) with respect to the basis \( \{e^1, e^2\} \).
From (4.5) we deduce that a Varley-Day deformation with constant principal stretches is neutrally stable if and only if

\[
[H''(Q_0)]^{-1} \leq \frac{\lambda_1}{\lambda_2} \leq H''(Q_0) \quad (4.7)
\]

and

\[
H''(Q_0) \left( \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \right) \leq 1. \quad (4.8)
\]

Since

\[
\{(\lambda_1, \lambda_2) | \lambda_1 + \lambda_2 = Q_0, \ H''(Q_0)(\lambda_1\lambda_2^{-1} + \lambda_2\lambda_1^{-1}) \leq 1, \ H''(Q_0) \geq 1\} = \emptyset, \quad (4.9)
\]

the condition (4.8) is violated in any Varley-Day deformation. In the case of Varley-Day deformations with constant principal stretches this fact precludes neutral stability as clearly, given any such deformation, an infinitesimal displacement \( u \) which renders \( K < 0 \) must always exist.

5. An example. The Harmonic material with

\[
H(Q) = \frac{1}{2} Q^2 + \frac{Q}{m-1} \left( \frac{2}{Q} \right)^m + \frac{1 + m}{1 - m}, \quad m \geq 0, \ m \neq 1, \quad (5.1)
\]

which was recently considered in [12], satisfies the restrictions (2.15), (2.19), (2.20), (2.23), and (2.27) with

\[
Q_0 = 2^{m/(m+1)} \quad (5.2)
\]

and

\[
H''(Q_0) = 1 + m. \quad (5.3)
\]

REFERENCES


\[\text{Footnote 6: See the remark following (2.27) and footnote 6.}\]